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TOPOLOGIES GENERATED BY DISCRETE SUBSPACES

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Abstract. A topological space X is called *discretely generated* if for every subset $A \subset X$ we have $\overline{A} = \cup\{\overline{D} : D \subset A$ and D is a discrete subspace of $X\}$. We say that X is *weakly discretely generated* if $A \subset X$ and $A \neq \overline{A}$ implies $\overline{D} \setminus A \neq \emptyset$ for some discrete $D \subset A$. It is established that sequential spaces, monotonically normal spaces and compact countably tight spaces are discretely generated. We also prove that every compact space is weakly discretely generated and under the Continuum Hypothesis any dyadic discretely generated space is metrizable.

1. Introduction

It is natural to say that the topology of a space X *is determined by discrete subspaces* if for every $A \subset X$ the closure of A is the union of the closures of discrete subspaces of A. We will also call such spaces *discretely generated*. There are two important classes of discretely generated topological spaces: Fréchet–Urysohn spaces and the scattered spaces. In a Fréchet–Urysohn space X every point x from a closure of a set $A \subset X$ is the limit of a convergent sequence $S \subset A$. Clearly, S is a discrete subspace of X. If X is scattered then every subspace of X has a dense discrete subspace and so every point of $\overline{A}$ is in the closure of a discrete subspace of A. The purpose of this paper is to study the classes of discretely generated and weakly discretely generated spaces both of which are wider than the class of Fréchet–Urysohn spaces and the class of scattered spaces.

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On one hand, these properties seem to be interesting in themselves due to their good categorical behaviour: (weak) discrete generability is (closed) hereditary; every compact space is weakly discretely generated and not necessarily discretely generated. On the other hand, discrete generability and weak discrete generability often have surprising relationships with classical properties which make it possible to obtain new information. For example, every compact space of countable tightness is discretely generated as well as any monotonically normal space. Thus, theorems on discretely generated compact spaces can be strengthenings of the results on countably tight compact spaces. For example, it turns out that under the Continuum Hypothesis any dyadic discretely generated compact space is metrizable.

In the third section of the paper we give examples which show that a pseudocompact Tychonoff space can fail to be weakly discretely generated. We show that the same can happen to a countably compact Hausdorff space. We prove that it is consistent with ZFC that there are countably compact Tychonoff spaces which are not weakly discretely generated. This result is related to the theory of remote points, because it is easy to see that the existence of countably compact Tychonoff non-weakly discretely generated spaces is equivalent to the existence of a countably compact space X with "discretely remote points" in βX, i.e., points of $\beta X \backslash X$ which are not in the closure of any discrete subspace of X.

2. Notation and terminology

All spaces under consideration are assumed to be Hausdorff. We will often abuse notation by saying "discrete subset" instead of "discrete subspace". Given a space X the family $\tau(X)$ is its topology and $\tau^*(X) = \tau(X)\backslash\{\emptyset\}$. For a point $x \in X$ the family $\tau(x, X)$ consists of all open subsets of X which contain the point x. A space X is scattered if every non-empty subspace of X has an isolated point. A space X is called Fréchet–Urysohn if for every $A \subset X$ and each $x \in \overline{A}$ there is a sequence $\{a_n : n \in \omega\} \subset A$ such that $a_n \to x$. We say that X is sequential if $A \subset X$ and $A \neq \overline{A}$ implies that there is a sequence $\{a_n : n \in \omega\} \subset A$ such that $a_n \to y$ for some $y \notin A$. By D we denote the two-point space $\{0, 1\}$ with the discrete topology. The rest of notation is standard and can be found in [En].

3. Discretely generated spaces: definitions and general facts

Any Fréchet–Urysohn space as well as any scattered space is discretely generated. Thus, discrete generability is a convergence-like property which at the same time has a global flavour. As a consequence, it exhibits quite a nontrivial categorical behaviour providing at the same time new information about some known classes of spaces.

Definition 3.1. *A space X is discretely generated if for every $A \subset X$ and any $x \in \overline{A}$ there exists a discrete $D \subset A$ such that $x \in \overline{D}$.*

Definition 3.2. *A space X is weakly discretely generated if for every $A \subset X$ with $\overline{A} \neq A$ there is a discrete $D \subset A$ such that $\overline{D} \backslash A \neq \emptyset$.*

Let us formulate the simplest properties of discrete and weak discrete generability. The evident proofs are left to the reader.

Proposition 3.3. *Let X be a space. Then*
(1) if X is Fréchet–Urysohn then it is discretely generated;
(2) if X is scattered then it is discretely generated;
(3) if X is sequential then it is weakly discretely generated;
(4) discrete generability is hereditary;
(5) weak discrete generability is closed-hereditary;
(6) every discretely generated space is weakly discretely generated;
(7) a space is discretely generated iff it is hereditarily weakly discretely generated.

Recall that a dense-in-itself space X is called *maximal* if any topology on X, stronger than $\tau(X)$, has isolated points.

Example 3.4. There exist countable regular spaces which are not weakly discretely generated. Therefore, countable tightness of a regular space does not necessarily imply weak discrete generability.

Proof. Consider van Douwen's example of a countable maximal space V [vD]. For any $x \in V$, let $A = V \backslash \{x\}$. Then x can not be in the closure of any discrete subset of A because every discrete subspace of V is closed. Therefore V is not weakly discretely generated. $\square$

Example 3.5. There exist discretely generated spaces of any given tightness κ.

Proof. There are scattered spaces of any given tightness κ: take for example a discrete space P of cardinality κ; add one point a and declare that the neighbourhoods of a in the space $X = P \cup \{a\}$ are the complements of the subsets of P of power less than κ. The points of P are isolated in X. Then X is scattered and $t(X) = \kappa$. Now apply Proposition 3.3(2). $\square$

Given a space X and $A \subset X$, let us call the set $D(A) = \bigcup\{\overline{D} : D \subset A$ and D is discrete$\}$ *the d-closure of A in X*. It is evident that $A \subset D(A) \subset \overline{A}$.

Proposition 3.6. *Suppose that a space X is weakly discretely generated and $t(X) \leqslant \kappa$. Then the closure of any $A \subset X$ is the union of $\leqslant \kappa^+$ iterations of the d-closure of A. In other words, let $A_0 = A$; if we have A_α, set $A_{\alpha+1} = D(A_\alpha)$. If we have the sets $\{A_\alpha : \alpha < \beta\}$, where $\beta < \kappa^+$ is a limit ordinal, let $A_\beta = \bigcup\{A_\alpha : \alpha < \beta\}$. Then $\overline{A} = \bigcup\{A_\alpha : \alpha < \kappa^+\}$.*

PROOF. It is clear that $\overline{A} \supset \bigcup\{A_\alpha : \alpha < \kappa^+\}$. To prove the equality, it is sufficient to establish that the set $B = \bigcup\{A_\alpha : \alpha < \kappa^+\}$ is closed. Suppose not. Then there is a discrete $C \subset B$ such that $z \in \overline{C}$ for some $z \in \overline{B}\backslash B$. Since $t(X)$ **6** κ, there exists an $E = \{e_\alpha : \alpha < \kappa\} \subset C$ with $z \in \overline{E}$. Now $E \subset B$ implies that for every $\alpha < \kappa$ we have $e_\alpha \in A_{\gamma_\alpha}$ for some $\gamma_\alpha < \kappa^+$. If $\gamma = \sup\{\gamma_\alpha : \alpha < \kappa\}$ then $E \subset A_\gamma$ and hence $z \in D(A_\gamma) \subset B$, which is a contradiction. □

Given a space X and $C \subset X$, we say that C is *strongly discrete* if there exists a disjoint family $\{U_x : x \in C\} \subset \tau(X)$ such that $x \in U_x$ for all $x \in C$. For an arbitrary $A \subset X$ let $SD(A) = \{x \in X :$ there is a strongly discrete $B \subset A$ such that $x \in \overline{B}\}$. Let us call $SD(A)$ *the sd-closure of the set A.*

PROPOSITION 3.7. *Given a cardinal κ, suppose that X is a space and $A \subset X$. Define a family $\{A(\alpha) : \alpha < \kappa\}$ as follows: $A(0) = A$ and if we have $A(\alpha)$, let $A(\alpha+1) = SD(A(\alpha))$. If $\beta < \kappa$ is a limit ordinal, and we have the family $\{A(\alpha) : \alpha < \beta\}$, let $A(\beta) = \bigcup\{A(\alpha) : \alpha < \beta\}$. Then $A(\kappa) = \bigcup\{A(\alpha) : \alpha < \kappa\}$ is contained in $D(A)$. In other words, any number of iterations of sd-closure of any set is contained in the d-closure of this set.*

PROOF. Induction on $\alpha < \kappa$. The case $\alpha = 0$ is clear. Suppose that for any $\alpha < \beta < \kappa$ we proved that $A(\alpha) \subset D(A)$. For any $x \in A(\beta)$ there is a strongly discrete $B \subset \bigcup\{A(\alpha) : \alpha < \beta\}$ such that $x \in \overline{B}$. Fix a disjoint family $\{U_b : b \in B\}$ of open sets of X such that $b \in U_b$ for any $b \in B$. The inductive hypothesis gives a discrete $A_b \subset A$ such that $b \in \overline{A_b}$ for every $b \in B$. Then $D = \bigcup\{U_b \cap A_b : b \in B\}$ is a discrete subset of A and $x \in \overline{D}$. □

THEOREM 3.8. *Let X be a regular space of countable tightness. If X is weakly discretely generated then it is discretely generated.*

PROOF. Since the tightness of X is countable, for any $A \subset X$ and $x \in D(A)$ we have a countable discrete $B \subset A$ with $x \in B$. It is an easy exercise to see that in a regular space any countable discrete subspace is strongly discrete. This proves that $D(A) = SD(A)$ for any $A \subset X$. For each $\alpha < \omega_1$ construct the sets A_α and $A(\alpha)$ for each $\alpha < \omega_1$ like in Propositions 3.6 and 3.7. Now, it follows immediately from the above mentioned propositions that

$$\overline{A} = \bigcup\{A_\alpha : \alpha < \omega_1\} = \bigcup\{A(\alpha) : \alpha < \omega_1\} \subset D(A),$$

and therefore $\overline{A} = D(A)$. □

THEOREM 3.9. *Every Hausdorff sequential space is discretely generated.*

PROOF. Any sequential space X is weakly discretely generated and has countable tightness. So, if X is regular, Theorem 3.8 finishes the proof. In the general case, for any $A \subset X$ let $A[\alpha]$ be the α-th iteration of the sequential closure of A, i.e., $A[0] = A$; if we have $A[\alpha]$, let $A[\alpha+1] = \{x \in X :$ there is a sequence in $A[\alpha]$ which converges to $x\}$. If for some limit ordinal $\beta < \omega_1$ we have the sets $\{A[\alpha] : \alpha < \beta\}$ let $A[\beta] = \bigcup\{A[\alpha] : \alpha < \beta\}$. The sequentiality of X implies $\overline{A} = \bigcup\{A[\alpha] : \alpha < \omega_1\}$. Note that in any Hausdorff space any convergent sequence is strongly discrete and hence $A[\alpha] \subset A(\alpha)$ for each $\alpha < \omega_1$ (see Proposition 3.7 for the definition of the sets $A(\alpha)$). Finally, apply Proposition 3.7 to conclude that

$$\overline{A} = \bigcup\{A[\alpha] : \alpha < \omega_1\} \subset \bigcup\{A(\alpha) : \alpha < \omega_1\} \subset D(A),$$

and hence $\overline{A} = D(A)$. □

Recall that a space X is *monotonically normal* if and only if there is a map

$$G : \{(x, U) : x \in U \in \tau(X)\} \to \tau(X)$$

such that $x \in G(x, U)$ and $G(x, U) \cap G(y, V) \neq \emptyset$ implies $y \in U$ or $x \in V$. It is an immediate consequence of the definition of the operator G, that $G(x, U) \subset U$.

