GLASNIK MATEMATIČKI
Vol. 37(57)(2002), 149 – 163

STRONG EXPANSIONS FOR TRIADS OF SPACES

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Abstract. Lisica and Mardešić introduced the notion of coherent expansion of a space to develop a strong shape theory for arbitrary topological spaces. Mardešić then introduced the notion of strong ANR-expansion of a space, which is an intermediate notion between ANR-resolution and ANR-expansion, and showed that this notion can be used to define the same strong shape category. The purpose of this paper is to generalize those notions to triads of spaces and show that resolutions of triads are strong expansions of triads and that strong expansions of triads are coherent expansions of triads. Hence the strong shape theory for triads is well-defined, and all notions and results on strong expansions generalize to triads of spaces. As an invariant, strong homotopy groups for triads are defined, and the excision property with respect to strong homotopy groups and Mayer-Vietoris sequences for strong homology groups are discussed.

1. Introduction

Lisica and Mardešić [3] defined a coherent expansion of an arbitrary topological space, and, using this notion, they defined the strong shape category whose objects are aribitrary topological spaces. In their theoy, spaces are represented as ANR-resolutions, which are shown to be coherent expansions definig the strong shape of the spaces. As an intermediate notion between ANR-resolution and ANR-expansion, Mardešić [11] introduced a strong ANR-expansion of an arbitrary topological space. Strong expansions are suitable for working in the strong shape category, while ANR-resolutions and ANR-expansions are suitable for working in the category of topological spaces and the shape category, respectively. In this paper, we develop the strong shape theory for triads of spaces by extending to triads of spaces the notions of

2000 *Mathematics Subject Classification.* 54C56, 55P55, 55Q07.
Key words and phrases. Triad, strong shape, resolution, strong expansion.

strong expansion and coherent expansion. More precisely, it is shown that every resolution of a triad is a strong expansion of a triad and that every strong expansion of a triad is a coherent expansion of a triad. Thus the strong shape category for triads of spaces is obtained. As an invariant, strong homotopy groups for triads are defined, and the excision property with respect to strong homotopy groups and Mayer-Vietoris sequences for strong homology groups are discussed.

The original work for the strong shape theory for single spaces is found in Lisica and Mardešić [3]. The extended work for the strong shape theory for pairs of spaces is found in Lisica and Mardešić [4, 5] and Mardešić [10]. This paper closely follows the ideas of those papers. A generalization of the Kuratowsky-Wojdislawski embedding theorem for triads, which is a key lemma in this paper, is obtained in the following section. The version for pairs is found in [6, §. 3]. ANR-resolutions for triads were studied by Mardešić [8] and used in the study of the excision property of strong homology by Lisica and Mardešić [5]. Related work for the ordinary shape theory for triads is found in Miyata [13, 14].

Throughout the paper, spaces mean topological spaces, and maps mean continuous maps. A *map of triads* $f : (X; X_0, X_1) \to (Y; Y_0, Y_1)$ means a map $f : X \to Y$ such that $f(X_0) \subseteq Y_0$ and $f(X_1) \subseteq Y_1$. A *homotopy of triads* means a map of triads $h : (X \times I; X_0 \times I, X_1 \times I) \to (Y; Y_0, Y_1)$. Let $f, g : X \to Y$ be functions between sets. For any covering $\mathcal{V}$ of Y, $(f, g) < \mathcal{V}$ means that f and g are $\mathcal{V}$-near. For any covering $\mathcal{U}$ of a set X, if A is a subset of X, then $\mathcal{U}|A$ means the covering $\{U \cap A : U \in \mathcal{U}\}$ of A, and the *star* of A in X with respect to $\mathcal{U}$ means the set $\mathrm{St}(A, \mathcal{U}) = \cup\{U \in \mathcal{U} : U \cap A \neq \emptyset\}$. Also the *star covering* of $\mathcal{U}$ means the covering $\mathrm{St}\,\mathcal{U} = \{\mathrm{St}(U, \mathcal{U}) : U \in \mathcal{U}\}$. For each space X, let $\mathrm{Cov}(X)$ denote the set of all normal open coverings of X.

2. Metric triads and ANR triads

In this section we prove a generalization of the Kuratowsky-Wojdislawski embedding theorem (see [15, Theorem 2, p. 35]) for triads and obtain a key lemma concerning a homotopy property of ANR triads. The version for pairs was obtained by Mardešić and Lisica [6, Theorem 5].

A *triad of spaces* $(X; X_0, X_1)$ means a space X and two subspaces X_0 and X_1 of X such that $X = X_0 \cup X_1$. A triad of spaces $(X; X_0, X_1)$ is an *ANR triad* if X_0 and X_1 are closed subsets of X and $X, X_0, X_1, X_0 \cap X_1$ are ANR's, and a triad of spaces $(X; X_0, X_1)$ is a *polyhedral triad* (resp., *CW triad*) if X is a polyhedron (resp., CW-complex) and X_0 and X_1 are subpolyhedra (resp., subcomplexes) of X.

The following theorem is a generalization of the Kuratowsky-Wojdislawski theorem for triads:

THEOREM 2.1. *Let $(X; X_0, X_1)$ be a triad of metric spaces such that X_0 and X_1 are closed. Then there exist a normed vector space L and an embedding $h : X \hookrightarrow L$ with the following properties:*

1.) *$h(X_0)$, $h(X_1)$ and $h(X_0 \cap X_1)$ are closed in their convex hulls K_0, K_1 and K_{01}, respectively;*
2.) *$h(X) \cap K_i = h(X_i)$, $i = 0, 1$, and $h(X) \cap K_{01} = h(X_0 \cap X_1)$;*
3.) *$K_0 \cap K_1 = K_{01}$; and*
4.) *K_0 and K_1 are closed in $K_0 \cup K_1$.*

For the proof, we need the two lemmas stated below. First of all, the following lemma is proved in [6, Lemma 5]:

LEMMA 2.2. *Let $f : X \to K$ and $f' : X \to K'$ be any maps between spaces. Then if f is an embedding, then the map $h = f \times f' : X \to K \times K'$ is also an embedding. Moreover, if $f(X)$ is closed in K and K' is a Hausdorff space, then $h(X)$ is closed in $K \times K'$.*

The second lemma is the following, which is essentially proved in [6, Lemma 6].

LEMMA 2.3. *Let $(X; X_0, X_1)$ be a triad of metric spaces such that X_0 and X_1 are closed. Then there exist a normed vector space L and an embedding $h : X \hookrightarrow L$ with the property that $h(X_0)$, $h(X_1)$ and $h(X_0 \cap X_1)$ are closed in their convex hulls K_0, K_1 and K_{01}, respectively.*

PROOF. Let L be the space of bounded real-valued functions with the sup-norm. For each $x \in X$, define the map $f_x : X \to \mathbb{R}$ by $f_x(z) = \mathrm{d}(z, x)$ for $z \in X$, where d is the metric on X, and then define the map $h : X \to L$ by $h(x) = f_x$. This map h is an isometric embedding. [6, Lemma 6] shows that for each closed subset A of X, $h(A)$ is closed in its convex hull, and hence the map h certainly satisfies the condition in the assertion. $\square$

