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SPAN MATES AND MESH

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Abstract. We identify a class of starlike curves with the span and the semispan equal to the infimum of the set of meshes of the covering chains. The same result is obtained for a class of indented circles as defined by West in [5].

1. Introduction

We begin by recalling the definitions of the span and the semispan introduced by A. Lelek in [2] and [3]. We omit the surjective varieties since they do not present different concepts for a simple closed curve.

Let X be a connected nonempty metric space. The span $\sigma(X)$ of X is the least upper bound of the set of real numbers $r \geq 0$ satisfying the following condition: there exists a connected space Y and a pair of continuous functions $f, g : Y \longrightarrow X$ such that $f(Y) = g(Y)$ and $\text{dist}[f(y), g(y)] \geq r$ for $y \in Y$. To obtain the definition of the semispan $\sigma_0(X)$ we replace the condition $f(Y) = g(Y)$ with the inclusion $f(Y) \supset g(Y)$. It was proven by Lelek in [3, p. 39] that when X is a continuum $\sigma_0(X) \leq \varepsilon(X)$, where $\varepsilon(X)$ is the infimum of the set of meshes of the chains that cover X.

The span of a simple closed curve, in general, has not been determined yet. Nevertheless, some progress has been made. This author has managed to determine the span and the semispan of the curves that constitute boundaries of convex domains [4]. T. West has computed the span and the semispan of a type of curve called indented circle [5]. We improve some of her results and show that the span she computed is, for the most part, equal to $\varepsilon(X)$. The results of this paper, together with those of [4], help to identify the class of simple closed curves X for which the span equals $\varepsilon(X)$.

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A *starlike* curve is a simple closed curve whose every point can be seen from a fixed point in the bounded component of its complement. Thus, X is starlike if there is a point P in the bounded component D of $\mathbf{C} \setminus X$ such that $PQ \setminus \{Q\} \subset D$ for each point $Q \in X$.

A simple closed polygonal path is a simple closed curve consisting of finitely many line segments.

Let X be a simple closed polygonal path, starlike with respect to the origin. A vertex $W \in X$ is considered to be *outer* if and only if the angle at W in the bounded component of $\mathbf{C} \setminus X$ is less than π. A vertex $W \in X$ is considered to be *inner* if the angle at W in the unbounded component of $\mathbf{C} \setminus X$ is less than π. A connected subset of X between two consecutive outer vertices is called a *segment*. Each segment inherits the positive orientation from X and hence has a uniquely determined *beginning* and *end*. A segment with the beginning A and the end B will be represented by AB. In contrast, AB^- denotes the line segment connecting A and B.

The distance $\mathrm{dist}(A, Y)$ from a point A to a set Y in the plane is defined, as usual, by letting $\mathrm{dist}(A, Y) = \inf_{P \in Y} \mathrm{dist}(A, P)$, where P is a point in Y.

Definition 1.1. *Let AB and CD be two segments of a starlike polygonal path X. The span distance between AB and CD, $s(AB, CD)$, is defined as*

$$s(AB,CD) = \\ = \max\{\min(\mathrm{dist}[A, CD], \mathrm{dist}[D, AB]), \min(\mathrm{dist}[B, CD], \mathrm{dist}[C, AB])\}.$$

Definition 1.2. *Let AB and CD be two different segments of a starlike polygonal path X. We say that AB is first with respect to CD if $s(AB, CD) = \min(\mathrm{dist}[B, CD], \mathrm{dist}[C, AB])$. We call AB second with respect to CD if $s(AB, CD) = \min(\mathrm{dist}[A, CD], \mathrm{dist}[D, AB])$.*

Definition 1.3. *Let V_{i-1}, V_i and V_{i+1} be three consecutive, in the positive direction, outer vertices on X, and let AB be a segment on X, $AB \neq V_{i-1}V_i$, $AB \neq V_iV_{i+1}$. We say that V_i is significant with respect to AB if $V_{i-1}V_i$ is first with respect to AB, V_iV_{i+1} is second with respect to AB and V_iV_{i+1} is not first with respect to AB.*

Notice that each segment AB has at least one significant vertex associated with it since the segment immediately following AB is first with respect to it and the segment immediately preceding AB is second, but not first, with respect to it.

Definition 1.4. *Let AB be a segment on X, let V_i be a significant vertex with respect to AB and let V_{i+1} be the next in the positive direction outer vertex on X. The segment V_iV_{i+1} is called a span mate of AB.*

2. Span mates and mesh

Throughout this section X will represent a simple closed polygonal path, starlike with respect to the origin. We shall denote the quadrilateral with vertices A, B, C and D by $ABCD$. For any point P in the plane, $|P|$ means the distance of P from the origin.

Let AB be a segment on X. Let $V_1, \dots, V_N$ be all outer vertices on $X \setminus AB$ in their consecutive positive order, so that V_1 immediately follows B and V_N immediately precedes A. Suppose V_i is the only significant vertex with respect to AB, for some i, $1 \le i \le N$. Notice that then $V_k V_{k+1}$ is first with respect to AB for all $k = 1, \dots, i-1$, and $V_k V_{k+1}$ is second (but not first) with respect to AB for all $k = i, \dots, N-1$.

The following lemma offers a chaining technique based on the span distance between the segments of X.