Theorem 3.10. *Any monotonically normal topological space X is discretely generated.*

Proof. Suppose that $A \subset X$ and $x \in \overline{A} \backslash A$. Denote by κ the cardinality of the set A. Choose an arbitrary $a_0 \in A$ and a set $U_0 \in \tau(a_0, X)$ such that $x \notin \overline{U}_0$. Suppose that $\alpha < \kappa^+$ and we have constructed the points $a_\beta \in A$ and sets $U_\beta \in \tau(a_\beta, X)$ in such a way that

(i) the family $\{W_\beta : \beta < \alpha\}$ is disjoint, where $W_\beta = G(a_\beta, U_\beta)$ for all $\beta < \alpha$;
(ii) $x \notin \overline{U}_\beta$ for every $\beta < \alpha$.

If $x \notin \overline{\bigcup\{W_\beta : \beta < \alpha\}}$, choose an $a_\alpha \in \Big(X \backslash \overline{\bigcup\{W_\beta : \beta < \alpha\}}\Big) \cap A$ and a set $U_\alpha \in \tau(a_\alpha, X)$ such that $x \notin \overline{U}_\alpha$ and $U_\alpha \cap \overline{\bigcup\{W_\beta : \beta < \alpha\}} = \emptyset$. It is clear that, if the step α is fulfilled, then the families $\{a_\beta : \beta \leqslant \alpha\}$ and $\{U_\beta : \beta \leqslant \alpha\}$ satisfy the conditions (i) and (ii).

Since $|A| = \kappa < \kappa^+$, we can not make κ^+ steps of our construction. Therefore $x \in \overline{\bigcup\{W_\beta : \beta < \alpha\}}$ for some $\alpha < \kappa^+$. The family $\{W_\beta : \beta < \alpha\}$ is disjoint, so the set $D = \{a_\beta : \beta < \alpha\}$ is discrete. Our proof will be finished if we establish that $x \in \overline{D}$.

Suppose not. Let $W = G(x, X \backslash \overline{D})$. Since $x \in \overline{\bigcup\{W_\beta : \beta < \alpha\}}$, there exists a $\beta < \alpha$ with $W_\beta \cap W \neq \emptyset$. Recalling the definition of monotone normality and the fact that $W_\beta = G(a_\beta, U_\beta)$, we conclude that either $a_\beta \in X \setminus \overline{D}$ or $x \in U_\beta$. The first inclusion contradicts the fact that $a_\beta \in D$ and the second one is impossible because $x \notin \overline{U}_\beta$. This proves that $x \in \overline{D}$. □

Since every stratifiable space is monotonically normal [Gr], we have the following fact.

Corollary 3.11. *Stratifiable spaces are discretely generated.*

Corollary 3.12. *Any subspace of a linearly ordered topological space is discretely generated.*

Proof. One has only to note that any LOTS is monotonically normal [Gr] and that discrete generability is hereditary. $\square$

Recall that a family $\mathcal{P}$ of sets is called *nested* if for any $A, B \in \mathcal{P}$ we have $A \subset B$ or $B \subset A$.

Theorem 3.13. *If a regular space X has a nested local base at every point then X is discretely generated.*

Proof. Let us establish first that every point $x \in X$ has a local base $\mathcal{B}_x$ which can be enumerated in the following way: $\mathcal{B}_x = \{U_\alpha : \alpha < \kappa\}$, where $\alpha < \beta$ implies $\overline{U}_\beta \subset U_\alpha$.

Indeed, let $\mathcal{B}$ be any nested local base at x. Choose a $U_0 \in \mathcal{B}$ arbitrarily. Suppose that we have chosen $\{U_\alpha : \alpha < \beta\} \subset \mathcal{B}$ in such a way that $\alpha < \alpha' < \beta$ implies $\overline{U}_{\alpha'} \subset U_\alpha$. If the family $\{U_\alpha : \alpha < \beta\}$ is a base at x, then we are done. If not, there exists a $W \in \mathcal{B}$ such that no U_α is contained in W. Being the base $\mathcal{B}$ a nested family, we have $W \subset U_\alpha$ for every $\alpha < \beta$. Since X is a regular space, there exists a $U_\beta \in \mathcal{B}$ for which $\overline{U}_\beta \subset W$. This shows that the inductive recursion can go on until the chosen sets form a local base at x.

Suppose that A is a subset of X and $x \in \overline{A} \backslash A$. Fix a local base $\mathcal{B}_x = \{U_\alpha : \alpha < \kappa\}$ such that $\overline{U}_\beta \subset U_\alpha$ as soon as $\alpha < \beta$. Take an $x_0 \in A \cap U_0$ and let $F_0 = \overline{U}_0$. Suppose that $\beta < \kappa$ and we have constructed points $\{x_\alpha : \alpha < \beta\}$ and closed sets $\{F_\alpha : \alpha < \beta\}$ with the following properties:

(i) for each $\alpha < \beta$ the set F_α is a closure of some U_γ where γ ¿ α;
(ii) $x_\alpha \in F_\alpha \cap A$ for every $\alpha < \beta$;
(iii) $F_\alpha \cap \overline{\{x_\delta : \delta < \alpha\}} = \emptyset$ for all $\alpha < \beta$;
(iv) $F_\alpha \subset F_\delta$ as soon as $\delta < \alpha < \beta$.

If $x \in \overline{\{x_\alpha : \alpha < \beta\}}$ then the inductive construction stops. If not, there is a δ ¿ β such that $\overline{\{x_\alpha : \alpha < \beta\}} \cap \overline{U}_\delta = \emptyset$. Let $F_\beta = \overline{U}_\delta$ and pick a point $x_\beta \in U_\delta \cap A$. It is immediate that the properties (i)-(iv) hold for $\{x_\alpha : \alpha$ **6** $\beta\}$ and $\{F_\alpha : \alpha$ **6** $\beta\}$.

Since $\mathcal{B}_x$ is a base at x, there is a β **6** κ for which $x \in \overline{\{x_\alpha : \alpha < \beta\}}$. The property (ii) implies that $D = \{x_\alpha : \alpha < \beta\} \subset A$ so it is sufficient to prove that the subspace D is discrete.

Given an $\alpha < \beta$ we have $x_\alpha \notin \overline{\{x_\delta : \delta < \alpha\}}$ due to $x_\alpha \in F_\alpha$ and property (iii). On the other hand, $\overline{\{x_\delta : \alpha < \delta\}} \subset F_{\alpha+1}$ and $x_\alpha \notin F_{\alpha+1}$. This shows that $x_\alpha \notin \overline{\{x_\delta : \delta \neq \alpha\}}$ and hence D is discrete. $\square$

4. Discrete generability in compact and similar spaces.

It turns out that any compact space is weakly discretely generated but not necessarily discretely generated. We will show that there are pseudocompact Tychonoff spaces in ZFC which are not weakly discretely generated. Any Tychonoff countably compact space of weight $\leqslant \omega_1$ is weakly discretely generated while there are models of ZFC with countably compact Tychonoff spaces of weight ω_2 which fail to be weakly discretely generated.

The following statement sounds surprising while having quite a short proof.

Proposition 4.1. *Any compact space is weakly discretely generated.*

Proof. Let X be a compact space. If $A \subset X$ and $A \neq \overline{A}$ then A is not compact. Now apply a theorem of Tkachuk [Tk]: if the closure of every discrete subset of a space is compact then the whole space is compact. There is a discrete $D \subset A$ such that $\mathrm{cl}_A(D)$ is not compact. Since $\overline{D}$ is compact we have $\overline{D} \backslash A \neq \emptyset$. □

Theorem 4.2. *Each compact space of countable tightness is discretely generated.*

Proof. This is an immediate consequence of Proposition 4.1 and Theorem 3.8. □

Example 4.3. (1) The space $\mathsf{D}^{\mathfrak{c}}$ is not discretely generated;
(2) weak discrete generability is not hereditary;
(3) there exist pseudocompact Tychonoff spaces which are not weakly discretely generated;
(4) there exist Hausdorff non-regular countably compact spaces (in ZFC) which are not weakly discretely generated.

Proof. The space V mentioned in Example 3.4 is countable and hence has weight $\leqslant \mathfrak{c}$. Therefore it can be embedded into $\mathsf{D}^{\mathfrak{c}}$. Now apply Proposition 4.1 and Proposition 3.3(4) to finish the proof of (1) and (2).

(3) Any Tychonoff space can be embedded as a closed subspace in a Tychonoff pseudocompact space [No]. Let X be a pseudocompact space which contains V as a closed subspace. It is clear that X is not weakly discretely generated.

(4) Take a dense $C \subset \beta\omega\backslash\omega$ of cardinality 2^{ω}. The subspace $B = (\beta\omega\backslash\omega)\backslash C$ is countably compact because any countable discrete subset of B has $2^{\mathfrak{c}}$ cluster points so all of them can not lie in C. Let μ be any maximal topology on C, stronger than the topology induced in C from $\beta\omega\backslash\omega$. Let ν be the topology generated by $\tau(\beta\omega\backslash\omega) \cup \mu$ as a subbase. Denote by X the space $(\beta\omega\backslash\omega, \nu)$.

The space X is Hausdorff since its topology is stronger than $\tau(\beta\omega\backslash\omega)$. Observe that X is countably compact. Indeed, for any infinite subset $A \subset X$ one of the sets $A \cap B$ or $A \cap C$ is infinite. If $A \cap B$ is infinite, then A has cluster points in B because B is countably compact as a subspace of $\beta\omega\backslash\omega$ and its topology did not change. If $|A \cap C|$ ¿ ω then the set A has $2^{\mathfrak{c}}$ cluster points in $\beta\omega\backslash\omega$ and therefore some point of B will be a cluster point for A in X due to the fact that the topology did not change at the points of B.