PROOF OF THEOREM 2.1. Let $h' : X \to L'$ be an embedding into a normed vector space L' such that $h'(X_0)$, $h'(X_1)$ and $h'(X_0 \cap X_1)$ are closed in their convex hulls K_0', K_1' and K_{01}', respectively (see Lemma 2.3), and let $\varphi_i : X_i \to [0, 1]$, $i = 0, 1$, be a map such that $\varphi_i^{-1}(0) = X_0 \cap X_1$. Now consider the normed vector space $L = L' \times \mathbb{R} \times \mathbb{R}$ where the norm is defined by $||(x, s, t)|| = ||x|| + |s| + |t|$. Then Lemma 2.2 implies that the maps $h_0 = h'|_{X_0} \times \varphi_0 \times 0 : X_0 \to L$ and $h_1 = h'|_{X_1} \times 0 \times \varphi_1 : X_1 \to L$ are respectively embeddings as closed subspaces of L. Since $h_0|_{X_0 \cap X_1} = h'|_{X_0 \cap X_1} \times 0 \times 0 = h_1|_{X_0 \cap X_1}$, h_0 and h_1 define a map $h : X \to L$. Indeed, h is an embedding. Let $K_{01} = K_{01}' \times 0 \times 0 \subseteq L$. Then K_{01} is the convex hull of $h(X_0 \cap X_1)$. Let K_i be the convex hull of $h(X_i)$ in L for $i = 0, 1$. So $K_0 \subseteq L' \times \mathbb{R} \times 0$ and $K_1 \subseteq L' \times 0 \times \mathbb{R}$. Then $h(X) \cap K_i = h(X_i)$, $i = 0, 1$. Indeed, for the case $i = 0$, if $x \in X \setminus X_0$, $\varphi_1(x) > 0$ and hence $h(x) \notin K_0$, and similarly for the other case. Also $h(X) \cap K_{01} = h(X_0 \cap X_1)$ since if $x \in X \setminus X_0 \cap X_1$, either $\varphi_0(x) > 0$ or

$\varphi_1(x) > 0$, so $h(x) \notin K_{01}$. Thus property 2) has been verified. By Lemma 2.2, $(h' \times \varphi_0)(X_i)$, $i = 0, 1$, and $(h' \times \varphi_0)(X_0 \cap X_1)$ are closed in $L' \times \mathbb{R}$, so $h(X_0)$, $h(X_1)$ and $h(X_0 \cap X_1)$ are closed in $L' \times \mathbb{R} \times 0$, $L' \times 0 \times \mathbb{R}$ and $L' \times \mathbb{R} \times 0$ and hence are closed in K_0, K_1 and K_{01}, respectively. Thus property 1) is verified. Since $N_0 = K_{01} \cup (K_0' \times (0,1] \times 0)$ and $N_1 = K_{01} \cup (K_1' \times 0 \times (0,1])$ are convex subsets of L containing $h(X_0)$ and $h_(X_1)$, respectively, then $K_i \subseteq N_i$, $i = 0, 1$. So, $K_{01} \subseteq K_0 \cap K_1 \subseteq N_0 \cap N_1 = K_{01}$, and hence $K_0 \cap K_1 = K_{01}$, verifying 3). This then implies $(K_0 \cup K_1) \setminus K_0 = K_1 \setminus K_{01}$ and $(K_0 \cup K_1) \setminus K_1 = K_0 \setminus K_{01}$. Since also K_{01} is closed in N_0 and N_1, and hence closed in K_0 and K_1, then K_0 and K_1 are closed in $K_0 \cup K_1$, verifying property 4). ∎

Lemma 2.4. *Let $(X; X_0, X_1)$ be a triad of metric spaces such that X_0 and X_1 are closed, let $(P; P_0, P_1)$ be an ANR triad, and let A be a closed subset of X. Then every map of triads $f : (A; A \cap X_0, A \cap X_1) \to (P; P_0, P_1)$ admits an extension $\overline{f} : (U; U \cap X_0, U \cap X_1) \to (P; P_0, P_1)$ for some open neighborhood U of A in X.*

Proof. See [14, Lemma 2.5]. ∎

Lemma 2.4 immediately implies

Theorem 2.5. *Let $(K; K_0, K_1)$ be a triad of metric spaces, and let $(P; P_0, P_1)$ be an ANR triad such that P is a closed subspace of K and $P \cap K_i = P_i$, $i = 0, 1$. Then there exist an open neighborhood U of P in K and a retraction of triads $r : (U; U \cap K_0, U \cap K_1) \to (P; P_0, P_1)$.*

The following theorem is a key result for later sections:

Theorem 2.6. *For each ANR triad $(P; P_0, P_1)$ and for each $\mathcal{U} \in \mathrm{Cov}(P)$, there exists $\mathcal{V} \in \mathrm{Cov}(P)$ such that any $\mathcal{V}$-near maps $g_0, g_1 : (Z; Z_0, Z_1) \to (P; P_0, P_1)$ admit a $\mathcal{U}$-homotopy of triads $H : (Z \times I; Z_0 \times I, Z_1 \times I) \to (P; P_0, P_1)$ such that $H(\ , 0) = g_0$ and $H(\ , 1) = g_1$.*

Proof. By Theorems 2.1 and 2.5, there is an embedding of P into a normed vector space L as a closed subspace with the following properties:

1.) P_i is closed in its convex hull K_i, $i = 0, 1$;
2.) $P \cap K_i = P_i$, $i = 0, 1$; and
3.) K_0 and K_1 are closed in $K = K_0 \cup K_1$,

and also there is a retraction of triads $r : (U; U \cap K_0, U \cap K_1) \to (P; P_0, P_1)$ for some open neighborhood U of P in K. Now let $\mathcal{U} \in \mathrm{Cov}(P)$, and take $\mathcal{U}' \in \mathrm{Cov}(U)$ such that $\mathcal{U}' < r^{-1}\mathcal{U}$ and $\mathcal{U}'|K_0$ and $\mathcal{U}'|K_1$ consist of convex sets. Let $\mathcal{V} = \mathcal{U}'|P \in \mathrm{Cov}(P)$. We wish to show that this $\mathcal{V}$ has the desired property. Let $g_0, g_1 : (X; X_0, X_1) \to (P; P_0, P_1)$ be $\mathcal{V}$-near maps of triads. For each $x \in X$, if $x \in X_i$, then $g_0(x), g_1(x) \in P_i \cap U_x'$ for some $U_x' \in \mathcal{U}'$, and since $P_i \cap U_x' \subseteq K_i \cap U_x'$ and $K_i \cap U_x'$ is convex, $g_0(x)$ and $g_1(x)$ can be joined by a line segment in $K_i \cap U_x'$. Thus we obtain a map $H : X \times I \to U$ such that

$H(\ ,0)=g_0$, $H(\ ,1)=g_1$ and $H(x\times I)\subseteq K_i\cap U'_x$. Moreover, this H defines a homotopy of triads $H:(X\times I;X_0\times I,X_1\times I)\to(U;U\cap K_0,U\cap K_1)$, and the map of triads $G=rH:(X\times I;X_0\times I,X_1\times I)\to(P;P_0,P_1)$ defines a desired homotopy of triads. ∎

3. Resolutions of triads are strong expansions

Let Top^T denote the category of triads of spaces and maps of triads.

Recall that a *resolution* of a triad $(X;X_0,X_1)$ is a morphism $\boldsymbol{p}=(p_\lambda):(X;X_0,X_1)\to(\boldsymbol{X};\boldsymbol{X}_0,\boldsymbol{X}_1)=((X_\lambda;X_{0\lambda},X_{1\lambda}),p_{\lambda\lambda'},\Lambda)$ in pro-Top^T with the following two properties [8]:

(R1) Let $(P;P_0,P_1)$ be an ANR triad, and let $\mathcal{V}\in\mathrm{Cov}(P)$. Then every map of triads $f:(X;X_0,X_1)\to(P;P_0,P_1)$ admits $\lambda\in\Lambda$ and a map of triads $g:(X_\lambda;X_{0\lambda},X_{1\lambda})\to(P;P_0,P_1)$ such that $(gp_\lambda,f)<\mathcal{V}$; and

(R2) Let $(P;P_0,P_1)$ be an ANR triad. Then for each $\mathcal{V}\in\mathrm{Cov}(P)$ there exists $\mathcal{V}'\in\mathrm{Cov}(P)$ such that whenever $\lambda\in\Lambda$ and $g,g':(X_\lambda;X_{0\lambda},X_{1\lambda})\to(P;P_0,P_1)$ are maps of triads such that $(gp_\lambda,g'p_\lambda)<\mathcal{V}'$, then $(gp_{\lambda\lambda'},g'p_{\lambda\lambda'})<\mathcal{V}$ for some $\lambda'\geq\lambda$.