Lemma 2.1. *Let X be a starlike polygonal path with the outer vertices $V_0, \dots, V_{N+1}$ in the consecutive positive order, $|V_0| = \cdots = |V_{N+1}|$. Suppose V_i is the only vertex significant with respect to $V_{N+1}V_0$, and let $\varepsilon > 0$. If $V_k V_{k+1}$ is first with respect to $V_n V_{n+1}$ for each n, k such that $0 \le n < k \le i$ or $i \le n < k \le N$ then there exists a chain of closed sets $\{C_j\}_{1 \le j \le M}$ with mesh not larger than $s(V_{N+1}V_0, V_i V_{i+1}) + \varepsilon$ such that $X \subset \bigcup_{1 \le j \le M} C_j$.*

Proof. For any outer vertex V and any segment PQ on X define $V(PQ)$ to be the point on PQ such that $\mathrm{dist}[V, V(PQ)] = \mathrm{dist}[V, PQ]$. Thus, $S(V_{N+1}V_0, V_i V_{i+1}) = \min\{\mathrm{dist}[V_i, V_i(V_{N+1}V_0)], \mathrm{dist}[V_0, V_0(V_i V_{i+1})]\}$. Suppose $\varepsilon > 0$.

We shall assume that $V_j(V_k V_{k+1}) \ne V_{j+1}(V_k V_{k+1})$ for any j, k. Whenever necessary, the equality can be eliminated by choosing two distinct points on $V_k V_{k+1}$ close enough to $V_j(V_k V_{k+1})$ so that $\mathrm{dist}[V_j, V_j(V_k V_{k+1})]$ and $\mathrm{dist}[V_{j+1}, V_{j+1}(V_k V_{k+1})]$ are lengthened at most by an arbitrarily fixed fraction of ε. Similarly, we assume that $V_j(V_k V_{k+1}), V_{j+1}(V_k V_{k+1}) \in \mathrm{int}(V_k V_{k+1})$.

We shall first consider the case when $s(V_{N+1}V_0, V_i V_{i+1}) = \mathrm{dist}[V_i, V_i(V_{N+1}V_0)]$. Let X_1 be the component of $X \setminus \{V_i, V_i(V_{N+1}V_0)\}$ that contains V_1, and let $X_2 = X \setminus X_1$. Since $V_{i-1}V_i$ is first with respect to $V_{N+1}V_0$, we have

$$s(V_{i-1}V_i, V_{N+1}V_0) = \min\{\mathrm{dist}[V_i, V_i(V_{N+1}V_0)], \mathrm{dist}[V_{N+1}, V_{N+1}(V_{i-1}V_i)]\}$$

and, consequently,

$$\begin{aligned} &\min\{\mathrm{dist}[V_0, V_0(V_{i-1}V_i)], \mathrm{dist}[V_{i-1}, V_{i-1}(V_{N+1}V_0)]\} \\ &\le s(V_{i-1}V_i, V_{N+1}V_0) \le \mathrm{dist}[V_i, V_i(V_{N+1}V_0)]. \end{aligned}$$

If

$$\begin{aligned} &\min\{\mathrm{dist}[V_0, V_0(V_{i-1}V_i)], \mathrm{dist}[V_{i-1}V_{i-1}(V_{N+1}V_0)]\} = \\ &\mathrm{dist}[V_{i-1}, V_{i-1}(V_{N+1}V_0)] \end{aligned}$$

then we define D_1 to be the quadrilateral $V_iV_i(V_{N+1}V_0)V_{i-1}(V_{N+1}V_0)V_{i-1}$. Otherwise, $D_1 = V_iV_i(V_{N+1}V_0)V_0V_0(V_{i-1}V_i)$, provided $V_iV_j(V_{N+1}V_0)^- \cap V_0V_0(V_{i-1}V_i)^- = \emptyset$. If the line segments $V_iV_i(V_{N+1}V_0)^-$ and $V_0V_0(V_{i-1}V_i)^-$ intersect we use the chaining technique of case III in the proof of Theorem 3 in [1], which we outline in the next paragraph for the sake of completeness of this paper.

Let P be the point of intersection of the line segment $V_iV_i(V_{N+1}V_0)^-$ and $\operatorname{int}(V_{i-1}V_i)$. If $\operatorname{dist}[P, V_0] \leq \operatorname{dist}[V_i, V_i(V_{N+1}V_0)]$ then we put $D_1 = V_iV_i(V_{N+1}V_0)V_0P'$, where P' is a point on PV_{i-1} chosen so that $\operatorname{dist}[V_0, P'] < \operatorname{dist}[V_0, P] + \varepsilon/2$. Suppose now that $\operatorname{dist}[P, V_0] > \operatorname{dist}[V_i, V_i(V_{N+1}V_0)]$ and choose a point $V_i(V_{N+1}V_0)'$ on $V_i(V_{N+1}V_0)V_0^-$ such that $\operatorname{dist}[V_i(V_{N+1}V_0)', V_i] < \operatorname{dist}[V_i(V_{N+1}V_0), V_i] + \varepsilon/2$. Next, choose a point V_i', on $V_iV_0(V_iV_{i+1})^-$ such that $\operatorname{dist}[V_i', V_i(V_{N+1}V_0)] < \operatorname{dist}[V_i, V_i(V_{N+1}V_0)] + \varepsilon/2$, and a point $V_0(V_{i-1}V_i)'$ on $V_0(V_{i-1}V_i)V_{i-1}^-$ such that $\operatorname{dist}[V_0(V_{i-1}V_i)', V_0] < \operatorname{dist}[V_0(V_{i-1}V_i), V_0] + \varepsilon/2$. Finally, choose a point $V_0(V_{i-1}V_i)''$ on $V_0(V_{i-1}V_i)'V_{i-1}^-$ such that $\operatorname{dist}[V_0(V_{i-1}V_i)'', V_0] < \operatorname{dist}[V_0(V_{i-1}V_i), V_0] + \varepsilon/2$. In lieu of D_1 we use the hexagon $V_i'V_0(V_{i-1}V_i)V_i(V_{N+1}V_0)V_i(V_{N+1}V_0)'V_0(V_{i-1}V_i)'V_i$ followed by the quadrilateral $V_0(V_{i-1}V_i)'V_i(V_{N+1}V_0)'V_0V_0(V_{i-1}V_i)''$.