Let us finally prove that X is not weakly discretely generated. Consider any $x \in C$. There is no discrete $A \subset X\backslash\{x\}$ with $x \in \overline{A}$. Indeed, if such an A existed then $x \in \overline{A \cap C}$ because C is an open neighbourhood of x. But the subspace C is maximal and hence every discrete subspace of C is closed in C, a contradiction. □

THEOREM 4.4. *(1) If X is a discretely generated dyadic compact space then $w(X) < \mathfrak{c}$.*
(2) If the space D^{ω_1} is not discretely generated then every dyadic compact discretely generated space is metrizable.

PROOF. (1) If $w(X)$ ¿ $\mathfrak{c}$ then $\mathsf{D}^{\mathfrak{c}}$ embeds in X [Ef]. Example 4.3(1) shows that $\mathsf{D}^{\mathfrak{c}}$ is not discretely generated. Now apply Proposition 3.3(4) to conclude that X is not discretely generated.

(2) If X is a dyadic compact space of uncountable weight, then D^{ω_1} embeds into X [Ef]. Thus, if X is discretely generated, then D^{ω_1} can not be a subspace of X and hence the weight of X is countable. □

COROLLARY 4.5. *Under the Continuum Hypothesis any discretely generated dyadic compact space is metrizable.*

A very natural question is whether it is possible to omit CH in Corollary 4.5. By Theorem 4.4(2) the statement "every dyadic discretely generated compact space is metrizable" is equivalent to the statement " D^{ω_1} is not discretely generated".

Recall that *an L-space* is a hereditary Lindelöf non-separable regular space. It is still an open question whether there exist models of ZFC in which L-spaces do not exist. Thus, the following theorem "almost proves" in ZFC that D^{ω_1} is not discretely generated, i.e., we can say that D^{ω_1} is not discretely generated in all known models of ZFC. Recall that a space X is called *left-separated* if it can be well-ordered in such a way that every initial segment is closed in X.

THEOREM 4.6. *If there exists an L-space then D^{ω_1} is not discretely generated.*

PROOF. Since $\mathsf{D}^{\mathfrak{c}}$ is not discretely generated in ZFC (see Example 4.3), there is nothing to prove if CH holds. Now, suppose that $\omega_1 < \mathfrak{c}$ and T is an L-space. The space T is not separable, so there exists a subspace $Z = \{z_\alpha : \alpha < \omega_1\} \subset T$ such that $z_\beta \notin \overline{\{z_\alpha : \alpha < \beta\}}$ for all $\beta < \omega_1$. Since Z is a Tychonoff space of cardinality $\omega_1 < \mathfrak{c}$, it is zero-dimensional, i.e., has a base $\mathcal{B}$ consisting of clopen sets. There exists a family $\mathcal{B}' \subset \mathcal{B}$ of cardinality ω_1, which separates points of Z and witnesses that Z is not separable, i.e., for any $\beta < \omega_1$ there is a $U \in \mathcal{B}'$ such that $x_\beta \in U$ and $\{x_\alpha : \alpha < \beta\} \cap U = \emptyset$. If we generate a topology μ by the family $\mathcal{B}' \cup \{Z \backslash U : U \in \mathcal{B}'\}$, then $Y = (Z, \mu)$ is a Tychonoff zero-dimensional L-space of weight and cardinality ω_1.

Consider the family $\mathcal{U} = \{U \in \tau(Y) : U \text{ is separable}\}$. Since Y is hereditarily Lindelöf and non-separable, the set $W = \bigcup \mathcal{U}$ is also separable and $Y' = Z \backslash \overline{W}$ is still an L-space. It is clear that in Y' every countable set is nowhere dense. For each $n \in \omega$ let X_n be a copy of Y'. Remembering that Y' is a subspace of Z which is left-separated, we can choose an enumeration $\{x(n, \alpha) : \alpha < \omega_1\}$ of the space X_n such that $\{x(n, \alpha) : \alpha < \beta\}$ is closed and nowhere dense for every $\beta < \omega_1$. Given $\alpha < \omega_1$ and $n \in \omega$, fix some clopen cover $\{U(\alpha, n, m) : m \in \omega\}$ of the set $\{x(n, \beta) : \beta \text{ ¿ } \alpha\}$.

For each $n \in \omega$ let $m(n, 0) = 0$ and $W_0 = \bigcup\{U(0, n, 0) : n \in \omega\}$. Assume that, for some $\beta < \omega_1$ we have constructed natural numbers $m(\alpha, n)$ and sets W_α for all $\alpha < \beta$ and $n \in \omega$ so that

(i) $W_\alpha = \bigcup\{U(\alpha, n, m(\alpha, n)) : n \in \omega\}$ for all $\alpha < \beta$;
(ii) for any $\alpha < \beta$ and any finite $F \subset \alpha$ the set $W_\alpha \cap \bigcap_{\gamma \in F} W_\gamma$ meets all but finitely many of the X_n's.

Fix some enumeration $\{\beta_n : n \in \omega\}$ of the ordinal $\beta = \{\alpha : \alpha < \beta\}$ and let $F_n = \{\beta_i : i \mathbf{6} n\}$ for all $n \in \omega$. By the inductive assumption, there is a natural k_0 such that $\bigcap\{W_\gamma : \gamma \in F_0\} \cap X_l \neq \emptyset$ for all $l \text{ ¿ } k_0$. If we have natural $k_0, \ldots, k_{n-1}$, choose a $k_n \in \omega$ such that $k_n > k_{n-1}$ and $\bigcap\{W_\gamma : \gamma \in F_n\} \cap X_l \neq \emptyset$ for any $l \text{ ¿ } k_n$. For each natural n choose $m(n, \beta)$ as follows: $m(n, \beta) = 0$ for $n < k_0$; if $n \in [k_i, k_{i+1})$ choose $m(n, \beta)$ so that $U(\beta, n, m(n, \beta))$ meets $\bigcap\{W_\gamma : \gamma \in F_i\}$. It is possible due to the fact that $\bigcap\{W_\gamma : \gamma \in F_i\} \cap X_n \neq \emptyset$ and $\bigcup\{U(\beta, n, m) : m \in \omega\}$ is dense in X_n. It is easy to see that the sequence $\{m(\beta, n) : n \in \omega\}$ and the set $W_\beta = \bigcup\{U(\beta, n, m(\beta, n)) : n \in \omega\}$ maintain the inductive conditions.

Our space X will be $\bigoplus\{X_n : n \in \omega\} \cup \{a\}$, where $a \notin \bigoplus\{X_n : n \in \omega\}$. The topology at the points of $\bigoplus\{X_n : n \in \omega\}$ is that of the free union and the base at a is generated by the family $\{W_\alpha : \alpha < \omega_1\}$. It is immediate that X is a Tychonoff zero-dimensional space of weight ω_1 in which all discrete subspaces are countable. Now, if D is a discrete subspace of $\bigoplus\{X_n : n \in \omega\}$, then there is an $\alpha < \omega_1$ such that $D \subset \{x(n, \beta) : \beta < \alpha,\ n \in \omega\}$. Therefore $W_\alpha \cap D = \emptyset$ and $a \notin \overline{D}$. To finish our proof, note that X is not discretely generated and embeds into D^{ω_1}. □

Observation 4.7. If we want to prove in ZFC that D^{ω_1} is not discretely generated, we must find a zero-dimensional space of weight ω_1 which is not discretely generated. This space can not be countable like the one of Example 3.4. Indeed, under Martin's Axiom and the negation of CH any countable space of weight ω_1 is Fréchet–Urysohn [Ar] and hence discretely generated. Thus, such a space X must have a point x which is not in the closure of any countable subset of $X\backslash\{x\}$. The following result shows that such a point must be a G_δ-set in X.

THEOREM 4.8. *Let X be a regular space. Suppose that $A \subset X$, $x \in \overline{A}$ and for any countable discrete $B \subset A$ we have $x \notin \overline{B}$. If the character of x in X is at most ω_1 and $\psi(x, \{x\} \cup A) > \omega$ then there is a discrete set $D \subset A$ with $x \in \overline{D}$.*

PROOF. Let $\{U_\alpha : \alpha < \omega_1\}$ be a local base of x in X. Observe that the uncountable pseudocharacter of x in $\{x\} \cup A$ implies that $F \cap A \neq \emptyset$ for any G_δ-set F containing x. Use the regularity of X to find a closed G_δ-set F_0 such that $x \in F_0 \subset U_0$ and let $d_0 \in F_0 \cap A$. Suppose that for some $\beta < \omega_1$ we have constructed points $\{d_\alpha : \alpha < \beta\}$ and closed G_δ-sets $\{F_\alpha : \alpha < \beta\}$ so that

(i) $x \in F_\alpha \subset U_\alpha$ for all $\alpha < \beta$;
(ii) $F_\alpha \subset F_{\alpha'}$ if $\alpha > \alpha'$;
(iii) $d_\alpha \in F_\alpha \cap A$ for any $\alpha < \beta$;
(iv) $\overline{\{d_\alpha : \alpha < \gamma\}} \cap F_\gamma = \emptyset$ for all $\gamma < \beta$.

Denote by D_β the set $\{d_\alpha : \alpha < \beta\}$. Note that for any ordinal $\alpha < \beta$ we have $(X\backslash\overline{\{d_\nu : \nu < \alpha\}}) \cap (X\backslash F_{\alpha+1}) \cap D_\beta = \{d_\alpha\}$, which shows that D_β is a discrete set.

Since x can not be in the closure of a countable discrete subset of A, we have $x \notin \overline{D}_\beta$, so there exists a closed G_δ-set F such that $x \in F \subset U_\beta\backslash\overline{D}_\beta$. Now let $F_\beta = F \cap \bigcap\{F_\alpha : \alpha < \beta\}$ and choose any $d_\beta \in F_\beta \cap A$. It is clear that the properties (i)-(iv) hold for all $\alpha \leqslant \beta$.

We claim that the set $D = \{d_\alpha : \alpha < \omega_1\} \subset A$ is discrete and $x \in \overline{D}$. Indeed, the property (i) implies $D \cap U_\alpha \neq \emptyset$ for all $\alpha < \omega_1$ and therefore $x \in \overline{D}$. Given an $\alpha \in \omega_1$ we have $(X\backslash\overline{\{d_\beta : \beta < \alpha\}}) \cap (X\backslash F_{\alpha+1}) \cap D = \{d_\alpha\}$, which shows that D is discrete. $\square$

COROLLARY 4.9. *Let $\Sigma = \{x \in \mathsf{D}^{\omega_1} : |x^{-1}(1)| \leqslant \omega\}$ be the Σ-product of D^{ω_1}. Then each point $y \in \mathsf{D}^{\omega_1}$ is an accumulation point of some discrete $D \subset \Sigma$.*

However, if we want a subset $A \subset \mathsf{D}^{\omega_1}$ and some $x \in \mathsf{D}^{\omega_1}$ which is not in the closure of any discrete subset of A, we must have $\psi(x, \{x\} \cup A) = \omega$ while no countable subset from A reaches x. Since D^{ω_1} is homogeneous, we may assume that $x(\alpha) = 1$ for all $\alpha < \omega_1$. A good candidate seems to be the set $\sigma = \{x \in \mathsf{D}^{\omega_1} : |x^{-1}(1)| < \omega\}$, but it turns out that any point of D^{ω_1} is in the closure of a discrete subset of σ. To establish this, we will need the following fact.