A resolution $\boldsymbol{p}$ is called an *ANR-resolution* (resp., *polyhedral resolution*) if all $(X_\lambda;X_{0\lambda},X_{1\lambda})$ are ANR triads (resp., polyhedral triads).

For each triad of spaces $(X;X_0,X_1)$, for each morphism $\boldsymbol{p}=(p_\lambda):(X;X_0,X_1)\to(\boldsymbol{X};\boldsymbol{X}_0,\boldsymbol{X}_1)=((X_\lambda;X_{0\lambda},X_{1\lambda}),p_{\lambda\lambda'},\Lambda)$ in pro-Top^T and for each class of triads $\mathcal{C}$, consider the following two properties:

(S1) For each $(P;P_0,P_1)\in\mathcal{C}$ and for each map $f:(X;X_0,X_1)\to(P;P_0,P_1)$ there exist $\lambda\in\Lambda$ and a map $g:(X_\lambda;X_{0\lambda},X_{1\lambda})\to(P;P_0,P_1)$ such that $gp_\lambda\simeq f$ as maps of triads; and

(S2) For each $(P;P_0,P_1)\in\mathcal{C}$ and for each $\lambda\in\Lambda$ and pair of maps of triads $f_0,f_1:(X_\lambda;X_{0\lambda},X_{1\lambda})\to(P;P_0,P_1)$ with a homotopy of triads $F:(X\times I;X_0\times I,X_1\times I)\to(P;P_0,P_1)$ such that $F(\ ,0)=f_0p_\lambda$ and $F(\ ,1)=f_1p_\lambda$, there exist $\lambda'\geq\lambda$ and a homotopy of triads $H:(X_{\lambda'}\times I;X_{0\lambda'}\times I,X_{1\lambda'}\times I)\to(P;P_0,P_1)$ such that $H(\ ,0)=f_0p_{\lambda\lambda'}$ and $H(\ ,1)=f_1p_{\lambda\lambda'}$, and $H(p_{\lambda'}\times 1_I)\simeq F\ \mathrm{rel}(X\times\partial I)$ as maps of triads.

A morphism $\boldsymbol{p}:(X;X_0,X_1)\to(\boldsymbol{X};\boldsymbol{X}_0,\boldsymbol{X}_1)$ is said to be a *strong expansion* if it satisfies conditions (S1) and (S2) with respect to the class of ANR triads.

Lemma 3.1. *Let $\mathcal{C}$ be a class of triads homotopy dominated by a class of triads $\mathcal{C}'$. Then for each $i=1,2$, condition (Si) for $\mathcal{C}$ implies condition (Si) for $\mathcal{C}'$.*

Proof. The proof for [12, Lemma 7.4] can be easily modified for the case of triads. ∎

Theorem 3.2. *(Miyata [13]) The following statements are equivalent:*

1.) $(X; X_0, X_1)$ *has the homotopy type of a polyhedral triad;*
2.) $(X; X_0, X_1)$ *has the homotopy type of an ANR triad; and*
3.) $(X; X_0, X_1)$ *has the homotopy type of a CW triad.*

By Lemma 3.1 and Theorem 3.2, for strong expansions, we can take the class of polyhedral triads (or CW triads) instead of the class of ANR triads.

Let $\mathrm{H}(\mathsf{Top}^T)$ denote the category of triads of spaces and the homotopy classes of maps of triads, and let HPol^T denote the full subcategory of $\mathrm{H}(\mathsf{Top}^T)$ whose objects are triads of spaces which have the homotopy type of a polyhedral triad (equivalently, an ANR triad). We call a strong expansion $\boldsymbol{p} : (X; X_0, X_1) \to (\boldsymbol{X}; \boldsymbol{X}_0, \boldsymbol{X}_1)$ a *strong* HPol^T-expansion if all triads $(X_\lambda; X_{0\lambda}, X_{1\lambda})$ are objects of HPol^T.

The following is the main theorem in this section:

Theorem 3.3. *Each resolution* $\boldsymbol{p} : (X; X_0, X_1) \to (\boldsymbol{X}; \boldsymbol{X}_0, \boldsymbol{X}_1)$ *is a strong expansion.*

Proof. We can prove the theorem as in [12, Theorem 7.6], using Theorem 2.6 and Lemma 3.4 below. ∎

Lemma 3.4. *Let*

$$\boldsymbol{p} = (p_\lambda) : (X; X_0, X_1) \to (\boldsymbol{X}; \boldsymbol{X}_0, \boldsymbol{X}_1) = ((X_\lambda; X_{0\lambda}, X_{1\lambda}), p_{\lambda\lambda'}, \Lambda)$$

be a resolution of a triad of spaces $(X; X_0, X_1)$*, and let* $(P; P_0, P_1)$ *be an ANR triad and* $\mathcal{U} \in \mathrm{Cov}(P)$*. Suppose* $f_0, f_1 : (X_\lambda; X_{0\lambda}, X_{1\lambda}) \to (P; P_0, P_1)$ *are maps of triads and* $F : (X \times I; X_0 \times I, X_1 \times I) \to (P; P_0, P_1)$ *is a homotopy of triads such that* $F(\ , 0) = f_0 p_\lambda$ *and* $F(\ , 1) = f_1 p_\lambda$*. Then there exist* $\lambda' \geq \lambda$ *and a homotopy of triads* $H : (X_{\lambda'} \times I; X_{0\lambda'} \times I, X_{1\lambda'} \times I) \to (P; P_0, P_1)$ *such that* $H(\ , 0) = f_0 p_{\lambda\lambda'}$*,* $H(\ , 1) = f_1 p_{\lambda\lambda'}$ *and* $(F, H(p_{\lambda'} \times 1)) < \mathcal{U}$*.*

Proof. We can prove the lemma as for [12, Lemma 7.10], using Theorem 2.6, Lemmas 3.5 and 3.6 in the below and the fact that every resolution of triads $\boldsymbol{p} : (X; X_0, X_1) \to (\boldsymbol{X}; \boldsymbol{X}_0, \boldsymbol{X}_1)$ induces a resolution $\boldsymbol{p}|_X : X \to \boldsymbol{X}$. ∎

Lemma 3.5. *Let* $(P; P_0, P_1)$ *be an ANR triad, and let* $\mathcal{V} \in \mathrm{Cov}(P)$*. Suppose* $K : (Y \times I; Y_0 \times I, Y_1 \times I) \to (P; P_0, P_1)$ *is a homotopy of triads from a triad of spaces* $(Y; Y_0, Y_1)$ *and* $L, M : (Y \times I; Y_0 \times I, Y_1 \times I) \to (P; P_0, P_1)$ *are* $\mathcal{V}$*-homotopies of triads such that* $K(\ , 0) = L(\ , 1)$ *and* $K(\ , 1) = M(\ , 1)$*. Then there exists a homotopy of triads* $H : (Y \times I; Y_0 \times I, Y_1 \times I) \to (P; P_0, P_1)$ *such that* $H(\ , 0) = L(\ , 0)$*,* $H(\ , 1) = M(\ , 0)$ *and* $(H, K) < \mathrm{St}\,\mathcal{V}$*.*

Proof. The same proof as for [12, Lemma 7.12] applies to our case. ∎

Lemma 3.6. *Let*

$$\boldsymbol{p} = (p_\lambda) : (X; X_0, X_1) \to (\boldsymbol{X}; \boldsymbol{X}_0, \boldsymbol{X}_1) = ((X_\lambda; X_{0\lambda}, X_{1\lambda}), p_{\lambda\lambda'}, \Lambda)$$

be a resolution of a triad $(X; X_0, X_1)$*, and let* Y *be a compact Hausdorff space. Then the morphism*

$$\begin{aligned}\boldsymbol{p} \times 1_Y = (p_\lambda \times 1_Y) : (X; X_0, X_1) \times Y &\to (\boldsymbol{X}; \boldsymbol{X}_0, \boldsymbol{X}_1) \times Y \\ &= ((X_\lambda \times Y; X_{0\lambda} \times Y, X_{1\lambda} \times Y), p_{\lambda\lambda'} \times 1_Y, \Lambda)\end{aligned}$$

is a resolution.