It is understood that this technique will be used automatically throughout the proof whenever needed.

To define D_2 we must consider the two above definitions of D_1 separately.

In the case when $D_1 = V_iV_i(V_{N+1}V_0)V_0V_0(V_{i-1}V_i)$ we appeal to the assumption that $V_{i-1}V_i$ is first with respect to V_0V_1. We have

$$s(V_{i-1}V_i, V_0V_1) = \min\{\operatorname{dist}[V_i, V_i(V_0V_1)], \operatorname{dist}[V_0, V_0(V_{i-1}V_i)]\}$$

and, consequently,

$$\begin{aligned}&\min\{\operatorname{dist}[V_{i-1}, V_{i-1}(V_0V_1)], \operatorname{dist}[V_1, V_1(V_{i-1}V_i)]\}\\ &\leq s(V_{i-1}V_i, V_0V_1) \leq \operatorname{dist}[V_0, V_0(V_{i-1}V_i)].\end{aligned}$$

If $\min\{\operatorname{dist}[V_{i-1}, V_{i-1}(V_0V_1)], \operatorname{dist}[V_1V_1(V_{i-1}V_i)]\} = \operatorname{dist}[V_{i-1}, V_{i-1}(V_0V_1)]$ we put $D_2 = V_0(V_{i-1}V_i)V_0V_{i-1}(V_0V_1)V_{i-1}$. Otherwise, we put $D_2 = V_0(V_{i-1}V_i)V_0V_1V_1(V_{i-1}V_i)$.

In the case when $D_1 = V_iV_i(V_{N+1}V_0)V_{i-1}(V_{N+1}V_0)V_{i-1}$ we appeal to the assumption that $V_{i-2}V_{i-1}$ is first with respect to $V_{N+1}V_0$. We have

$$\begin{aligned}&s(V_{i-2}V_{i-1}, V_{N+1}V_0) =\\ &\min\{\operatorname{dist}[V_{i-1}, V_{i-1}(V_{N+1}V_0)], \operatorname{dist}[V_{N+1}, V_{N+1}(V_{i-2}V_{i-1})]\}\end{aligned}$$

and, consequently,

$$\begin{aligned}&\min\{\operatorname{dist}[V_{i-2}, V_{i-2}(V_{N+1}V_0)], \operatorname{dist}[V_0, V_0(V_{i-2}V_{i-1})]\}\\ &\leq s(V_{i-2}V_{i-1}, V_{N+1}V_0) \leq \operatorname{dist}[V_{i-1}, V_{i-1}(V_{N+1}V_0)].\end{aligned}$$

If

$$\min\{\operatorname{dist}[V_{i-2}, V_{i-2}(V_{N+1}V_0)], \operatorname{dist}[V_0V_0(V_{i-2}V_{i-1})]\} = \operatorname{dist}[V_{i-2}, V_{i-2}(V_{N+1}V_0)]$$

we put $D_2 = V_{i-1}V_{i-1}(V_{N+1}V_0)V_{i-2}(V_{N+1}V_0)V_{i-2}$. Otherwise, we put $D_2 = V_{i-1}V_{i-1}(V_{N+1}V_0)V_0V_0(V_{i-2}V_{i-1})$.

We continue this construction of the sequence $D_1, D_2, \dots$ until we reach the set D_K such that there is at most one outer vertex V on $X_1 \setminus \bigcup_{1\le j\le K} D_j$. If there is such V then we define D_{K+1} to be the triangle connecting V with the two vertices of D_K disjoint from D_{K-1}. Otherwise, $D_{K+1} = \emptyset$. Notice that $X_1 \subset \bigcup_{1\le j\le K+1} D_j$. Furthermore, neither of the sides of D_j with endpoints on X_1 and the interior in the bounded component of $\mathbf{C}\setminus X$ exceeds $\operatorname{dist}[V_i, V_i(V_{N+1}V_0)]$ in length, $j = 1, \dots, K$.

We now proceed to cover X_2 with a similar sequence of closed sets. Since $s(V_{N+1}V_0, V_iV_{i+1}) = \operatorname{dist}[V_i, V_i(V_{N+1}V_0)]$ we have

$$\min\{\operatorname{dist}[V_{N+1}, V_{N+1}(V_iV_{i+1})], \operatorname{dist}[V_{I=1}, V_{i+1}(V_{N+1}V_0)]\} \le \operatorname{dist}[V_i, V_i(V_{N+1}V_0)].$$

If

$$\min\{\operatorname{dist}[V_{N+1}, V_{N+1}(V_iV_{i+1})], \operatorname{dist}[V_{i+1}V_{i+1}(V_{N+1}V_0)]\} = = \operatorname{dist}[V_{N+1}, V_{N+1}(V_iV_{i+1})]$$

then we define D_{-1} to be the quadrilateral $V_{N+1}(V_iV_{i+1})V_{N+1}V_i(V_{N+1}V_0)V_i$. Otherwise, $D_{-1} = V_{i+1}V_{i+1}(V_{N+1}V_0)V_i(V_{N+1}V_0)V_i$.

In order to define D_{-2} we consider the two above definitions of D_{-1} separately.