PROPOSITION 4.10. *Let X be a space of character $\leqslant \omega_1$. Suppose that $A \subset X$, $x \in \overline{A}$ and no countable subset of A contains x in its closure. Then the following conditions are equivalent:*

(1) There exists a discrete $D \subset A$ such that $x \in \overline{D}$;
(2) There exists a family γ of open subsets of A such that $x \in \overline{(\bigcup \gamma) \backslash (\bigcup \mu)}$ for any countable $\mu \subset \gamma$.

PROOF. It is clear that $\chi(x, A \cup \{x\}) = \omega_1$. If $D \subset A$ is a discrete subset of A we can assume, without loss of generality, that $|D| = \omega_1$. Let $D = \{d_\alpha : \alpha < \omega_1\}$. For any $\alpha < \omega_1$ take a $V_\alpha \in \tau(X)$ such that $V_\alpha \cap D = \{d_\alpha\}$. Then the family $\gamma = \{V_\alpha \cap A : \alpha < \omega_1\}$ has property (2). Thus, we have proved that (1) $\Longrightarrow$ (2).

To show (2) $\Longrightarrow$ (1), suppose that γ satisfies the condition (2) from the lemma. Let $\{O_\alpha : \alpha < \omega_1\}$ be a local base at the point x. Assume that $\alpha < \omega_1$ and we have defined a set $\{x_\beta : \beta < \alpha\} \subset A$ and a family $\{U_\beta : \beta < \alpha\} \subset \gamma$ so that the following conditions are satisfied for every $\beta < \alpha$:

(a) $x_\beta \in U_\beta \cap O_\beta$;
(b) $x_\beta \notin U_\delta$ if $\delta < \beta$;
(c) $x_\beta \notin \overline{\{x_\delta : \delta < \beta\}}$.

Consider the set $X_\alpha = \{x_\beta : \beta < \alpha\}$ and the family $\mu = \{U_\beta : \beta < \alpha\}$. Since $x \notin \overline{X}_\alpha$, there exists a point $x_\alpha \in (O_\alpha \cap (\bigcup \gamma \backslash \bigcup \mu)) \backslash \overline{X}_\alpha$. Denote by U_α any element of γ which contains the point x_α. Clearly, $D = \{x_\beta : \beta < \omega_1\}$ and $\lambda = \{U_\beta : \beta < \omega_1\}$ satisfy (a)-(c) for all $\beta < \omega_1$. From (a) it follows that $x \in \overline{D}$. For every $\alpha < \omega_1$ let $W_\alpha = U_\alpha \backslash \overline{X}_\alpha$. Then by (a) and (c) the set W_α is an open neighbourhood of x_α and $W_\alpha \cap D = \{x_\alpha\}$ for each α, i.e., D is discrete. $\square$

THEOREM 4.11. *For any point $y \in \mathsf{D}^{\omega_1}$ there is a discrete subspace of σ which contains y in its closure.*

PROOF. First, let us prove this for the point $x \in \mathsf{D}^{\omega_1}$ such that $x(\alpha) = 1$ for all $\alpha \in \omega_1$. It suffices to construct a family γ of open subsets of σ as in Proposition 4.10. Given a finite $F \subset \omega_1$ and a function $f : F \to \mathsf{D}$, we have a standard open subset $U(f, F) = \{y \in \mathsf{D}^{\omega_1} : y|F = f\}$ of the space D^{ω_1}.

Call a standard open set $U(f, F)$ *admissible* if it satisfies the following two conditions:

(i) there exist distinct limit ordinals (the zero is also considered a limit) $\alpha_1, \ldots, \alpha_n < \omega_1$ and positive integers $k_1, \ldots, k_n$ such that $F = \bigcup_{i=1}^{n} \{[\alpha_i, \alpha_i + 2k_i]$;
(ii) for every $i \in \{1, \ldots, n\}$ we have $f(\alpha_i + j) = 0$ for any $j \in \{0, \ldots, k_i\}$ and $f(\alpha_i + j) = 1$ for all $j \in \{k_i + 1, \ldots, 2k_i\}$.

Denote by $\mathcal{U}$ the family of all admissible open sets. Let $\gamma = \{U \cap \sigma : U \in \mathcal{U}\}$. We claim that the family γ is as required. Indeed, let μ be countable subfamily of γ and U a neighbourhood of x in D^{ω_1}. One can assume without loss of generality that $U = U(f, K)$ where $f(\alpha) = 1$ for any $\alpha \in K$. By definition of γ, there exists a countable subfamily $\mathcal{V} \subset \mathcal{U}$ such that $\mu =$

$\{V \cap \sigma : V \in \mathcal{V}\}$. Since μ is countable, there exists a limit ordinal $\alpha < \omega_1$ such that $K \subset \alpha$ and $P \subset \alpha$ whenever $U(f, P) \in \mathcal{V}$. In addition, we can find distinct limit ordinals $\beta_1, \ldots, \beta_m < \alpha$ and positive integers $l_1, \ldots, l_m$ each greater than or equal to 2 for which $K \subset \bigcup_{j=1}^m [\beta_j, \beta_j + l_j]$. Consider the set $F = (\bigcup_{j=1}^m [\beta_j, \beta_j + l_j + 2]) \cup [\alpha, \alpha + 2]$. It is clear that $K \subset F$. Define a function $h : F \to \mathsf{D}$ by $h(\beta) = 1$ if and only if $\beta \in \bigcup_{j=1}^m [\beta_j, \beta_j + l_j]$ or $\beta = \beta_j + l_j + 2$ for some $j \mathbf{6} m$ or $\beta \in \{\alpha, \alpha + 1\}$. Then $h|K = f$ whence $U(h, F) \subset U(f, K)$. Define the point $z \in U(h, F)$ by $z|_F = h|_F$ and $z(\beta) = 0$ for all $\beta \in \omega_1 \backslash F$. It remains to show that $z \in \bigcup \gamma \backslash \bigcup \mu$.

Note that if $h^* = h|[\alpha, \alpha + 2]$ and $H = [\alpha, \alpha + 2]$ then $W = U(h^*, H) \in \mathcal{U}$ and hence $z \in W' \in \gamma$ where $W' = W \cap \sigma$. Therefore $z \in \bigcup \gamma$. Now, if $z \in U(g, G) \in \mu$ then z coincides with g on G. Since g is admissible and $G \subset \alpha$, there are distinct limit ordinals $\alpha_1, \ldots, \alpha_n < \alpha$ and positive integers $k_1, \ldots, k_n$ such that $G = \bigcup_{i=1}^n \{[\alpha_i, \alpha + 2k_i]$ and $g(\nu) = 1$ if and only if $\nu \in \bigcup_{i=1}^n \{[\alpha_i, \alpha + k_i]$. We have $z(\beta) = 0$ if $\beta \in \omega_1 \backslash F$, which implies $\bigcup_{i=1}^n [\alpha_i, \alpha_i + k_i] \subset F$. Therefore there exists $j \mathbf{6} m$ such that $[\alpha_1, \alpha_1 + k_1] \subset [\beta_j, \beta_j + l_j]$. This immediately implies that $\alpha_1 = \beta_j$ and $k_1 \mathbf{6} l_j$. If $k_1 < l_j$ then the inequalities $\beta_j < \alpha_1 + k_1 + 1 \mathbf{6} \beta_j + l_j$ imply $z(\alpha_1 + k_1 + 1) = 1$, while from $z \in U(g, G)$ it follows that $z(\alpha_1 + k_1 + 1) = 0$, a contradiction. This proves that $k_1 = l_j$. By definition of z we have $z(\alpha_1 + k_1 + 2) = 1$, while the inequalities $\beta_j + l_j < \beta_j + l_j + 2 \mathbf{6} \beta_j + 2l_j$ and $z \in U(g, G)$ imply that $z(\alpha_1 + k_1 + 2) = 0$, which is a contradiction. As a consequence, $z \notin U(g, G)$ for all $U(g, G) \in \mathcal{V}$, i.e., $z \notin \bigcup \mu$.

Now, let y be an arbitrary point of D^{ω_1}. If $y \in \Sigma$ then there is a sequence from σ converging to y, so let us suppose that the set $B = y^{-1}(1)$ is uncountable.

Denote by $\pi_B : \mathsf{D}^{\omega_1} \to \mathsf{D}^B$ the natural projection. By the above argument, there exists a discrete set $D \subset \pi_B(\sigma)$ such that $\pi_B(x) \in \mathrm{cl}_{\mathsf{D}^B}(D)$. Given a point $d \in D$, let $d^*(\nu) = d(\nu)$ for all $\nu \in B$ and $d^*(\nu) = 0$ for all $\nu \in \omega_1 \backslash B$. The set $D^* = \{d^* : d \in D\}$ is a discrete subspace of σ and $y \in \overline{D}$. $\square$

Example 4.12. We saw that any point of D^{ω_1} is reachable from Σ by a discrete subset. However this can not be proved for all dense subsets of Σ, because under CH there exists a dense subspace $L \subset \Sigma$ such that no point from $\mathsf{D}^{\omega_1} \backslash \Sigma$ can be reached from L by a discrete subset.

Proof. Under CH, there exists a dense Luzin subspace L of the space Σ [Ar]. If D is a discrete subspace of L then D is countable, so $\overline{D} \subset \Sigma$. $\square$

Remark 4.13. The argument of Example 4.12 is not applicable to σ because it is a union of a countably many discrete subspaces, so any subspace of σ of countable spread is countable and hence can not be dense in σ. So, the question arises as to whether for any dense subspace $X \subset \sigma$ any point of D^{ω_1} is in the d-closure of X. It turns out that the answer is not trivial at all.

THEOREM 4.14. *Assume MA+*$(\omega_1 < \mathfrak{c})$. *Then for any dense subspace X of the space σ any point $x \in \mathsf{D}^{\omega_1}$ is in the closure of a discrete subspace of X.*

PROOF. Take a sequence $\{x_n : n \in \omega\} \subset \mathsf{D}^{\omega_1} \setminus \{x\}$ with $x_n \to x$. Choose disjoint clopen subsets $\{U_n : n \in \omega\}$ of D^{ω_1} such that $x_n \in U_n$ for all $n \in \omega$ and let $X_n = X \cap U_n$. It is easy to see that any neighbourhood of x hits all but finitely many of the sets X_n. Each X_n is a union of countably many discrete subsets, because so is σ. It is easy to see that if we have two discrete subsets A and B in a space Y, then $A \cup B$ is scattered and hence there exists a discrete $C \subset Y$ such that $\overline{A} \cup \overline{B} \subset \overline{C}$. This makes it possible to choose for each natural m a discrete set $D(m,n) \subset X_n$ so that $\bigcup\{D(m,n) : m \in \omega\}$ is dense in X_n and $\overline{D(m,n)} \subset \overline{D(m+1,n)}$ for every $m \in \omega$.