Proof. First note that if $(P; P_0, P_1)$ is an ANR triad and if Y is a compact Hausdorff space, then $(P^Y; P_0^Y, P_1^Y)$ is an ANR triad. We will verify (R1) and (R2) for $\boldsymbol{p} \times 1_Y$. For (R1), let $f : (X; X_0, X_1) \times Y \to (P; P_0, P_1)$ be a map of triads into an ANR triad, and let $\mathcal{V} \in \mathrm{Cov}(P)$. Then f induces a map of triads $f' : (X; X_0, X_1) \to (P^Y; P_0^Y, P_1^Y)$. Let $\mathcal{V}' \in \mathrm{Cov}(P)$ such that $\mathrm{St}\,\mathcal{V}' < \mathcal{V}$, and let $U(\mathcal{V}')$ be the open covering of P^Y consisting of the subsets $U(f, \mathcal{V}') = \{g \in P^Y : (f, g) < \mathcal{V}'\}$ for $f \in P^Y$. (R1) for $\boldsymbol{p} : (X; X_0, X_1) \to (\boldsymbol{X}; \boldsymbol{X}_0, \boldsymbol{X}_1)$ implies that there exist $\lambda \in \Lambda$ and a map of triads $g' : (X_\lambda; X_{0\lambda}, X_{1\lambda}) \to (P^Y; P_0^Y, P_1^Y)$ such that $(f', g'p_\lambda) < U(\mathcal{V}')$. g' induces a map of triads $g : (X_\lambda \times Y; X_{0\lambda} \times Y, X_{1\lambda} \times Y) \to (P; P_0, P_1)$. Then $(f, g(p_\lambda \times 1_Y)) < \mathrm{St}\,\mathcal{V}' < \mathcal{V}$. Indeed, let $x \in X$ and $y \in Y$. Then $f'(x), g'(p_\lambda(x)) \in U(h, \mathcal{V}')$ for some $h \in P^Y$, so $f(x, y) = f'(x)(y), h(y) \in V_1$ and $g(p_\lambda(x), y) = g'(p_\lambda(x))(y), h(y) \in V_2$ for some $V_1, V_2 \in \mathcal{V}'$.

For (R2), let $(P; P_0, P_1)$ be an ANR triad, and let $\mathcal{V} \in \mathrm{Cov}(P)$. Let $\mathcal{V}''$ be an open covering of P such that $\mathrm{St}\,\mathcal{V}'' < \mathcal{V}$. By (R2) for $\boldsymbol{p} : (X; X_0, X_1) \to (\boldsymbol{X}; \boldsymbol{X}_0, \boldsymbol{X}_1)$, there exists $\mathcal{V}' \in \mathrm{Cov}(P)$ so that whenever $f_0', f_1' : (X_\lambda; X_{0\lambda}, X_{1\lambda}) \to (P^Y; P_0^Y, P_1^Y)$ are maps of triads such that $(f_0' p_\lambda, f_1' p_\lambda) < U(\mathcal{V}')$, then $(f_0' p_{\lambda\lambda'}, f_1' p_{\lambda\lambda'}) < U(\mathcal{V}'')$ for some $\lambda' \geq \lambda$. Suppose that $f_0, f_1 : (X_\lambda \times Y; X_{0\lambda} \times Y, X_{1\lambda} \times Y) \to (P; P_0, P_1)$ are maps of triads such that $(f_0(p_\lambda \times 1_Y), f_1(p_\lambda \times 1_Y)) < \mathcal{V}'$. Then f_0 and f_1 induce maps of triads $f_0', f_1' : (X_\lambda; X_{0\lambda}, X_{1\lambda}) \to (P^Y; P_0^Y, P_1^Y)$ such that $(f_0' p_\lambda, f_1' p_\lambda) < U(\mathcal{V}')$. So, $(f_0' p_{\lambda\lambda'}, f_1' p_{\lambda\lambda'}) < U(\mathcal{V}'')$ for some $\lambda' > \lambda$, which implies that $(f_0(p_{\lambda\lambda'} \times 1_Y), f_1(p_{\lambda\lambda'} \times 1_Y)) < \mathrm{St}\,\mathcal{V}'' < \mathcal{V}$. ∎

We have the following existence theorem for a resolution of triads:

Theorem 3.7. 1.) *(Mardešić [8]) Every triad* $(X; X_0, X_1)$ *of spaces admits an ANR-resolution* $\boldsymbol{p} = (p_\lambda) : (X; X_0, X_1) \to (\boldsymbol{X}; \boldsymbol{X}_0, \boldsymbol{X}_1) = ((X_\lambda; X_{0\lambda}, X_{1\lambda}), p_{\lambda\lambda'}, \Lambda)$ *such that* Λ *is cofinite and* $X_\lambda = \mathrm{Int}(X_{0\lambda}) \cup \mathrm{Int}(X_{1\lambda})$ *for each* $\lambda \in \Lambda$*.*

2.) *(Miyata [13]) Every triad* $(X; X_0, X_1)$ *of spaces admits a polyhedral resolution*

$$\boldsymbol{p} = (p_\lambda) : (X; X_0, X_1) \to (\boldsymbol{X}; \boldsymbol{X}_0, \boldsymbol{X}_1) = ((X_\lambda; X_{0\lambda}, X_{1\lambda}), p_{\lambda\lambda'}, \Lambda)$$

such that Λ *is cofinite.*

Theorems 3.3 and 3.7 imply

THEOREM 3.8. *Every triad of spaces* $(X; X_0, X_1)$ *admits a cofinite strong* HPol^T*-expansion.*

We also need the following result for later sections:

THEOREM 3.9. *(Miyata [14]) Every triad* $(X; X_0, X_1)$ *of spaces such that* X_0 *and* X_1 *are closed and* $X_0 \cap X_1$ *is normally embedded in* X *admits a polyhedral resolution*

$$\boldsymbol{p} = (p_\lambda) : (X; X_0, X_1) \to (\boldsymbol{X}; \boldsymbol{X}_0, \boldsymbol{X}_1) = ((X_\lambda; X_{0\lambda}, X_{1\lambda}), p_{\lambda\lambda'}, \Lambda)$$

with Λ *being cofinite such that the following restrictions are resolutions:*

$$\begin{cases} \boldsymbol{p}|_X = (p_\lambda|_X) : X \to \boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}|_{X'_\lambda}, \Lambda) \\ \boldsymbol{p}|_{X_i} = (p_\lambda|_{X_i}) : X_i \to \boldsymbol{X}_i = (X_{i\lambda}, p_{\lambda\lambda'}|_{X'_{i\lambda}}, \Lambda),\ i = 0, 1 \\ \boldsymbol{p}|_{X_0 \cap X_1} = (p_\lambda|_{X_0 \cap X_1}) : X_0 \cap X_1 \to \boldsymbol{X}_0 \cap \boldsymbol{X}_1 \\ \qquad\qquad = (X_{0\lambda} \cap X_{1\lambda}, p_{\lambda\lambda'}|_{X_{0\lambda'} \cap X_{1\lambda'}}, \Lambda) \end{cases}.$$

4. COHERENT HOMOTOPY FOR TRIADS

In this section we define the coherent homotopy category CH(pro-Top^T) for triads. This is completely analogous to the case of single spaces, which was introduced by Lisica and Mardešić [3] (see also [12, Chapters 7, 8]). Here we will recall the definitions and outline the important results for triads.