In the case when $D_{-1} = V_{N+1}(V_iV_{i+1})V_{N+1}V_i(V_{N+1}V_0)V_i$ we appeal to the assumption that V_iV_{i+1} is second with respect to V_NV_{N+1}. It follows that $s(V_iV_{i+1}, V_NV_{N+1}) = \min\{\operatorname{dist}[V_i, V_i(V_NV_{N+1}), \operatorname{dist}[V_{N+1}, V_{N+1}(V_iV_{i+1})]$. Therefore,

$$\begin{aligned}\min\{\operatorname{dist}[V_{i+1}, V_{i+1}(V_NV_{N+1})], \operatorname{dist}[V_N, V_N(V_iV_{i+1})]\} &\le \\ &\le s(V_iV_{i+1}, V_NV_{N+1}) \\ &\le \operatorname{dist}[V_{N+1}, V_{N+1}(V_iV_{i+1})] \\ &\le \operatorname{dist}[V_i, V_i(V_{N+1}V_0)].\end{aligned}$$

If

$$\min\{\operatorname{dist}[V_{i+1}, V_{i+1}(V_NV_{N+1})], \operatorname{dist}[V_NV_N(V_iV_{i+1})]\} = = \operatorname{dist}[V_{i+1}, V_{i+1}(V_NV_{N+1})]$$

we put $D_{-2} = V_{i+1}V_{i+1}(V_NV_{N+1})V_{N+1}V_{N+1}(V_iV_{i+1})$. Otherwise, $D_{-2} = V_N(V_iV_{i+1})V_NV_{N+1}V_{N+1}(V_iV_{i+1})$.

In the case when $D_{-1} = V_{i+1}V_{i+1}(V_{N+1}V_0)V_i(V_{N+1}V_0)V_i$ we appeal to the assumption that $V_{i+1}V_{i+2}$ is second with respect to $V_{N+1}V_0$. It follows that $s(V_{i+1}V_{i+2}, V_{N+1}V_0) = \min\{\mathrm{dist}[V_{i+1}, V_{i+1}(V_{N+1}V_0), \mathrm{dist}[V_0, V_0(V_{i+1}V_{i+2})]$. Hence

$$\begin{aligned}\min\{\mathrm{dist}[V_{i+2},V_{i+2}(V_{N+1}V_0)], \mathrm{dist}[V_{N+1}, V_{N+1}(V_{i+1}V_{i+2})]\}&\\ \le s(V_{i+1}V_{i+2}, V_{N+1}V_0)&\\ \le \mathrm{dist}[V_{i+1}, V_{i+1}(V_{N+1}V_0)]&\\ \le \mathrm{dist}[V_i, V_i(V_{N+1}V_0)].&\end{aligned}$$

If

$$\min\{\mathrm{dist}[V_{i+2},V_{i+2}(V_{N+1}V_0)], \mathrm{dist}[V_{N+1}V_{N+1}(V_{i+1}V_{i+2})]\} = \\ = \mathrm{dist}[V_{i+2}, V_{i+2}(V_{N+1}V_0)]$$

we put $D_{-2} = V_{i+2}V_{i+2}(V_{N+1}V_0)V_{i+1}(V_{N+1}V_0)V_{i+1}$. Otherwise, $D_{-2} = V_{N+1}(V_{i+1}V_{i+2})V_{N+1}V_{i+1}(V_{N+1}V_0)V_{i+1}$.

The construction of the sequence $D_{-1}, D_{-2}, \dots$ stops with a definition of the set D_J, $-\infty < J < -1$, such that there exists at most one outer vertex V on $X_2 \setminus \bigcup_{J \le j \le -1} D_j$. If such V exists then D_{J-1} is defined to be the triangie connecting V with the two vertices of D_J disjoint from D_{J+1}. Otherwise, $D_{J-1} = \emptyset$. Clearly, $X_2 \subset \bigcup_{J-1 \le j \le -1} D_j$. Moreover, neither of the sides of D_j with endpoints on X_2 and the interior in the bounded component of $\mathbf{C} \setminus X$ exceeds $\mathrm{dist}[V_i, V_i(V_{N+1}V_0)]$ in length, $j = J-1, \dots, -1$.

We now turn to the other possible starting point, i. e. $s(V_{N+1}V_0, V_iV_{i+1}) = \mathrm{dist}[V_0, V_0(V_iV_{i+1})]$. Let X_2 be the component of $X \setminus \{V_0, V_0(V_iV_{i+1})\}$ that contains V_{N+1}, and put $X_1 = X \setminus (X_2 \cup \{V_0, V_0(V_iV_{i+1}\})$. The construction of the sequences $\{D_j\}_{1 \le j \le K+1}$ and $\{D_j\}_{J-1 \le j \le -1}$ covering X_2 and X_1, respectively, is analogous.

It follows from the construction that both sides of D_j whose interiors lie in the bounded component of $\mathbf{C} \setminus X$ are not longer than $s(V_{N+1}V_0, V_iV_{i+1})$, $j = J, \dots, -1, 1, \dots, K$. In order to combine $\{D_j\}_{1 \le j \le K+1}$ and $\{D_j\}_{J-1 \le j \le -1}$ put $Q_1 = D_{j-1}, \dots, Q_{1-J} = D_{-1}, Q_{2-J} = D_1, \dots, Q_{K-J} = D_{K+1}$.

For every quadrilateral Q_j, $1 < j < K - J$, let e_j, f_j, g_j and h_j be its consecutive vertices chosen so that $\mathrm{int}(e_jf_j)$ and $\mathrm{int}(g_jh_j)$ are contained in the bounded component of $\mathbf{C} \setminus X$. If $\mathrm{diam}(Q_j) > s(V_{N+1}V_0, V_iV_{i+1}) + \varepsilon$ we partition e_jh_j and f_jg_j with sequences of points $\{p_n\}_{1 \le n \le n(j)}$ and $\{r_n\}_{1 \le n \le n(j)}$, respectively, such that $p_1 = e_j$, $p_{n(j)} = h_j$, $r_1 = f_j$, $r_{n(j)} = g_j$ and $\mathrm{diam}(p_kr_kr_{k+1}p_{k+1}) \le s(V_{N+1}V_0, V_iV_{i+1}) + \varepsilon$ for every k, $1 \le k < n(j)$. We define $C_{k,j}$ to be the quadrilateral $p_kr_kr_{k+1}p_{k+1}$ for every k, $1 \le k < n(j)$. If $\mathrm{diam}(Q_j) \le s(V_{N+1}V_0, V_iV_{i+1}) + \varepsilon$ we put $C_{1,j} = Q_j$, $b(j) = 1$.