Let $\{W_\alpha : \alpha < \omega_1\}$ be a neighbourhood base for x in D^{ω_1}. For any $\alpha < \omega_1$ there exists a function $g_\alpha \in \omega^\omega$ such that for all but finitely many n we have $W_\alpha \cap D(g_\alpha(n), n) \neq \emptyset$. Use Martin's Axiom to find a function $g \in \omega^\omega$ such that g is eventually larger than each g_α. It is not difficult to check that $D = \bigcup\{D(g(n), n) : n \in \omega\}$ is a discrete set and $x \in \overline{D}$. $\square$

Recall that the axiom of Jensen $(\diamondsuit)$ states that for any $\alpha < \omega_1$ there is a set $A_\alpha \subset \alpha$ such that for every $A \subset \omega_1$ the set $\{\alpha < \omega_1 : A \cap \alpha = A_\alpha\}$ is stationary, i.e., meets every cofinal subset of ω_1, which is closed in the interval topology on ω_1.

THEOREM 4.15. *Under the axiom of Jensen $(\diamondsuit)$ there exists a dense subspace X of the space σ such that the point $x \equiv 1 \in \mathsf{D}^{\omega_1}$ is not in the closure of any discrete subset of X.*

PROOF. A standard argument shows that there exists a $\diamondsuit$-sequence of functions $\{f_\lambda : \lambda < \omega_1\}$ such that

(Ω_1) $f_\lambda : \lambda \to [\lambda]^{<\omega} \times \mathsf{D}$ for any $\lambda < \omega_1$;

(Ω_2) for any function $f : \omega_1 \to [\omega_1]^{<\omega} \times \mathsf{D}$ the set $\{\lambda < \omega_1 : f|\lambda = f_\lambda\}$ is stationary.

For every $\lambda < \omega_1$, let $(F^\lambda_\beta, e^\lambda_\beta) = f_\lambda(\beta)$ and $J_\lambda = \{\beta < \lambda : e^\lambda_\beta = 1\}$. Denote by L the set of limit ordinals of ω_1 and by $\mathcal{F}$ the set of all finite partial functions from ω_1 to D; let $\{s_\alpha : \alpha < \omega_1\}$ be an enumeration (possibly with repetitions) of $\mathcal{F}$ for which $\mathrm{dom}(s_\alpha) \subset \alpha$ for each $\alpha < \omega_1$. Given an $f \in \mathcal{F}$ let $[f] = \{s \in \sigma : f \subset s\}$ and define $e(f) \in \sigma$ as follows: $e(f)(\alpha) = f(\alpha)$ for all $\alpha \in \mathrm{supp}(f)$ and $e(f)(\alpha) = 0$ for every $\alpha \notin \mathrm{supp}(f)$.

Our subset $X \subset \sigma$ will have the form $\{x_\alpha : \alpha < \omega_1\}$. To start with, for every $n \in \omega$, let $x_n = e(s_n)$ for all $n \in \omega$. Suppose that for some α ¿ ω we have constructed the points $\{x_\beta : \beta < \alpha\}$. For each $\lambda < \alpha$, let $D_\lambda = \{x_\beta : \beta \in J_\lambda\}$.

Claim. There exists an $x_\alpha \in \sigma$ with the following properties

$(U(\alpha))$ $\mathrm{dom}(x_\alpha) \subset \alpha + 1,\ s_\alpha \subset x_\alpha$;

$(V(\alpha))$ if $\mu \leqslant \alpha$ is some limit ordinal with $x_\alpha(\mu) = 1$, then either there is some finite $H \subset \mu$ such that $[x_\alpha|H]$ misses D_μ or there is a $\beta \in J_\mu$ such that $x_\beta|F^\mu_\beta \subset x_\alpha$.

Proof of the claim. Let $t_0 = s_\alpha$. We have two possibilities:

(P_0) $t_0^{-1}(1) \cap L = \emptyset$ or

(Q_0) $t_0^{-1}(1) \cap L \neq \emptyset$.

If (P_0) holds, let $x_\alpha = e(t_0)$. If not, then (Q_0) is true and we can define the ordinal $\mu_0 = \max\{\mu : \mu \in L \cap t_0^{-1}(1)\}$. Note that $\mu_0 \leqslant \alpha$. Suppose that we have constructed $t_0, \dots, t_n \in \mathcal{F}$ and $\mu_0, \dots, \mu_n \in \omega_1$ so that

(a) $\mu_0 > \dots > \mu_n$;

(b) $t_0 \subset \dots \subset t_n$.

To construct t_{n+1}, check if the following statement holds:

(R_n) there exists a $\beta_n < \mu_n$ such that $x_{\beta_n} \in D_{\mu_n}$ and t_n is compatible with $x_{\beta_n}|F^{\mu_n}_{\beta_n}$.

If (R_n) holds, set $t_{n+1} = t_n \cup x_{\beta_n}|F^{\mu_n}_{\beta_n}$. If not, let $t_{n+1} = t_n$. We have two possibilities:

(P_{n+1}) $t^{-1}_{n+1}(1) \cap L \cap \mu_n = \emptyset$.

(Q_{n+1}) $t^{-1}_{n+1}(1) \cap L \cap \mu_n \neq \emptyset$.

If (P_{n+1}) is true, let $x_\alpha = e(t_{n+1})$. If not, then (Q_{n+1}) holds and we can define the ordinal $\mu_{n+1} = \max\{\mu : \mu \in L \cap \mu_n \cap t^{-1}_{n+1}(1)\}$. It is clear that the properties (a) and (b) also hold for $\mu_0, \dots, \mu_{n+1}$ and $t_0, \dots, t_{n+1}$. By (a), the sequence $\{\mu_i\}$ can not be infinite, the property (P_{n+1}) holds for some $n \in \omega$ and therefore $x_\alpha = e(t_{n+1})$.

It is clear that $s_\alpha \subset x_\alpha$. Suppose that $\mu \leqslant \alpha$ is a limit ordinal such that $x_\alpha(\mu) = 1$. Assume first that $\mu = \mu_i$ for some i. If (R_i) is true then for $\beta = \beta_i$ we have $x_\beta|F^\mu_\beta \subset x_\alpha$. If not, then for $H = \mathrm{dom}(t_i) \cap \mu$ we have $[x_\alpha|H] \cap D_\mu = \emptyset$.

Now, if $\mu \neq \mu_i$ for all $i \leqslant n$ then it is impossible that $\mu < \mu_n$ because $t^{-1}_{n+1}(1) = x^{-1}_\alpha(1)$ and $t^{-1}_{n+1}(1) \cap L \cap \mu_n = \emptyset$. It is also impossible that $\mu_0 < \mu$ due to the fact that $\mu_0 = \max\{x_\alpha^{-1}(1) \cap L\}$. If $\mu_{i+1} < \mu < \mu_i$ for some i, then $\max\{t^{-1}_{i+1}(1) \cap L \cap \mu_i\} = \mu \neq \mu_{i+1}$, which is a contradiction proving our claim.

The claim shows that we can construct the set $X = \{x_\alpha : \alpha < \omega_1\}$ so that the conditions $(U(\alpha))$ and $(V(\alpha))$ are satisfied for all $\alpha < \omega_1$. Let us prove that the point $x \equiv 1$ is not in the closure of any discrete $D \subset X$. For any $\beta < \omega_1$ with $x_\beta \in D$, let $e_\beta = 1$ and choose a finite $F_\beta \subset \omega_1$ such that $W_\beta = [x_\beta|F_\beta]$ witnesses that x_β is isolated in D, i.e., $W_\beta \cap D = \{x_\beta\}$. If $x_\beta \notin D$, let $e_\beta = 0$ and choose a finite $F_\beta \subset \omega_1$ such that $W_\beta \cap D = \emptyset$ if

$x_\beta \notin \overline{D}$. If $x_\beta \in \overline{D}$, then it does not matter what F_β is, so let $F_\beta = \{0\}$ and define the function $f : \omega_1 \to [\omega_1]^{<\omega} \times \mathsf{D}$ by $f(\beta) = (F_\beta, e_\beta)$.

Let $B_1 = \{\lambda \in \omega_1 :$ for every $\beta < \lambda$ we have $F_\beta \subset \lambda\}$ and $B_2 = \{\lambda < \omega_1 :$ for any $H \subset [\lambda]^{<\omega}$ and any $s : H \to \mathsf{D}$ such that $[s] \cap D \neq \emptyset$ there is a $\beta < \lambda$ such that $x_\beta \in [s] \cap D\}$. A standard argument shows that the sets B_1 and B_2 are closed and unbounded in ω_1. Apply (Ω_2) to conclude that there is a $\lambda \in B_1 \cap B_2$ such that $f|\lambda = f_\lambda$. As a consequence, $F^\lambda_\beta = F_\beta$ for each $\beta < \lambda$ and $D \cap \{x_\beta : \beta < \lambda\} = D_\lambda$.

The set $W = \{y \in \mathsf{D}^{\omega_1} : y(\lambda) = 1\}$ is an open neighbourhood of the point x. Let us prove that $W \cap D = \emptyset$. Indeed, if $d \in D_\lambda$, then $(U(\lambda))$ implies $d(\lambda) = 0$. Now, suppose that $x_\gamma \in D$ for some γ ¿ λ such that $x_\gamma(\lambda) = 1$. Observe that for any finite $H \subset \lambda$ we have $x_\gamma \in [x_\gamma|H] \cap D$ and hence $[x_\gamma|H] \cap D \neq \emptyset$. Now, $\lambda \in B_2$ implies $[x_\gamma|H] \cap D_\lambda \neq \emptyset$. Applying $(V(\gamma))$ to the limit ordinal λ **6** γ we can conclude that there exists an $x_\beta \in D_\lambda$ such that $x_\beta|F_\beta \subset x_\gamma$ and therefore $x_\gamma \in W_\beta \cap D = \{x_\beta\}$, which is a contradiction. □

COROLLARY 4.16. *The statement "for any dense $X \subset \sigma$ any point from D^{ω_1} is in the closure of a discrete subset of X" is independent of ZFC.*

The last group of results describes the situation with regard to the weak generability of regular countably compact spaces.

THEOREM 4.17. *Any countably compact regular space X of character* **6** ω_1 *is weakly discretely generated.*

PROOF. Take a non-closed subset $A \subset X$ and suppose that the closure of every discrete subset of A is contained in A. This implies, in particular, that A is countably compact. Take any $p \in \overline{A} \setminus A$. For any G_δ-set $G \ni p$ we have $G \cap A \neq \emptyset$ for otherwise it is easy to construct a decreasing sequence of closed non-empty subsets of A with empty intersection which contradicts countable compactness of A. Hence $\psi(p, \{p\} \cup A) > \omega$. Since p does not belong to the closure of any (countable) discrete $B \subset A$, Theorem 4.8 is applicable, so we can conclude that there is a discrete $D \subset A$ with $p \in \overline{D}$, a contradiction. □

COROLLARY 4.18. *Any countably compact regular space of weight* **6** ω_1 *is weakly discretely generated.*

We will now show that there are models of ZFC where not all countably compact Tychonoff spaces of weight ω_2 are weakly discretely generated. Since the proofs depend heavily on the construction of certain special trees, let us recall some basic notions and facts about trees.