A *coherent map* of inverse systems $\boldsymbol{f} = (f, f_\mu) : (\boldsymbol{X}; \boldsymbol{X}_0, \boldsymbol{X}_1) \to (\boldsymbol{Y}; \boldsymbol{Y}_0, \boldsymbol{Y}_1)$ consists of an increasing function $f : M \to \Lambda$ and maps $f_{\boldsymbol{\mu}} : (X_{f(\mu)}; X_{0f(\mu)}, X_{1f(\mu)}) \times \Delta^n \to (Y_{\mu_0}; Y_{0\mu_0}, Y_{1\mu_0})$ for $\boldsymbol{\mu} = (\mu_0, \dots, \mu_n) \in M_n$ with the following property:

$$\text{(C)} \quad f_{\boldsymbol{\mu}}(x, d_j t) = \begin{cases} q_{\mu_0\mu_1} f_{d^0\boldsymbol{\mu}}(x, t) & j = 0; \\ f_{d^j\boldsymbol{\mu}}(x, t) & 0 < j < n; \\ f_{d^n\boldsymbol{\mu}}(p_{f(\mu_{n-1})f(\mu_n)}(x), t) & j = n, \end{cases}$$

$$f_{\boldsymbol{\mu}}(x, s_j t) = f_{s^j\boldsymbol{\mu}}(x, t)\ 0 \le j \le n.$$

Here for $n \ge 0$, let

$$M_n = \{\boldsymbol{\mu} = (\mu_0, \mu_1, \dots, \mu_n) : \mu_i \in M, \mu_0 \le \mu_1 \le \dots \le \mu_n\},$$

and let $d_n^j : M_n \to M_{n-1}$ and $s_n^j : M_n \to M_{n+1}$ be respectively the face operator and the degeneracy operator defined by

$$d_n^j(\mu_0, \dots, \mu_n) = (\mu_0, \dots, \mu_{j-1}, \mu_{j+1}, \dots, \mu_n); \text{ and}$$
$$s_n^j(\mu_0, \dots, \mu_n) = (\mu_0, \dots, \mu_j, \mu_j, \dots, \mu_n).$$

Also, for $n \geq 0$, let Δ^n be the standard n-simplex, and let $d_j^n : \Delta^{n-1} \to \Delta^n$ and $s_j^n : \Delta^{n+1} \to \Delta^n$ be respectively the face operator and the degeneracy operator defined by

$$\begin{aligned} &d_j^n(t_0, \ldots, t_{n-1}) = (t_0, \ldots, t_{j-1}, 0, t_{j+1}, \ldots, t_{n-1}); \text{ and} \\ &s_j^n(t_0, \ldots, t_{n+1}) = (t_0, \ldots, t_{j-1}, t_j + t_{j+1}, t_{j+2}, \ldots, t_{n+1}). \end{aligned}$$

Let $\boldsymbol{f} = (f, f_{\boldsymbol{\mu}}), \boldsymbol{f} = (f', f'_{\boldsymbol{\mu}}) : (\boldsymbol{X}; \boldsymbol{X}_0, \boldsymbol{X}_1) \to (\boldsymbol{Y}; \boldsymbol{Y}_0, \boldsymbol{Y}_1)$ be coherent maps. Then a *coherent homotopy* from $\boldsymbol{f}$ to $\boldsymbol{f}'$ is a coherent map $\boldsymbol{F} = (F, F_\mu) : (\boldsymbol{X}; \boldsymbol{X}_0, \boldsymbol{X}_1) \times I \to (\boldsymbol{Y}; \boldsymbol{Y}_0, \boldsymbol{Y}_1)$ such that

1.) $F \geq f, f'$; and
2.) for each $x \in X_{F(\mu_n)}$ and $t \in \Delta^n$,

$$\begin{cases} F_{\boldsymbol{\mu}}(x, 0, t) = f_{\boldsymbol{\mu}}(p_{f(\mu_n)F(\mu_n)}(x), t); \\ F_{\boldsymbol{\mu}}(x, 1, t) = f_{\boldsymbol{\mu}}(p_{f'(\mu_n)F(\mu_n)}(x), t). \end{cases}$$

In this case we write $\boldsymbol{f} \simeq \boldsymbol{f}'$, and denote by $[\boldsymbol{f}]$ the homotopy class of $\boldsymbol{f}$.

For each map of inverse systems $\boldsymbol{f} = (f, f_\mu) : (\boldsymbol{X}; \boldsymbol{X}_0, \boldsymbol{X}_1) \to (\boldsymbol{Y}; \boldsymbol{Y}_0, \boldsymbol{Y}_1)$ we define a coherent map $C(\boldsymbol{f}) = (f, f_{\boldsymbol{\mu}}) : (\boldsymbol{X}; \boldsymbol{X}_0, \boldsymbol{X}_1) \to (\boldsymbol{Y}; \boldsymbol{Y}_0, \boldsymbol{Y}_1)$ as follows: for each $\boldsymbol{\mu} = (\mu_0, \ldots, \mu_n) \in M_n$, the map $f_{\boldsymbol{\mu}} : (X_{f(\mu)}; X_{0f(\mu)}, X_{1f(\mu)}) \times \Delta^n \to (Y_{\mu_0}; Y_{0\mu_0}, Y_{1\mu_0})$ is defined by

$$f_{\boldsymbol{\mu}}(x, t) = f_{\mu_0} p_{f(\mu_0)f(\mu_n)}(x).$$

Now the *coherent homotopy category* CH(pro-Top^T) for triads is defined as follows: The objects are inverse systems consisting of triads of spaces, maps of triads and cofinite directed index sets. The morphisms are the homotopy classes of coherent maps, and the identity morphism is $[C(1_{(\boldsymbol{X}; \boldsymbol{X}_0, \boldsymbol{X}_1)})] : (\boldsymbol{X}; \boldsymbol{X}_0, \boldsymbol{X}_1) \to (\boldsymbol{X}; \boldsymbol{X}_0, \boldsymbol{X}_1)$. Then the composition $[\boldsymbol{g}][\boldsymbol{f}] = [\boldsymbol{g}\boldsymbol{f}]$ is well-defined (see [12, Lemma 2.4]), and all the properties required for a category can be verified (see [12, Theorems 2.8, 2.11]). The *coherence functor* C : pro-$\mathsf{Top}^T \to$ CH(pro-Top^T) is defined by $C(\boldsymbol{X}; \boldsymbol{X}_0, \boldsymbol{X}_1) = (\boldsymbol{X}; \boldsymbol{X}_0, \boldsymbol{X}_1)$ for each object $(\boldsymbol{X}; \boldsymbol{X}_0, \boldsymbol{X}_1)$ of pro-Top^T and $C[\boldsymbol{f}] = [C(\boldsymbol{f})]$ for each morphism $\boldsymbol{f}$ of pro-Top^T. The well-definedness is proved as for [12, Lemma 1.17].

A coherent map $\boldsymbol{p} : (X; X_0, X_1) \to (\boldsymbol{X}; \boldsymbol{X}_0, \boldsymbol{X}_1)$ is said to be a *coherent expansion* of $(X; X_0, X_1)$ provided the following condition holds:

(CE) for each morphism $[\boldsymbol{f}] : (X; X_0, X_1) \to (\boldsymbol{Y}; \boldsymbol{Y}_0, \boldsymbol{Y}_1)$ of CH(pro-Top^T) there exists a unique morphism $[\boldsymbol{h}] : (\boldsymbol{X}; \boldsymbol{X}_0, \boldsymbol{X}_1) \to (\boldsymbol{Y}; \boldsymbol{Y}_0, \boldsymbol{Y}_1)$ such that $[\boldsymbol{h}]C(\boldsymbol{p}) = [\boldsymbol{f}]$.