If $Q_1 \ne \emptyset$ and $\mathrm{diam}(Q_1) > s(V_{N+1}V_0, V_iV_{i+1}) + \varepsilon$ then a similar partition will replace it by a chain of polygons $\{C_{k,1}\}_{1 \le k \le n(1)}$ such that $\mathrm{diam}(C_{k,1}) \le$

$s(V_{N+1}V_0, V_iV_{i+1}) + \varepsilon$ for each k, $1 \le k \le n(1)$, and $\bigcup_{1\le k\le n(1)} C_{k,1} = Q_1$. If $Q_1 \neq \emptyset$ and $\operatorname{diam}(Q_1) \le s(V_{N+1}V_0, V_iV_{i+1}) + \varepsilon$ we put $C_{1,j} = Q_j$, $n(j) = 1$.

If $Q_{K-J} \neq \emptyset$ and $\operatorname{diam}(Q_{K-J}) > s(V_{N+1}, V_0, V_iV_{i+1}) + \varepsilon$ then Q_{K-J} is partitioned into an analogous sequence $\{C_{k,K-J}\}_{1\le k\le n(K-J)}$. If $Q_{K-J} \neq \emptyset$ and $\operatorname{diam}(Q_{K-J}) \le s(V_{N+1}V_0, V_iV_{i+1}) + \varepsilon$ we put $C_{1,K-J} = Q_{K-J}$, $n(K-J) = 1$.

The chain

$$\bigcup_{1\le k\le n(1)} C_{k,1} \cup \cdots \cup \bigcup_{1\le k\le n(j)} C_{k,j} \cup \cdots \cup \bigcup_{1\le k\le n(K-J)} C_{k,K-J}$$

has the desired properties. This concludes the proof of the lemma. □

It is convenient to denote the vertices of the starlike polygonal line X in the following theorem by $V_0, V_1, \ldots, V_{N+1}$ in their consecutive positive order. Whenever an arbitrary V_jV_{j+1} is considered it will be understood that $j+1$ is taken modulo $N+2$.

We shall suppose that no segment on X has more than one span mate. We represent the span mate of V_jV_{j+1}, by β_j the significant vertex with respect to V_jV_{j+1} by B_j, and the other endpoint of β_j by A_j for $j = 0, \ldots, N+1$.

For any two line segments CD and PQ in the plane $L[CD \longrightarrow PQ]$ shall represent an affine transformation of CD onto PQ with P and Q corresponding to C and D, respectively.

Theorem 2.2. *Let X be a starlike polygonal path with the outer vertices $V_0, \ldots, V_{N+1}$ in the consecutive positive order, $|V_0| = \cdots = |V_{N+1}|$, and suppose each segment on X has exactly one span mate. If for every segment V_jV_{j+1} for which $B_j \neq B_{j+1}$ either V_{j+1} is significant with respect to B_jA_j or V_{j+2} is significant with respect to B_jA_j, and the latter implies that $B_{j+1} = A_j$ then $\sigma(X) = \sigma_0(X) = \varepsilon(X)$.*

Proof. Let $0 \le j \le N+1$. It follows from the assumptions that every segment V_kV_{k+1} contained in the positive arc $V_{j+1}B_j$ must have its significant vertex B_k on the positive arc B_jV_{k-1}. We shall show that, in addition, B_k must lie on the positive arc B_jV_j, as long as $V_{k+1} \neq B_j$.

Suppose not. Then there exists a segment, which for the notational convenience we assume to be $V_{N+1}V_0$, and an i, $0 < i < N+1$, such that

$$B_{N+1} = V_i \tag{2.1}$$

and

$$B_{i-1} = V_n \quad \text{for some } n,\ 1 \le n \le i-2. \tag{2.2}$$

Moving in the negative direction from V_{N+1}, we look for the first segment whose significant vertex is not V_i or the first vertex which is significant, whichever comes first.

Suppose the former comes first and let m be the number, $i < m < N+1$, such that $B_m \neq V_i$ while $B_k = V_i$ for all $m < k < N+1$. Since any two adjacent segments have their significant vertices at most two segments apart, it follows that $B_m = V_{i-1}$ or $B_m = V_{i-2}$.

If $B_m = V_{i-1}$ then $B_{i-1} = V_n$ for $n = m+1$ or $n = m+2$. Since $m < N+1$, we have $m < n \leq N+1$ or $n = 0$. This contradicts (2.2).

If $B_m = V_{i-2}$ then, by the same token, $B_{i-2} = V_n$ for some $m < n \leq N+1$ or $n = 0$. Since $B_{m+1} = V_i$ the case $B_{i-2} = V_{m+1}$ implies that $B_{i-1} = V_{m+2}$ which contradicts (2.2). Suppose now that $B_{i-2} = V_{m+2}$ and consider the cases $m < N$ and $m = N$ separately.

If $m < N$ then, in view of $B_{m+2} = V_i$, we have $B_{i-1} = V_n$ for $n = m+1$ or $n = m+2$. Hence, $B_{i-1} = V_n$ for some n, $m < n \leq N+1$, and (2.2) is contradicted again.