A tree is a partially ordered set $(S, <)$ such that for any $s \in S$ the set $\{t \in S : t < s\}$ is well ordered by $<$. We write S instead of $(S, <)$. A subset $S' \subset S$ is called *a subtree* of the tree S if $\{t \in S : t < s\} \subset S'$ for any $s \in S'$. A subset $C \subset S$ of a tree S is called *a chain* if $s < s'$ or

$s' < s$ for any distinct $s, s' \in C$. A set $A \subset S$ is *an antichain* if any distinct $a, b \in A$ are incomparable, i.e., neither $a < b$ nor $b < a$ is true. A tree S is *countably closed, or ω-closed* if for any countable chain $C \subset S$ there is an $s \in S$ such that $t \leqslant s$ for all $t \in C$. If S is a tree and $s \in S$ then $\mathrm{ht}(s)$ is the order type of the well ordered set $\{t \in S : t < s\}$. An element s of a tree S is called *a successor* if $\mathrm{ht}(s)$ is a successor ordinal. Given a tree S and an $s \in S$ let $T_S(s) = \{t \in S : s \leqslant t\}$. For any ordinal α and a tree S let $S_\alpha = \{s \in S : \mathrm{ht}(s) = \alpha\}$. The set S_α is called *the α-th level of S*. A tree S is *ever branching* if for any $s \in S$ there are incomparable $a, b \in S$ such that $s < a$ and $s < b$.

If S is a tree and $\{s_n : n \in \omega\} \subset S_\alpha$ for some α, let $T = \prod_* \{T_S(s_n) : n \in \omega\} = \{f : \omega \to S : f(\omega) \subset S_\beta$ for some β and $f(n) \geqslant s_n$ for all $n \in \omega\}$. If $f, g \in T$ then $f < g$ if $f(n) < g(n)$ for all $n \in \omega$. The pair $(T, <)$ is called *the tree product of the trees $\{T_S(s_n) : n \in \omega\}$*.

In this paper we will work with the tree $\omega_1^{<\omega_2} = \{f : f$ is a function from α to ω_1 for some $\alpha < \omega_2\}$ and its subtrees with the order defined by $f < g$ if g extends f. Given an $s \in \omega_1^{<\omega_2}$ with $\mathrm{dom}(s) = \beta$ and $\alpha < \omega_1$ the function $t = s^\frown\alpha$ has domain $\beta + 1$ and $t \restriction \beta = s,\ t(\beta) = \alpha$.

Recall that a subset $C \subset \omega_2$ is *closed unbounded* if it is closed in the order topology on ω_2 and cofinal in ω_2. A subset $B \subset \omega_2$ is called *stationary* if it intersects any closed unbounded subset of ω_2. Let $\omega_2^1 = \{\alpha < \omega_2 : \mathrm{cf}(\alpha) = \omega_1\}$. Observe that we do not use the standard notation S_1^2 for the set of ordinals from ω_2 of cofinality ω_1 to avoid possible confusion of this statement with the square of the first level S_1 of the tree S which will be constructed in 3.19. *The set-theoretic principle* $\diamondsuit(\omega_2^1)$ says that for each $\alpha < \omega_2$ there exists a set $A_\alpha \subset \alpha$ such that for any $A \subset \omega_2$ the set $\{\alpha \in \omega_2^1 : A \cap \alpha = A_\alpha\}$ is stationary. It is well known that $\diamondsuit(\omega_2^1)$ is consistent with CH and the usual axioms of ZFC. It is not difficult to prove that $\diamondsuit(\omega_2^1)$ is equivalent to the following statement:

For an arbitrary set A of cardinality ω_2 and any $\alpha < \omega_2$ there exists a function $f_\alpha : \alpha \to A$ such that for any map $f : \omega_2 \to A$ the set $\{\alpha \in \omega_2^1 : f \restriction \alpha = f_\alpha\}$ is stationary. The family $\{f_\alpha : \alpha < \omega_2\}$ is called $\diamondsuit(\omega_2^1)$-sequence for A.

THEOREM 4.19. *Suppose that CH and $\diamondsuit(\omega_2^1)$ hold. Then there exists a tree S with the following properties;*
(1) S is a countably closed subtree of the tree $\omega_1^{<\omega_2}$ and $|S| = \omega_2$;
(2) S has neither chains nor antichains of cardinality ω_2;
(3) every member of S has exactly ω_1 successors at every higher level of S;
(4) if $\{s_i : i \in \omega\} \subset S \cap (\omega_1)^\alpha$ for some $\alpha < \omega_2$, then the tree $\prod_* \{T_S(s_i) : i \in \omega\}$ has no antichains of cardinality ω_2.

PROOF. Since $|(\omega_1^{<\omega_2})^\omega| = \omega_2$, we can fix a sequence $\{g_\alpha : \alpha < \omega_2\}$ such that

($\diamondsuit_1$) $g_\alpha : \alpha \to (\omega_1^{<\omega_2})^\omega$ for each $\alpha \in \omega_2$;
($\diamondsuit_2$) for each $g : \omega_2 \to (\omega_1^{<\omega_2})^\omega$ the set $\{\alpha : \mathrm{cf}(\alpha) = \omega_1 \text{ and } g^- \alpha = g_\alpha\}$ is stationary.

Our tree S will have the form $S = \bigcup\{S_\alpha : \alpha < \omega_2\}$, where $S_\alpha \subset (\omega_1)^\alpha$ for each $\alpha < \omega_2$. We are going to construct the sets S_α by tranfinite induction. To start with, let $S_\alpha = (\omega_1)^\alpha$ for all $\alpha < \omega_1$.

Suppose that for some $\beta < \omega_2$ we have constructed the sets S_α for all $\alpha < \beta$ so that
($\Omega_1(\beta)$) the set $S(\beta) = \bigcup\{S_\alpha : \alpha < \beta\}$ is an ω-closed tree;
($\Omega_2(\beta)$) if $s \in S(\beta)$ and $\mathrm{ht}(s) + 1 < \beta$, then $s^\frown\gamma \in S(\beta)$ for all $\gamma < \omega_1$;
($\Omega_3(\beta)$) if $\alpha < \gamma < \beta$ then for every $s \in S_\alpha$ there is $t \in S_\gamma$ such that $s < t$;
($\Omega_4(\beta)$) $|S_\alpha| = \omega_1$ for each $\alpha < \beta$.

If $\beta = \gamma + 1$, for every $s \in S_\gamma$ let $P_s = \{s^\frown\alpha : \alpha < \omega_1\}$ and $S_\beta = \bigcup\{P_s : s \in S_\gamma\}$. If β is an ordinal of countable cofinality, let $S_\beta = \{s \in (\omega_1)^\beta : s^-\alpha \in S_\alpha \text{ for all } \alpha < \beta\}$. It is clear that in both cases the properties $(\Omega_1(\beta + 1)) - (\Omega_4(\beta + 1))$ hold. Moreover, whatever we do at the ordinals of uncountable cofinality, after the construction is finished, the conditions $(\Omega_1(\omega_2))$ and $(\Omega_2(\omega_2))$ will be satisfied.

Now assume that $\mathrm{cf}(\beta) = \omega_1$. It follows easily from the Continuum Hypothesis, that the set $E = S(\beta) \times (\omega_1)^\omega$ has cardinality ω_1; let $\{(x_\alpha, f_\alpha) : \alpha < \omega_1\}$ be some enumeration of E in which every pair $(x, f) \in S(\beta) \times (\omega_1)^\omega$ occurs ω_1 times. In addition, let $\{\beta_\alpha : \alpha < \omega_1\}$ be any continuous increasing sequence cofinal in β. The set S_β will be constructed as $\{s_\mu : \mu < \omega_1\} \subset (\omega_1)^\beta$. First, look at the function g_β to check if the following statement $P(\beta)$ holds:

$P(\beta)$: "There exists an $\alpha < \beta$ such that $\{g_\beta(0)(n) : n \in \omega\} \subset S_\alpha$ and the sequence $\{g_\beta(\gamma) : 0 < \gamma < \beta\}$ is a maximal antichain of $\prod_*\{T_{S(\beta)}(g_\beta(0)(n)) : n \in \omega\}$".

If $P(\beta)$ is not true, then note that, by ω-closedness of $S(\beta)$, for each $\mu < \omega_1$ there exists a chain $C_\mu \subset S(\beta)$ such that $x_\mu \in C_\mu$ and $C_\mu \cap S_\gamma \neq \emptyset$ for all $\gamma < \beta$. Let $S_\beta = \{s_\mu : \mu < \omega_1\}$, where $s_\mu = \bigcup C_\mu$ for each $\mu < \omega_1$.

However, if $P(\beta)$ is true, more work is required. In this case we will construct $S_\beta = \{s_\mu : \mu < \omega_1\}$ by transfinite induction of length ω_1. For $\nu_1 = \max\{\beta_1, \mathrm{ht}(x_0)\}$ choose any $s \in S_{\nu_1}$ for which $x_0 \leqslant s$ and let $s_1^0 = s$.

Suppose that for some $\xi < \omega_1$ we constructed the sequences $\{s_\gamma^\mu : \mu < \gamma < \xi\}$ and $\{\nu_\gamma : 0 < \gamma < \xi\}$ so that the following conditions hold:
(i) ν_γ is an ordinal and $\beta_\gamma \leqslant \nu_\gamma < \beta$ for each $\gamma < \xi$;
(ii) $s_\gamma^\mu \in (\omega_1)^{\nu_\gamma}$ for all $\mu < \gamma < \xi$;
(iii) $\nu_\gamma < \nu_{\gamma'}$ and $s_{\gamma'}^{\mu\,-}\, \nu_\gamma = s_\gamma^\mu$ for any $\mu < \gamma < \gamma' < \xi$.

If ξ is a limit ordinal, let $\nu_\xi = \sup\{\nu_\gamma : \gamma < \xi\}$ and $s^\mu_\xi = \bigcup\{s^\mu_\gamma : \mu < \gamma < \xi\}$ for all $\mu < \xi$. It is clear that the properties (i)–(iii) hold for all $\mu < \gamma \leqslant \xi$.

Now, if $\xi = \delta + 1$, check whether the following statement $Q(\delta)$ is true:

$Q(\delta)$: "$f_\delta(n) < \delta$ for every $n \in \omega$ and the ω-tuple $\langle s^{f_\delta(n)}_\delta : n \in \omega\rangle$ belongs to the tree $\prod_*\{T_{S(\beta)}(g_\beta(0)(n)) : n \in \omega\}$."

If $Q(\delta)$ does not hold, let $\nu_\xi = \max\{\beta_\xi, \nu_\delta, \mathrm{ht}(x_\delta)\}$. Define s^δ_ξ to be any element s from S_{ν_ξ} for which $x_\delta \leqslant s$. If $\mu < \delta$, define s^μ_ξ to be any element $t \in S_{\nu_\xi}$ such that $s^\mu_\delta \leqslant t$.