Then we have

Theorem 4.1. *Every strong expansion of a triad of spaces is a coherent expansion of a triad of spaces.*

The proof is analogous to the case for a single space (see [12, Theorem 8.1]) as outlined in the following:

LEMMA 4.2. *Every strong expansion* $\boldsymbol{p} : (X; X_0, X_1) \to (\boldsymbol{X}; \boldsymbol{X}_0, \boldsymbol{X}_1)$ *of a triad* $(X; X_0, X_1)$ *satisfies the following property for every* $n \geq 1$ *and ANR triad* $(P; P_0, P_1)$*:*

(S2)^n *For each* $\lambda \in \Lambda$ *and for any maps* $f : (X_\lambda; X_{0\lambda}, X_{1\lambda}) \times \partial\Delta^n \to (P; P_0, P_1)$ *and* $F : (X; X_0, X_1) \times \Delta^n \to (P; P_0, P_1)$ *such that* $F|_{(X;X_0,X_1)\times\partial\Delta^n} = f(p_\lambda \times 1_{\partial\Delta^n})$, *there exist* $\lambda' \geq \lambda$ *and a map of triads* $H : (X_{\lambda'}; X_{0\lambda'}, X_{1\lambda'}) \times \Delta^n \to (P; P_0, P_1)$ *such that*

$$H|_{(X_{\lambda'};X_{0\lambda'},X_{1\lambda'})\times\partial\Delta^n} = f(p_{\lambda\lambda'} \times 1_{\partial\Delta^n}) \text{ and}$$
$$H(p_{\lambda'} \times 1_{\Delta^n}) \simeq F \text{ rel } X \times \partial\Delta^n \text{ as maps of triads.}$$

PROOF. The proof is similar to that for [12, Lemma 8.3]. In the proof, the following lemma is used in an appropriate place. □

LEMMA 4.3. *Let*

$$\boldsymbol{p} = (p_\lambda) : (X; X_0, X_1) \to (\boldsymbol{X}; \boldsymbol{X}_0, \boldsymbol{X}_1) = ((X_\lambda; X_{0\lambda}, X_{1\lambda}), p_{\lambda\lambda'}, \Lambda)$$

be a strong expansion and let Y *be a compact Hausdorff. Then the morphism*

$$\boldsymbol{p} \times 1_Y : (X; X_0, X_1) \times Y \to (\boldsymbol{X}; \boldsymbol{X}_0, \boldsymbol{X}_1) \times Y$$

is a strong expansion.

PROOF. The proof for [12, Theorem 7.5] can be easily modified for the case of triads, using the fact that for any ANR triad $(P; P_0, P_1)$ and compact Hausdorff space Y, $(P^Y; P_0^Y, P_1^Y)$ is an ANR triad. □

PROOF OF THEOREM 4.1. Lemmas 1.14, 2.12, 1.13 of [12] hold for triads, and, using those lemmas together with Lemma 4.2, we can prove Theorem 4.1 similarly to [12, Lemmas 8.5, 8.6]. □

Theorems 3.3 and 4.1 imply

COROLLARY 4.4. *Every resolution for a triad of spaces is a coherent expansion for a triad of spaces.*

All the results in Sections 3 and 4 are true for pointed triads.

5. STRONG SHAPE CATEGORY FOR TRIADS AND INVARIANTS

In this section, we define the strong shape category for triads of spaces, using the same inverse system approach as for [3, 11].

We define the *strong shape category for triads* $\mathrm{SSh}(\mathsf{Top}^T)$ as follows: Its objects are all triads of spaces. The morphisms $F : (X; X_0, X_1) \to (Y; Y_0, Y_1)$ are the equivalence classes of triples $(\boldsymbol{p}, \boldsymbol{q}, [\boldsymbol{f}])$ where $\boldsymbol{p} : (X; X_0, X_1) \to$

$(\boldsymbol{X};\boldsymbol{X}_0,\boldsymbol{X}_1)$ and $\boldsymbol{q}:(Y;Y_0,Y_1)\to(\boldsymbol{Y};\boldsymbol{Y}_0,\boldsymbol{Y}_1)$ are cofinite strong HPol^T-expansions of $(X;X_0,X_1)$ and $(Y;Y_0,Y_1)$, respectively, and $[\boldsymbol{f}]:(\boldsymbol{X};\boldsymbol{X}_0,\boldsymbol{X}_1)\to(\boldsymbol{Y};\boldsymbol{Y}_0,\boldsymbol{Y}_1)$ is a morphism of CH(pro-Top^T). Note that every triad of spaces admits such a strong expansion (Theorem 3.8). The equivalence relation $\sim$ between triples $(\boldsymbol{p},\boldsymbol{q},[\boldsymbol{f}])$ and $(\boldsymbol{p}',\boldsymbol{q}',[\boldsymbol{f}'])$ is defined as follows: $(\boldsymbol{p},\boldsymbol{q},[\boldsymbol{f}])\sim(\boldsymbol{p}',\boldsymbol{q}',[\boldsymbol{f}'])$ provided $[\boldsymbol{f}'][\boldsymbol{i}]=[\boldsymbol{j}][\boldsymbol{f}]$ where $[\boldsymbol{i}]:(\boldsymbol{X};\boldsymbol{X}_0,\boldsymbol{X}_1)\to(\boldsymbol{X}';\boldsymbol{X}'_0,\boldsymbol{X}'_1)$ and $[\boldsymbol{j}]:(\boldsymbol{Y};\boldsymbol{Y}_0,\boldsymbol{Y}_1)\to(\boldsymbol{Y}';\boldsymbol{Y}'_0,\boldsymbol{Y}'_1)$ are the unique isomorphisms such that $C(\boldsymbol{p}')=[\boldsymbol{i}]C(\boldsymbol{p})$ and $C(\boldsymbol{q}')=[\boldsymbol{j}]C(\boldsymbol{q})$, respectively. The identity morphism $1_{(X;X_0,X_1)}:(X;X_0,X_1)\to(X;X_0,X_1)$ of SSh(Top^T) and the composition of morphisms $F:(X;X_0,X_1)\to(Y;Y_0,Y_1)$ and $G:(Y;Y_0,Y_1)\to(Z;Z_0,Z_1)$ are defined as for the strong shape category SSh(Top) for single spaces. The strong shape functor $\overline{S}:\mathrm{H}(\mathsf{Top}^T)\to\mathrm{SSh}(\mathsf{Top}^T)$ and the *forgetful functor* $\overline{E}:\mathrm{SSh}(\mathsf{Top}^T)\to\mathrm{Sh}(\mathsf{Top}^T)$ are defined analogously to the strong shape category for single spaces, and we have $\overline{E}\circ\overline{S}=S$ where Sh(Top^T) is the shape category for triads of spaces, and $S:\mathrm{H}(\mathsf{Top}^T)\to\mathrm{Sh}(\mathsf{Top}^T)$ is the shape functor (see [13]).

Strong homology groups. For each abelian group G and space X, let $\overline{H}_n(X;G)$ denote the strong homology groups of X with coefficients in G (see [4, 12]). For the class of normal pairs, i.e., pairs (X,X_0) of spaces such that X_0 is normally embedded in X, the strong homology groups are invariants in the strong shape category, and satisfy all the Eilenberg-Steenrod axioms. In particular, for each normal pair (X,X_0), there is an exact sequence:

$$\cdots\to\overline{H}_m(X_0;G)\xrightarrow{i_*}\overline{H}_m(X;G)\xrightarrow{j_*}\overline{H}_m(X,X_0;G)\xrightarrow{\partial}\overline{H}_{m-1}(X_0;G)\to\cdots$$

and for each triad of spaces $(X;X_0,X_1)$, the excision maps $i_0:(X_0,X_0\cap X_1)\hookrightarrow(X,X_1)$ and $i_1:(X_1,X_0\cap X_1)\hookrightarrow(X,X_0)$ induce isomorphisms $(i_0)_*:\overline{H}_m(X_0,X_0\cap X_1;G)\to\overline{H}_m(X,X_1;G)$ and $(i_1)_*:\overline{H}_m(X_1,X_0\cap X_1;G)\to\overline{H}_m(X,X_0;G)$, respectively. Those facts together with Theorem 3.9 imply the following theorem (see [1, Chap. I., §15]):

THEOREM 5.1. *For each triad of spaces $(X;X_0,X_1)$ such that X_0 and X_1 are closed in X and $X_0\cap X_1$ is normally embedded in X, there is a natural exact sequence*

$$\cdots\to\overline{H}_m(X_0\cap X_1;G)\to\overline{H}_m(X_0;G)\oplus\overline{H}_m(X_1;G)\to$$
$$\overline{H}_m(X;G)\to\overline{H}_{m-1}(X_0\cap X_1;G)\to\cdots$$

This exact sequence is called the Mayer-Vietoris sequence for $(X;X_0,X_1)$ and denoted by $\mathcal{MV}_{(X;X_0,X_1)}$.