If $m = N$ then, since $B_m = V_{i-2}$ and $B_{i-2} = V_0$, it follows that $B_{N+1} = V_{i-1}$. This contradicts (2.1).

Suppose now we first encounter a significant vertex while moving in the positive direction from V_{N+1}. Let M be a number, $i < M \leq N+1$, such that V_M is a significant vertex, V_j is not significant for all j, $M < j \leq N+1$, and $B_j = V_i$ for all j, $M \leq j \leq N+1$. It follows that $V_M = B_{i-1}$ or $V_M = B_{i-2}$. However, (2.2) excludes the former and we have $V_M = B_{i-2}$. The latter and $B_M = V_i$ imply that $B_{i-1} = V_{M+1}$ and contradict (2.2).

We have shown that, given an arbitrary j, every segment V_kV_{k+1}, contained in the positive arc $V_{j+1}B_j$ must have its significant vertex B_k on the positive arc B_jV_j provided $V_{k+1} \neq B_j$. Note that if $V_{k+1} = B_j$ then either $B_k = V_{j+1}$ or B_k lies on the positive arc B_jV_j.

It follows that

$$\text{(2.3)} \qquad \begin{array}{l} V_kV_{k+1} \text{ is first with respect to } V_nV_{n+1} \\ \text{for each } n, k \text{ such that } 0 \leq n < k \leq i. \end{array}$$

Next we claim that for every j, $0 \leq j \leq N+1$, if $V_j = B_m$ for some m then V_mV_{m+1} must be contained in the positive arc $V_{j+1}B_j$. Indeed, uniqueness of B_j implies that all segments on the positive arc B_jV_j are second, and not first, with respect to V_jV_{j+1}. This would be contradicted if V_mV_{m+1} were located on that arc since $V_j = B_m$ implies that V_jV_{j+1} is second, and not first with respect to V_mV_{m+1}.

Consider an arbitrary significant vertex V_j, on the positive arc V_iV_{N+1}. Since B_j lies on the positive arc V_0V_i, the above claim implies that $V_j = B_m$ for some m, $0 \leq m \leq i$. It follows that

$$\text{(2.4)} \qquad \begin{array}{l} V_kV_{k+1} \text{ is first with respect to } V_nV_{n+1} \\ \text{for each } n, k \text{ such that } i \leq n < k \leq N. \end{array}$$

In view of (2.3) and (2.4), $\varepsilon(X) \leq s(V_{N+1}V_0, B_{N+1}A_{N+1})$ by virtue of Lemma 2.1, and since V_0 was an arbitrarily chosen outer vertex on X we conclude that $\varepsilon(X) \le s(V_jV_{j+1}, B_jA_j)$ for every j, $0 \le j \le N+1$. Hence,

$$\varepsilon(X) \le \min_{0\le j\le N+1} s(V_jV_{j+1}, B_jA_j). \tag{2.5}$$

We now partition $[0,1]$ into $2(N+2)$ line segments and define two mappings $f, g : [0,1] \longrightarrow X$ in the following way.

Let $P_0 = (0,0), P_1 = (1/2(N+2), 0), \dots, P_n = (n/2(N+2), 0), \dots, P_{2(N+2)} = (1,0)$. For $t \in [0, 1/2(N+2)]$ set $f(t) = L[P_0P_1 \to V_{N+1}V_0](t)$ and $g(t) = B_{N+1}$. For $t \in [1/2(N+2), 1/N+2]$ set $f(t) = V_0$ and $g(t) = L[P_1P_2 \to B_{N+1}A_{N+1}](t)$, provided $B_0 \ne B_{N+1}$; otherwise set $f(t) = L[P_1P_2 \to V_0V_1]$ and $g(t) = B_{N+1}$.

Suppose m, $m > 0$, is the smallest number such that $B_m \ne B_{N+1}$. For $t \in [0, (m+1)/2(N+2)]$ we set $g(t) = B_{N+1}$, while $f(t) = L[P_jP_{j+1} \to V_{j-1}V_j](t)$ for $t \in [j/2(N+2), (j+1)/2(N+2)]$, $j = 1, \dots, m$. Then, for $t \in [(m+1)/2(N+2), (m+2)/2(N+2)]$ we put $f(t) = V_m$ and $g(t) = L[P_{m+1}P_{m+2} \to B_{N+1}A_{N+1}](t)$.

In general, for an arbitrary n, $0 < n < 2(N+2)$, such that $f(t) = V_k$ and $g(t) = L[P_{n-1}P_n \to V_jV_{j+1}](t)$ on $[(n-1)/2(N+2), n/2(N+2)]$ for some k, j, we set $f(t) = L[P_nP_{n+1} \to V_kV_{k+1}](t)$ and $g(t) = V_{j+1}$ on $[n/2(N+2), (n+1)/2(N+2)]$ provided $B_{j+1} \ne V_k$. Otherwise, $f(t) = V_k$ and $g(t) = L[P_nP_{n+1} \to V_{j+1}V_{j+2}](t)$ for $t \in [n/2(N+2), (n+1)/2(N+2)]$.

Note that each time g covers a segment V_jV_{j+1} while $f(t) = V_k$, at least one of the following holds:

1) $V_j = B_{k-1}$, i. e. V_jV_{j+1} is the span mate of the last segment covered by f
2) $B_j = V_k$, i. e. the next segment covered by f is the span mate of V_jV_{j+1}.