Now, if $Q(\delta)$ is satisfied, then the ω-tuple $\langle s^{f_\delta(n)}_\delta : n \in \omega\rangle$ is compatible with some $g_\beta(\nu)$ due to the fact that $\{g_\beta(\gamma) : 0 < \gamma < \beta\}$ is a maximal antichain of the tree product $\prod_*\{T_{S(\beta)}(g_\beta(0)(n)) : n \in \omega\}$. If $g_\beta(\nu) = \langle t_n : n \in \omega\rangle$ then all elements of the set $\{t_n : n \in \omega\}$ belong to some level S_μ of the tree $S(\beta)$. Analogously, $\{s^{f_\delta(n)}_\delta : n \in \omega\} \subset S_{\mu'}$ for some $\mu' < \beta$. Let $\nu_\xi = \max\{\beta_\xi, \nu_\delta, \mathrm{ht}(x_\delta), \mu, \mu'\}$. Define s^δ_ξ to be any element $s \in S_{\nu_\xi}$ for which $x_\delta \leqslant s$. Since $\langle s^{f_\delta(n)}_\delta : n \in \omega\rangle$ and $\langle t_n : n \in \omega\rangle$ are compatible, there exists a sequence $\{u_n : n \in \omega\} \subset S_{\nu_\xi}$ such that the ω-tuple $\langle u_n : n \in \omega\rangle$ is an extension of $\langle t_n : n \in \omega\rangle$ and $\langle s^{f_\delta(n)}_\delta : n \in \omega\rangle$ at the same time. For each $n \in \omega$, let $s^{f_\delta(n)}_\xi = u_n$; if $\gamma \notin \{\delta\} \cup \{f_\delta(n) : n \in \omega\}$, take any $w \in S_{\nu_\xi}$ with $s^\gamma_\delta \leqslant w$ and let $s^\gamma_\xi = w$. This ends our inductive construction and gives us the sequences $\{s^\mu_\gamma : \mu < \gamma < \omega_1\}$ and $\{\nu_\gamma : 0 < \gamma < \omega_1\}$ for which the conditions (i)–(iii) are satisfied.

For each $\mu < \omega_1$, let $s_\mu = \bigcup\{s^\mu_\gamma : \mu < \gamma < \omega_1\}$ and $S_\beta = \{s_\mu : \mu < \omega_1\}$. Note that the properties $\Omega_1(\beta+1)$, $\Omega_2(\beta+1)$ and $\Omega_4(\beta+1)$ hold trivially. To prove $\Omega_3(\beta+1)$, fix an $s \in S_\alpha$. There is nothing to prove if $\gamma < \beta$. Now if $\gamma = \beta$, take a $\xi < \omega_1$ such that $x_\xi = s$ and note that $x_\xi \leqslant s^\xi_{\xi+1} < s_\xi \in S_\beta$. Thus, $\Omega_3(\beta+1)$ also holds.

This concludes the construction of the tree $S = \bigcup\{S_\alpha : \alpha < \omega_2\}$. Let us show that S has the properties (1)–(4).

Note first, that the properties $(\Omega_1(\beta)) - (\Omega_4(\beta))$ hold for each $\beta < \omega_2$ and therefore $(\Omega_1(\omega_2)) - (\Omega_4(\omega_2))$ are fulfilled as well. That (1) is satisfied follows immediately from $\Omega_1(\omega_2)$ and $\Omega_4(\omega_2)$. If $\alpha < \beta < \omega_2$ and $s \in S_\alpha$ then $\alpha + 1 \leqslant \beta$ and there are ω_1 many successors of s in $S_{\alpha+1}$ by $(\Omega_2(\omega_2))$. Since every $t \in S_{\alpha+1}$ has a successor in S_β by $(\Omega_3(\omega_2))$, this proves (3). Since S is an ever branching tree, to prove (2) it suffices to establish that there are no antichains of cardinality ω_2 in S.

Suppose that C is an antichain in S with $|C| = \omega_2$. Since $|S_1| = \omega_1$, there is an $s \in S_1$ such that the set $C_1 = \{f \in C : f^{-}1 = s\}$ has cardinality ω_2. Take a faithful enumeration $\{f_\alpha : \alpha < \omega_2\}$ of the set C_1. Let $D = \{\langle f_n^\alpha : n \in \omega\rangle : \alpha < \omega_2\}$, where $f_n^\alpha = f_\alpha$ for all $n \in \omega$. It is evident that D is an antichain in $\prod_* \{T_S(s_i) : i \in \omega\}$ where $s_i = s$ for all $i \in \omega$. As a consequence, the property (4) implies (2) so we only have to prove (4).

Take an arbitrary $\alpha < \omega_2$ and $\{s_n : n \in \omega\} \subset S_\alpha$. Suppose that a function $g : \omega_2 \backslash \{0\} \to \prod_* \{T_S(s_n) : n \in \omega\}$ enumerates a maximal antichain in the tree product $U = \prod_* \{T_S(s_n) : n \in \omega\}$. Set $g(0) = \langle s_n : n \in \omega\rangle$. Given a $u = \langle t_n : n \in \omega\rangle \in U$ let $l(u) = \mathrm{ht}(u(0))$. Evidently, $l(u) = \mathrm{ht}(u(n))$ for any natural n. For any $\beta \in \omega_2 \backslash \{0\}$ let $p(\beta) = l(g(\beta))$. Since for each $\gamma < \omega_2$ the set $\{p(\beta) : \beta < \gamma\}$ has cardinality ω_1, for the ordinal $q(\gamma) = \min\{\delta$ ¿ $\gamma : \{p(\beta) : \beta < \gamma\} \subset \delta\} < \omega_2$ we have $\{p(\beta) : \beta < \gamma\} \subset q(\gamma)$. It takes a standard routine proof to show that there is a closed unbounded $B_1 \subset \omega_2$ such that $q(\gamma) = \gamma$ for any $\gamma \in B_1$.

Since CH holds, for any $\beta < \omega_2$ the set $U(\beta) = \{u \in U : l(u)$ **6** $\beta\}$ has cardinality at most ω_1. Let $r(\beta) = \min\{\gamma$ ¿ $\beta : \{\lambda : p(\lambda) < \beta\} \subset \gamma\}$. Then $p(\lambda)$ ¿ β for each λ ¿ $r(\beta)$. This shows that it can be proved in a standard way that there exists a closed unbounded $B_2 \subset \omega_2$ such that for each $\lambda \in B_2$ we have $r(\lambda) = \lambda$. The set $\{g(\lambda) : 0 < \lambda < \omega_2\}$ is a maximal antichain in U so for each $u \in U$ there is a $\lambda(u) < \omega_2$ for which $g(\lambda(u))$ is compatible with u. For each $\beta < \omega_1$ consider $w(\beta) = \min\{\gamma : \lambda(U(\beta)) \subset \gamma\}$. Then $\lambda(u) < w(\beta)$ for all $u \in U(\beta)$. As a result, there exists a closed unbounded $B_3 \subset \omega_2$ such that $w(\beta) = \beta$ for every $\beta \in B_3$. Since the set $B = B_1 \cap B_2 \cap B_3$ is closed unbounded, we can apply $\diamondsuit_2$ to conclude that there is a $\beta \in B$ of cofinality ω_1 such that $g^{-}\beta = g_\beta$.

Since $\beta \in B_2$ we have $p(\beta)$ ¿ β. The level S_β was constructed as the set $\{s_\mu : \mu < \omega_1\}$. For each natural n we have $g(\beta)(n)^{-}\beta = s_{\mu_n}$ for some $\mu_n < \omega_1$. Then $f = \langle \mu_n : n \in \omega\rangle \in (\omega_1)^\omega$ and hence there are ω_1 many $\xi < \omega_1$ such that $f = f_\xi$. Take any $\delta < \omega_1$ with $f = f_\delta$ and $\mu_n < \delta$ for every $n \in \omega$.

Observe that $\beta \in B_3$ implies that for every $u \in U$ there is a $\gamma < \beta$ such that $g(\gamma)$ is compatible with u. This means that $\{g(\gamma) : 0 < \gamma < \beta\} = \{g_\beta(\gamma) : 0 < \gamma < \beta\}$ is a maximal antichain in U, i.e., the condition $P(\beta)$ holds. Now remember the $(\delta+1)$-th step of the construction of S_β. Since $\mu_n = f(n) = f_\delta(n) < \delta$ and $s_\delta^{f_\delta(n)} = g(\beta)(n)^{-}\nu_\delta$ for each $n \in \omega$, the condition $Q(\delta)$ is satisfied. This means that there is a $\gamma < \beta$ such that $g_\beta(\gamma)$ is compatible with $\langle s_\delta^{f_\delta(n)} : n \in \omega\rangle$ which is evidently compatible with $\langle \mu_n : n \in \omega\rangle$ which in its turn is compatible with $g(\beta)$.

Thus, $g(\beta)$ is compatible with $g(\gamma) \neq g(\beta)$, a contradiction with the fact that $\{g(\lambda) : 0 < \lambda < \beta\} = \{g_\beta(\lambda) : 0 < \lambda < \beta\}$ is an antichain. $\square$

Recall that a space X is called *ω-bounded* if for any countable $A \subset X$ the subspace $\overline{A}$ is compact. Clearly, each ω-bounded space is countably compact.

THEOREM 4.20. *Under CH and $\diamondsuit(\omega_2^1)$ there is an ω-bounded Tychonoff space X of weight ω_2 and a point $w \in \beta X \setminus X$ such that $w \notin cl_{\beta X}(D)$ for any discrete subspace $D \subset X$. Therefore $X \cup \{w\}$ is a countably compact ω-bounded space which is not weakly discretely generated.*

PROOF. Consider the tree $S = \bigcup\{S_\alpha : \alpha < \omega_2\}$ constructed in Theorem 4.19. Let τ be the topology generated on S by the family

$$\mathcal{S} = \{T_S(s) : s \in S_{\alpha+1},\ \alpha \in \omega_2\} \cup \{S \setminus T_S(s) : s \in S_{\alpha+1},\ \alpha \in \omega_2\}$$

as a subbase. Clearly, for each $\alpha < \omega_2$ and $s \in S_{\alpha+1}$ the set $T_S(s)$ is clopen in (S, τ). It is an easy exercise to see that (S, τ) is a Tychonoff space. In what follows we identify S with the topological space (S, τ) and the subsets of S with the respective subspaces of (S, τ). Let $Y = S \setminus S_0$ and $X = \{z \in \beta Y :$ there is a countable $A_z \subset Y$ such that $z \in \overline{A_z}\}$. It is clear that X is ω-bounded. Observe that, for any $s \in S$, the family $\{T_S(s^\frown n) : n \in \omega\}$ is a local π-base of s in S.

Claim. Each discrete subset of X has cardinality at most ω_1.