Let SSh(Top^T_N) denote the full subcategory of SSh(Top^T) whose objects are triads of spaces $(X;X_0,X_1)$ such that X_0 and X_1 are closed in X and $X_0\cap X_1$ is normally embedded in X, and let $\mathcal{MV}$ denote the category whose objects are all Mayer-Vietoris sequences $\mathcal{MV}_{(X;X_0,X_1)}$ of strong homology groups for triads of spaces $(X;X_0,X_1)$ such that X_0 and X_1 are closed in X and $X_0\cap X_1$ is normally embedded in X and whose morphisms $\Phi:\mathcal{MV}_{(X;X_0,X_1)}\to\mathcal{MV}_{(Y;Y_0,Y_1)}$ are homomorphisms of Mayer-Vietoris sequences (see [1, p. 9]).

Then we have

Theorem 5.2. *There exists a covariant functor $\mathcal{F}$ from* $\mathrm{SSh}(\mathsf{Top}_N^T)$ *to* $\mathcal{MV}$.

Proof. First, for each $(X;X_0,X_1)\in \operatorname{ob}\mathrm{SSh}(\mathsf{Top}_N^T)$, let $\mathcal{F}(X;X_0,X_1)$ be the Mayer-Vietoris sequence $\mathcal{MV}_{(X;X_0,X_1)}$. Let F be a morphism from $(X;X_0,X_1)$ to $(Y;Y_0,Y_1)$ in $\mathrm{SSh}(\mathsf{Top}_N^T)$, and let F be represented by the morphism $\boldsymbol{f}=(f,f_{\boldsymbol{\mu}}):(\boldsymbol{X};\boldsymbol{X}_0,\boldsymbol{X}_1)\to(\boldsymbol{Y};\boldsymbol{Y}_0,\boldsymbol{Y}_1)$ in $\mathrm{CH}(\text{pro-}\mathsf{Top}^T)$ where $\boldsymbol{p}:(X;X_0,X_1)\to(\boldsymbol{X};\boldsymbol{X}_0,\boldsymbol{X}_1)$ and $\boldsymbol{q}:(Y;Y_0,Y_1)\to(\boldsymbol{Y};\boldsymbol{Y}_0,\boldsymbol{Y}_1)$ are strong HPol^T-expansions such that the restrictions

$$\begin{cases}\boldsymbol{p}|_X : X\to\boldsymbol{X}\\ \boldsymbol{p}|_{X_i}:X_i\to\boldsymbol{X}_i,\ i=0,1\\ \boldsymbol{p}|_{X_0\cap X_1}:X_0\cap X_1\to\boldsymbol{X}_0\cap\boldsymbol{X}_1\end{cases}\quad\text{and}\quad\begin{cases}\boldsymbol{q}|_Y : Y\to\boldsymbol{Y}\\ \boldsymbol{q}|_{Y_i}:Y_i\to\boldsymbol{Y}_i,\ i=0,1\\ \boldsymbol{q}|_{Y_0\cap Y_1}:Y_0\cap Y_1\to\boldsymbol{Y}_0\cap\boldsymbol{Y}_1\end{cases}$$

are strong expansions. Then $\boldsymbol{f}$ induces morphisms in $\mathrm{CH}(\text{pro-}\mathsf{Top})$

$$\begin{cases}\boldsymbol{f}|_{\boldsymbol{X}}=(f,f_{\boldsymbol{\mu}}|_{\boldsymbol{X}}):\boldsymbol{X}\to\boldsymbol{Y}\\ \boldsymbol{f}|_{\boldsymbol{X}_i}=(f,f_{\boldsymbol{\mu}}|_{\boldsymbol{X}_i}):\boldsymbol{X}_i\to\boldsymbol{Y}_i,\ i=0,1\\ \boldsymbol{f}|_{\boldsymbol{X}_0\cap\boldsymbol{X}_1}=(f,f_{\boldsymbol{\mu}}|_{\boldsymbol{X}_0\cap\boldsymbol{X}_1}):\boldsymbol{X}_0\cap\boldsymbol{X}_1\to\boldsymbol{Y}_0\cap\boldsymbol{Y}_1\end{cases}$$

and these morphisms induce the following commutative diagram in the strong shape category $\mathrm{SSh}(\mathsf{Top})$ for $i=0,1$:

$$\begin{array}{ccccc} X_0\cap X_1 & \xrightarrow{j} & X_i & \xrightarrow{k} & X\\ {\scriptstyle F|_{X_0\cap X_1}}\downarrow & & {\scriptstyle F|_{X_i}}\downarrow & & \downarrow{\scriptstyle F|_X}\\ Y_0\cap Y_1 & \xrightarrow{j'} & Y_i & \xrightarrow{k'} & Y\end{array}$$

Here the horizontal morphisms are induced by the inclusions. This diagram induces a homomorphism of Mayer-Vietoris sequences $\mathcal{F}(F):\mathcal{MV}_{(X;X_0,X_1)}\to\mathcal{MV}_{(Y;Y_0,Y_1)}$. It is easy to verify this defines a functor. ∎

Strong homotopy groups. Let Top_*^T denote the category of pointed triads of spaces and base point preserving maps of pointed triads. Analogously to $\mathrm{SSh}(\mathsf{Top}^T)$, we have the strong shape category $\mathrm{SSh}(\mathsf{Top}_*^T)$ for pointed triads. Let HPol_*^T denote the category whose objects are pointed triads of spaces which have the homotopy type of a pointed polyhedral triad and whose morphisms are homotopy classes. For each pointed triad of spaces $(X;X_0,X_1,x_0)$, we wish to define the *strong homotopy groups* $\overline{\pi}_m(X;X_0,X_1,x_0)$. For this purpose, we consider the triple of spaces together with a base point $(E^m; E_+^{m-1}, E_-^{m-1}, s_0)$ where E^m is the unit m-cell in $\mathbb{R}^m$ with the boundary S^{m-1}, and

$$\begin{cases} E_+^{m-1} = \{t = (t_1, \ldots, t_m) \in S^{m-1} : t_m \geq 0\} \\ E_-^{m-1} = \{t = (t_1, \ldots, t_m) \in S^{m-1} : t_m \leq 0\} \\ s_0 = (1, 0, \ldots, 0) \end{cases}$$

and maps of $(E^m; E_+^{m-1}, E_-^{m-1}, s_0)$ to pointed triads of spaces. Note here that $(E^m; E_+^{m-1}, E_-^{m-1}, s_0)$ is a more general type of pointed triad $(Z; Z_0, Z_1, z_0)$ where not necessarily $Z = Z_0 \cup Z_1$. For such pointed triads, the notions of coherent map and coherent homotopy and the coherent homotopy category are well-defined.