It follows that $\operatorname{dist}(f(t), g(t)) \ge \min_{0\le j\le N+1} s(V_jV_{j+1}, B_jA_j)$, $t \in [0,1]$. Hence,

$$\sigma(X) \ge \min_{0\le j\le N+1} s(V_jV_{j+1}, B_jA_j). \tag{2.6}$$

In (2.5) and (2.6) we have shown that

$$\varepsilon(X) \le \min_{0\le j\le N+1} s(V_jV_{j+1}, B_jA_j) \le \sigma(X).$$

Since it is known that $\sigma(X) \le \sigma_0(X) \le \varepsilon(X)$, (see [3] or [1]), we conclude that $\sigma(X) = \sigma_0(X) = \varepsilon(X)$. ∎

3. The chaining of an indented circle

In [5] West constructed a simple closed curve by endowing a circle with a number of wedge-like indentations at the angles $\theta_0, \dots, \theta_{N-1}$, $0 < \theta_0 < \dots < \theta_{N-1} < \pi$ and at $\theta_0 + \pi, \dots \theta_{N-1} + \pi$. Thus, the indented circle X is a union of circle arcs and segments containing at most one inner vertex each. We shall represent the segments by $V_0V_1, V_2V_3, \dots, V_{2N-2}V_{2N-1}, W_0W_1, W_2W_3, \dots, W_{2N-2}W_{2N-1}$. Note that V_jV_{j+1} and W_jW_{j+1} are opposite each other, $j = 0, 2, \dots, 2N-2$. The circle arcs are $V_1V_2 \sim, V_3V_4 \sim, \dots, V_{2N-1}V_{2N} \sim, W_1W_2 \sim, \dots, W_{2N-1}W_{2N} \sim$.

We assume that the indentations V_jV_{j+1}, W_jW_{j+1} are symmetric with respect to the line $\theta = \theta_{j/2}, \theta_{j/2} + \pi$, $j = 0, 2, \dots, 2N-2$. However, we do not need to assume that each indentation contains at most one inner vertex. We allow finitely many inner vertices on each $V_jV_{j+1}(W_jW_{j+1})$.

West determined the span and the semispan of X in [5]. Using the terminology and notation of this paper we can express her result by the following equation:

$$\sigma(X) = \sigma_0(X) = \min_{j=0,2,\dots,2N-2} s(V_jV_{j+1}, W_jW_{j+1}). \tag{3.7}$$

We shall show that $\sigma(X) = \varepsilon(X)$.

Without loss of generality assume that

$$\min_{j=0,2,\dots,2N-2} s(V_jV_{j+1}, W_jW_{j+1}) = s(V_0V_1, W_0W_1).$$

We make the following claim.

(3.8)

For even n, k, $0 \le n < k \le 2N-2$, V_kV_{k+1}, is first with respect to V_nV_{n+1}.

To prove it, we let L_n be the line $\theta = \theta_{n/2}, \theta_{n/2+\pi}$ and observe that every point on V_kV_{k+1} lies in the same half-plane of $\mathbf{C} \setminus L_n$ as V_{n+1}. Hence, $\text{dist}[V_n, V_kV_{k+1}] > \text{dist}[V_{n+1}, V_kV_{k+1}]$. Furthermore,

$$\begin{aligned}\min\{\text{dist}[V_n,V_kV_{k+1}], \text{dist}[V_{k+1}, V_nV_{n+1}]\} >\\ > \min\{\text{dist}[V_k, V_nV_{n+1}], \text{dist}(V_{n+1}, V_kV_{k+1}]\}.\end{aligned}$$

It follows that $s(V_nV_{n+1}, V_kV_{k+1}) = \min\{\text{dist}[V_n, V_kV_{k+1}], \text{dist}[V_{k+1}, V_nV_{n+1}]\}$ and so V_kV_{k+1} is first with respect to V_nV_{n+1}.

Analogous arguments yield the following observations.

$$\begin{gathered}\text{For even } j,\ 2 \le j \le 2N-2,\ W_0W_1,\\ \text{is first with respect to } V_jV_{j+1}\end{gathered} \tag{3.9}$$

$$\begin{gathered}\text{For even } n,k,\ 0 \le n < k \le 2N-2,\\ W_kW_{k+1},\ \text{is first with respect to } W_nW_{n+1}\end{gathered} \tag{3.10}$$

$$\begin{gathered}\text{For even } j,\ 2 \le j \le 2N-2,\\ V_0V_1,\ \text{is first with respect to } W_jW_{j+1}.\end{gathered} \tag{3.11}$$

As in section 2 we use $V_j(V_iV_{i+1})$ to represent the point on the segment V_iV_{i+1} such that $\text{dist}(V_j, V_j(V_iV_{i+1})] = \text{dist}[V_j, V_iV_{i+1}]$.

We begin by defining a sequence $\{D_j\}$ of closed sets whose union covers X. We assume, without loss of generality, that $\operatorname{dist}[W_1, V_0V_1] \le \operatorname{dist}[V_0, W_0W_1]$ and $\operatorname{dist}[V_1, W_0W_1] \le \operatorname{dist}[W_0, V_0V_1]$. The three remaining cases can be handled in a similar manner. As in Lemma 2.1 we ensure that $W_1(V_0V_1) \in \operatorname{int}(V_0V_1)$ and $V_1(W_0W_1) \in \operatorname{int}(W_0W_1)$. Let D_0 be the quadrilateral $W_1W_1(V_0V_1)V_1V_1(W_0W_1)$. The case when $W_1W_1(V_0V_1)^-$ and $V_1V_1(W_0W_1)^-$ intersect is handled the same way as in Lemma 2.1.

If $V_1 = V_2$ then, by virtue of (3.9), we have

$$\min\{\operatorname{dist}[W_0, V_2V_3], \operatorname{dist}[V_3, W_0W_1]\} \le \operatorname{dist}[V_2, W_0W_1].$$

Note also that $\operatorname{dist}[V_2, W_0W_1] \le s(V_0V_1, W_0W_1)$. We define D_1 to be the quadrilateral $V_1V_1(W_0W_1)V_3V_3(W_0W_1)$, provided

$$\min\{\operatorname{dist}[W_0, V_2V_3], \operatorname{dist}(V3, W_0W_1]\} = \operatorname{dist}[V_3, W_0W_1].$$

If $V_2(W_0W_1) = V_3(W_0W_1)$ we apply the same remedy as in Lemma 2.1. Otherwise, $D_1 = V_1V_1(W_0W_1)W_0(V_2V_3)\ W_0$.

If $V_1 \ne V_2$ then D_1 is the wedge whose boundary consists of $V_1(W_0W_1)V_1^-$, the arc $V_1V_2 \sim$ and $V_2V_1(W_0W_1)^-$. Note that in this case $\operatorname{diam} D_1 = \operatorname{dist}[V_1, V_1(W_0W_1)] \le s(V_0V_1, W_0W_1)$.

In order to define D_2 we must consider the three above definitions of D_1 separately.

In the latter case, when D_1 is a wedge, we apply (3.9) to V_2V_3 and define D_2 as one of the two quadrilaterals $V_1(W_0W_1)V_2W_0(V_2V_3)W_0$ or $V_1(W_0W_1)V_2V_3V_3(W_0W_1)$, depending on $\min\{dist[W_0, V_2V_3], \operatorname{dist}[V_3, W_0W_1]\}$.

In the case when $V_1 = V_2$ and $D_1 = V_2V_2(W_0W_1)V_3V_3(W_0W_1)$ we consider the cases $V_3 = V_4$ and $V_3 \ne V_4$ separately, and define D_2 as either a quadrilateral or a wedge according to the procedure we used to obtain D_1.

In the case when $V_1 = V_2$ and $D_1 = V_2V_2(W_0W_1)W_0(V_2V_3)W_0$ we define D_2 to be the wedge whose boundary consists of $W_0W_0(V_2V_3)^-$, $W_0(V_2V_3)V_{2N-1}^-$ and the arc $V_{2N-1}W_0 \sim$, provided $V_{2N-1} \ne W_0$. Otherwise, we use (3.8) to conclude that $V_{2N-2}V_{2N-1}$ is first with respect to V_2V_3 and define D_2 as one of the two quadrilaterals $V_{2N-1}V_{2N-1}(V_2V_3)V_{2N-2}(V_2V_3)V_{2N-2}$ or $V_{2N-1}V_{2N-1}(V_2V_3)V_3V_3(V_{2N-2}V_{2N-1})$, depending on $\min\{\operatorname{dist}[V_{2N-2}, V_2V_3], \operatorname{dist}[V_3, V_{2N-2}V_{2N-1}]\}$.

We continue the construction of the sequence $\{D_j\}_{j\ge 0}$ until the positive arc on X from $W_1(V_0V_1)$ to W_1 is covered. The last set, say D_m, could be a triangle, as in Lemma 2.1, or a quadrilateral if all V_i, $1 \le i \le 2N-1$, are covered by $\bigcup_{0\le j\le M-1} D_j$ and a subarc of the positive arc from V_1, to W_1 is not.

While the diameter of each D_j which is a wedge does not exceed $s(V_0V_1, W_0W_1)$, the sides of each quadrilateral D_j with endpoints on X and interiors in the bounded component of $\mathbf{C} \setminus X$ do not exceed $s(V_0V_1, W_0W_1)$ as well.

Appealing to (3.10) and (3.11) we construct a similar sequence $\{D_j\}_{-n\leq j<0}$, which, as in Lemma 2.1, covers the remaining portion of X, the positive arc from W_1, to $W_1(V_0V_1)$.

Let $\delta > 0$. Suppose D_j is an arbitrary wedge in the sequence $\{D_j\}_{-n\leq j\leq m}$. Without loss of generality we consider the wedge whose boundary consists of $V_kV_k(V_iV_{i+1})^-$, $V_kV_{k+1} \sim$ and $V_{k+1}V_k(V_iV_{i+1})^-$, for some k, i. We choose a point V_i' on $V_k(V_iV_{i+1})V_{k+1}(V_iV_{i+1})^-$ such that $\mathrm{dist}[V_k(V_iV_{i+1}), V_i'] < \delta$ and enlarge D_j by adding the triangle $V_k(V_iV_{i+1})V_kV_i'$. The quadrilateral D_{j+1} is modified accordingly so that $D_j \cap D_{j+1} = \partial D_j \cap \partial D_{j+1}$. After this cosmetic change the sequence $\{D_j\}_{-n\leq j\leq m}$ is a chain.

Next, we partition all quadrilaterals with diameter larger than $s(V_0V_1, W_0W_1) + \delta$ in a manner described in Lemma 2.1, and thus obtain the chain $\{C_k\}_{1\leq k\leq M}$ of closed sets such that $\mathrm{diam}\, C_k < s(V_0V_1, W_0W_1) + \delta$ for each k, $1 \leq k \leq M$, and $X \subset \bigcup_{1\leq k\leq M} C_k$. Hence, $\varepsilon(X) \leq s(V_0V_1, W_0W_1)$.

It follows that $\varepsilon(X) \leq \min_{0\leq j\leq 2N-2} s(V_jV_{j+1}, W_jW_{j+1})$.

The latter inequality, $\sigma_0(X) \leq \sigma(X)$, and (3.7) imply that $\sigma(X) = \varepsilon(X)$.

Remark 3.1. The reader will observe that the indentation V_jV_{j+1} need not be symmetric with respect to L_j, as long as $\mathrm{dist}[V_j, L_j] = \mathrm{dist}[V_{j+1}, L_j]$ and X is starlike with respect to 0, $j = 0, 2, \ldots, 2N-2$.

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