For each $m \geq 2$ and for each pointed triad $(X; X_0, X_1, x_0)$, denote by $\overline{\pi}_m(X; X_0, X_1, x_0)$ the set of all homotopy classes of coherent maps

$$\boldsymbol{f} : (E^m; E_0^{m-1}, E_1^{m-1}, s_0) \to (\boldsymbol{X}; \boldsymbol{X}_0, \boldsymbol{X}_1, \boldsymbol{x}_0)$$

where $\boldsymbol{p} : (X; X_0, X_1, x_0) \to (\boldsymbol{X}; \boldsymbol{X}_0, \boldsymbol{X}_1, \boldsymbol{x}_0)$ is a strong HPol_*^T-expansion of $(\boldsymbol{X}; \boldsymbol{X}_0, \boldsymbol{X}_1, \boldsymbol{x}_0)$. This definition is well-defined. For, if $\boldsymbol{p'} : (X; X_0, X_1, x_0) \to (\boldsymbol{X'}; \boldsymbol{X}'_0, \boldsymbol{X}'_1, \boldsymbol{x}'_0)$ is another strong HPol_*^T-expansion of $(X; X_0, X_1, x_0)$, then there is an isomorphism

$$[\boldsymbol{i}] : (\boldsymbol{X}; \boldsymbol{X}_0, \boldsymbol{X}_1, \boldsymbol{x}_0) \to (\boldsymbol{X'}; \boldsymbol{X}'_0, \boldsymbol{X}'_1, \boldsymbol{x}'_0)$$

in $\mathrm{CH}(\text{pro-}\mathsf{Top}_*^T)$ such that $[\boldsymbol{i}]C(\boldsymbol{p}) = C(\boldsymbol{p'})$, which gives a one-to-one correspondence between the homotopy classes of coherent maps

$$\boldsymbol{f} : (E^m; E_0^{m-1}, E_1^{m-1}, s_0) \to (\boldsymbol{X}; \boldsymbol{X}_0, \boldsymbol{X}_1, \boldsymbol{x}_0)$$

and the homotopy classes of coherent maps

$$\boldsymbol{f'} : (E^m; E_0^{m-1}, E_1^{m-1}, s_0) \to (\boldsymbol{X'}; \boldsymbol{X}'_0, \boldsymbol{X}'_1, \boldsymbol{x}'_0).$$

For $m \geq 3$, let $\overline{\pi}_m(X; X_0, X_1, x_0)$ has a group structure by the H-cospace structure on $(E^m; E_0^{m-1}, E_1^{m-1}, s_0)$. For each morphism $F : (X; X_0, X_1, x_0) \to (Y; Y_0, Y_1, y_0)$ in $\mathrm{SSh}(\mathsf{Top}_*^T)$, let a morphism

$$\boldsymbol{f} : (\boldsymbol{X}; \boldsymbol{X}_0, \boldsymbol{X}_1, \boldsymbol{x}_0) \to (\boldsymbol{Y}; \boldsymbol{Y}_0, \boldsymbol{Y}_1, \boldsymbol{y}_0)$$

in $\mathrm{CH}(\text{pro-}\mathsf{Top}_*^T)$ represent F where $\boldsymbol{p} : (X; X_0, X_1, x_0) \to (\boldsymbol{X}; \boldsymbol{X}_0, \boldsymbol{X}_1, \boldsymbol{x}_0)$ and $\boldsymbol{q} : (Y; Y_0, Y_1, y_0) \to (\boldsymbol{Y}; \boldsymbol{Y}_0, \boldsymbol{Y}_1, \boldsymbol{y}_0)$ are strong HPol_*^T-expansions of $(X; X_0, X_1, x_0)$ and $(Y; Y_0, Y_1, y_0)$, respectively. Then for each $\alpha \in \overline{\pi}_m(X; X_0, X_1, x_0)$, if α is represented by a morphism

$$\boldsymbol{g} : (E^m; E_0^{m-1}, E_1^{m-1}, s_0) \to (\boldsymbol{X}; \boldsymbol{X}_0, \boldsymbol{X}_1, \boldsymbol{x}_0)$$

in $\mathrm{CH}(\text{pro-}\mathsf{Top}_*^T)$, $F_*\alpha \in \overline{\pi}_m(Y; Y_0, Y_1, y_0)$ is defined as

$$[\boldsymbol{fg}] : (E^m; E_0^{m-1}, E_1^{m-1}, s_0) \to (\boldsymbol{Y}; \boldsymbol{Y}_0, \boldsymbol{Y}_1, \boldsymbol{y}_0).$$

This is well-defined. Indeed, it is easy to see that the definition does not depend on the choice of the representatives $\boldsymbol{f}$ and $\boldsymbol{g}$. Then we can easily show that $\overline{\pi}$ defines a functor from $\mathrm{SSh}(\mathsf{Top}_*^T)$ to the category Set of sets and functions for $m \geq 2$ and to the category Gp of groups and homomorphisms for $m \geq 3$.

Theorem 5.3. *For each pointed triad of spaces $(X; X_0, X_1, x_0)$ such that X_0 and X_1 are closed and $X_0 \cap X_1$ is normally embedded in X, there are natural exact sequences:*

$$\cdots \to \overline{\pi}_{m+1}(X; X_0, X_1, x_0) \xrightarrow{\partial} \overline{\pi}_m(X_0, X_0 \cap X_1, x_0) \xrightarrow{(i_0)_*} \overline{\pi}_m(X, X_1, x_0)$$

$$\xrightarrow{(j_0)_*} \overline{\pi}_m(X; X_0, X_1, x_0) \xrightarrow{\partial} \overline{\pi}_{m-1}(X_0, X_0 \cap X_1, x_0) \to \cdots$$

and

$$\cdots \to \overline{\pi}_{m+1}(X; X_0, X_1, x_0) \xrightarrow{\partial} \overline{\pi}_m(X_1, X_0 \cap X_1, x_0) \xrightarrow{(i_1)_*} \overline{\pi}_m(X, X_0, x_0)$$

$$\xrightarrow{(j_1)_*} \overline{\pi}_m(X; X_0, X_1, x_0) \xrightarrow{\partial} \overline{\pi}_{m-1}(X_1, X_0 \cap X_1, x_0) \to \cdots$$

Proof. Let $\boldsymbol{p} = (p_\lambda) : (X; X_0, X_1, x_0) \to (\boldsymbol{X}; \boldsymbol{X}_0, \boldsymbol{X}_1, \boldsymbol{x}_0)$ be a polyhedral resolution of $(X; X_0, X_1, x_0)$. By [2, Lemma 2.7] X_0 and X_1 are normally embedded in X, and by [8, Section 5] the restricted morphisms

$$\begin{cases} \boldsymbol{p}|_{(X_i, X_0 \cap X_1)} = (p_\lambda|_{(X_i, X_0 \cap X_1)}) : (X_i, X_0 \cap X_1) \to (\boldsymbol{X}_i, \boldsymbol{X}_0 \cap \boldsymbol{X}_1), \ i = 0, 1 \\ \boldsymbol{p}|_{(X, X_i)} = (p_\lambda|_{(X, X_i)}) : (X, X_i) \to (\boldsymbol{X}, \boldsymbol{X}_i), \ i = 0, 1 \end{cases}$$

are resolutions. These resolutions for pairs induce strong expansions for pairs (see [6, 10]) and hence induce the strong homotopy groups for pairs. By an argument similar to the case for ordinary homotopy groups, we have the desired natural homotopy sequences. ∎

Using Theorem 5.3, we immediately have

Theorem 5.4. *For each pointed triad of spaces $(X; X_0, X_1, x_0)$ such that X_0 and X_1 are closed and $X_0 \cap X_1$ is normally embedded in X, the following statements are equivalent:*

1.) *The excision map $i_0 : (X_0, X_0 \cap X_1, x_0) \hookrightarrow (X, X_1, x_0)$ induces an isomorphism $(i_0)_* : \overline{\pi}_m(X_0, X_0 \cap X_1, x_0) \to \overline{\pi}_m(X, X_1, x_0)$ for $2 \leq m < n$ and an epimorphism for $m = n$;*
2.) *The excision map $i_1 : (X_1, X_0 \cap X_1, x_0) \hookrightarrow (X, X_0, x_0)$ induces an isomorphism $(i_1)_* : \overline{\pi}_m(X_1, X_0 \cap X_1, x_0) \to \overline{\pi}_m(X, X_0, x_0)$ for $2 \leq m < n$ and an epimorphism for $m = n$; and*
3.) *$\overline{\pi}_m(X; X_0, X_1, x_0) \cong 0$ for $2 \leq m \leq n$.*

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Received: 29.08.2000.