Proof of the claim. Suppose that $D \subset X$ is discrete and has cardinality ω_2. For each $d \in D$ fix a clopen neighbourhood U_d of the point d such that $U_d \cap D = \{d\}$. Using the remark about the π-bases, for each $d \in D$, and $s \in A_d \cap U_d$ choose a countable $P_s \subset S$ such that $\{T_S(p) : p \in P_s\}$ is a π-base at s and $\bigcup\{T_S(p) : p \in P_s\} \subset U_d$. Let $B_d = \bigcup\{P_s : s \in A_d \cap U_d\}$. Then the sets B_d have the following properties for each $d \in D$:

$P(d) : \bigcup\{T_S(s) : s \in B_d\} \subset U_d$;
$Q(d)$: the family $\{T_S(s) : s \in B_d\}$ is a π-base at d.

It is clear that if $d, e \in D$ and $d \neq e$ then $d \notin \overline{\bigcup\{T_S(t) : t \in B_e\}}$. Observe that if s **6** t then $T_S(t) \subset T_S(s)$. As a consequence, if $d \in D$ and for every $s \in B_d$ we choose in a non-limit level of S an $f(s)$ ¿ s then $\{T_S(f(s)) : s \in B_d\}$ is still a π-base at d and $\bigcup\{T_S(f(s)) : s \in B_d\} \subset \bigcup\{T_S(s) : s \in B_d\}$. The property (3) for S implies that for any $d \in D$ and any $s \in B_d$ there exists an $f(s) \in S_{\alpha+1} \cap T_S(s)$, where $\alpha = \sup\{ht(s) : s \in B_d\}$. Therefore $B'_d = \{f(s) : s \in B_d\} \subset S_{\alpha+1}$, the family $\{T_S(s) : s \in B'_d\}$ is a π-base at d and $\bigcup\{T_S(s) : s \in B'_d\} \subset \bigcup\{T_S(s) : s \in B_d\} \subset U_d$. This shows that, without loss of generality, we can assume that each B_d is contained in some $S_{\alpha+1}$. Let $\mu(d) = \min\{\alpha < \omega_2 : B_d \subset S_{\alpha+1}\}$.

Our plan is to find distinct $d, e \in D$ such that for each $s \in B_d$ there is a $t \in B_e$ such that $t < s$. This will imply $\bigcup\{T_S(s) : s \in B_d\} \subset \bigcup\{T_S(t) : t \in B_e\}$ and hence $d \in \overline{\bigcup\{T_S(t) : t \in B_e\}}$ which is a contradiction.

By CH, there only ω_1 countable subsets contained in each level of S and therefore the set $\{\mu(d) : d \in D\}$ is cofinal in ω_2. This makes it possible to choose a sequence $\{\gamma_\alpha : \alpha < \omega_2\}$ so that the following properties hold:

(a) $\gamma_\alpha = \mu(d_\alpha)$ for some $d_\alpha \in D$;
(b) $\gamma_\beta > \sup\{\gamma_\alpha : \alpha < \beta\}$ for each $\beta < \omega_2$.

Note that it follows from (a) and (b) that $d_\alpha \neq d_\beta$ if $\alpha \neq \beta$. Let C be the closure of $\{\gamma_\alpha : \alpha < \omega_2\}$ in ω_2 (considered with the interval topology). Then C is a closed unbounded subset of ω_2 and therefore $E = C \cap \omega_2^1$ is stationary. For each $\lambda \in E$ let $\nu(\lambda) = \min\{\gamma_\alpha : \lambda < \gamma_\alpha\}$. Let $F_\lambda = B_{d_\alpha}$ and $e_\lambda = d_\alpha$, where α is determined by the condition $\gamma_\alpha = \nu(\lambda)$. Note that $\lambda, \delta \in E,\ \lambda < \delta$ implies that $\nu(\lambda) < \nu(\delta)$ and therefore $e_\lambda \neq e_\delta$.

For any $\lambda \in \omega_2$ and $s \in S$ denote by $\pi_\lambda : S \to S_\lambda$ the projection: $\pi_\lambda(s) = s^{-}\lambda$. For each $\lambda \in E$ choose a $G_\lambda \subset F_\lambda$ such that $\pi_\lambda{}^{-}G_\lambda : G_\lambda \to \pi_\lambda(F_\lambda)$ is a bijection. Since λ has cofinality ω_1, there exists a $\beta(\lambda) < \lambda$ such that the restriction $\pi_{\beta(\lambda)} : G_\lambda \to S_{\beta(\lambda)}$ is a bijective map. By pressing down lemma, there is a $\delta < \omega_1$ such that the set $\{\lambda \in E : \beta(\lambda) = \delta\}$ is stationary. Using CH find a set $P = \{s_n : n \in \omega\} \subset S_\delta$ such that the set $R = \{\lambda \in E : \pi_\delta(G_\lambda) = P\}$ has cardinality ω_2. Let $f_\lambda : \omega \to G_\lambda$ be any surjection. Then $F = \{f_\lambda : \lambda \in R\} \subset \prod_*\{T_S(s_n) : n \in \omega\}$ can not be an antichain in $\prod_*\{T_S(s_n) : n \in \omega\}$ by property (4) of the tree S. Thus, there are distinct $\lambda, \beta \in R$, say, $\lambda < \beta$ such that $f_\lambda(n) < f_\beta(n)$ for all $n \in \omega$. If $s \in F_\beta$, then $s^{-}\beta = f_\beta(n)^{-}\beta$ for some $n \in \omega$ and therefore $t = f_\lambda(n) \in F_\lambda$ and $t = f_\lambda(n) < f_\beta(n)^{-}\beta \ \mathbf{6}\ s$. Remember that $F_\beta = B_d$ and $F_\lambda = B_e$ for some distinct d and e and for each $s \in B_d$ there is a $t \in B_e$ such that $t < s$. We saw already that this is a contradiction which finishes the proof of our claim.

Returning to the proof of our theorem, note that for any discrete $D \subset X$ we have $|D|\ \mathbf{6}\ \omega_1$. For each $d \in D$ there exists a countable $B_d \subset S$ such that $\{T_S(s) : s \in B_d\}$ is a π-base at d. If $\alpha = \sup\{\mathrm{ht}(s) : s \in B_d,\ d \in D\} + 1$ then by the property (3) of the tree S for each $d \in D$ and for every $s \in B_d$ there is an $h(s) \in S_\alpha \cap T_S(s)$. It is clear that $d \in \overline{\{h(s) : s \in B_d\}}$ and therefore $D \subset \overline{S}_\alpha$. Thus, to construct the promised point, it is sufficient to find a point $w \in \beta X \setminus X$ such that $w \notin \overline{S}_\alpha$ for all $\alpha < \omega_2$.

To do this, we will construct a family $\mathcal{W} = \{W_\gamma : \gamma < \omega_2\}$ such that

(i) W_γ is a clopen subset of Y;
(ii) $W_\gamma \cap S_\gamma = \emptyset$ for each $\gamma < \omega_2$;
(iii) the family $\mathcal{W}$ has finite intersection property.

To see that it is sufficient, observe that $\beta Y = \beta X$ and the family $\mathcal{U} = \{\mathrm{cl}_{\beta Y}(W_\gamma) : \gamma < \omega_2\}$ consists of compact open sets. Since $\mathcal{U}$ has finite intersection property and $\mathrm{cl}_{\beta Y}(W_\gamma) \cap \mathrm{cl}_{\beta Y}(S_\gamma) = \emptyset$ for each $\gamma < \omega_2$, any point $w \in \bigcap\{\mathrm{cl}_{\beta Y}(W_\gamma) : \gamma < \omega_2\}$ will be as promised.

Let $\{s_\alpha : \alpha < \omega_1\}$ be a faithful enumeration of the first level S_1 of the tree S. For each $\gamma < \omega_2$ we will construct a function $f_\gamma : \omega_1 \to S' = \bigcup\{S_{\lambda+1} : \lambda < \omega_2\}$ with the following properties:

(c) $f_\gamma(\alpha) \geqslant s_\alpha$ for all $\gamma < \omega_2$ and $\alpha < \omega_1$;
(d) $\mathrm{ht}(f_\gamma(\alpha)) > \gamma$ for every $\gamma < \omega_2$ and $\alpha < \omega_1$;
(e) if $\gamma < \lambda < \omega_2$ then there exists a $\beta < \omega_1$ such that $f_\lambda(\alpha) \geqslant f_\gamma(\alpha)$ for all $\alpha \geqslant \beta$.

To start with, let $f_0(\alpha) = s_\alpha$ for all $\alpha < \omega_1$. Assume that, for some $\delta < \omega_2$, we have constructed the functions $\{f_\gamma : \gamma < \delta\}$. There are three cases to consider.

1) $\delta = \mu + 1$. Then for each α, let $f_\delta(\alpha) = f_\mu(\alpha)^\frown 0$. It is clear that the properties (c)-(e) hold for all $\gamma \leqslant \delta$.

2) $\mathrm{cf}(\delta) = \omega$. Take a strictly increasing sequence $\{\delta_n : n \in \omega\}$ cofinal in δ. There exists a $\beta < \omega_1$ such that $f_{\delta_{n+1}}(\alpha) \geqslant f_{\delta_n}(\alpha)$ for all $\alpha \geqslant \beta$ and $n \in \omega$. Since the tree S is countably closed, for each $\alpha \geqslant \beta$ there is a $t_\alpha \in S'$ such that $f_{\delta_n}(\alpha) < t_\alpha$ for all $n \in \omega$; set $f_\delta(\alpha) = t_\alpha$. If $\alpha < \beta$, let $f_\delta(\alpha) = f_{\delta_0}(\alpha)$. We omit the routine and straightforward verification that (c)-(e) hold for all $\gamma \leqslant \delta$.

3) $\mathrm{cf}(\delta) = \omega_1$. Choose a strictly increasing sequence $\{\delta_\nu : \nu < \omega_1\}$ cofinal in δ. The property (e) guarantees the existence of a cofinal in ω_1 strictly increasing sequence $\{\beta_\nu : \nu < \omega_1\}$ such that $\beta_0 = 0$ and for any $\nu < \omega_1$ and $\alpha \geqslant \beta_\nu$ we have $f_{\delta_\nu}(\alpha) \geqslant f_{\delta_\mu}(\alpha)$ for all $\mu < \nu$. Let $\xi = \sup\{\mathrm{ht}(f_\gamma(\alpha)) : \gamma < \delta$ and $\alpha < \omega_1\}$. Now, for each $\alpha < \omega_1$, choose a $\nu < \omega_1$ such that $\beta_\nu \leqslant \alpha < \beta_{\nu+1}$ and select $f_\delta(\alpha) \in S_{\xi+1}$ so that $f_\delta(\alpha) > f_{\beta_\nu}(\alpha)$. It is again straightforward to see that (c)-(e) hold for all $\gamma \leqslant \delta$.

Once we have the sequence $\{f_\gamma : \gamma < \omega_2\}$, let $W_\gamma = \bigcup\{T_S(f_\gamma(\alpha)) : \alpha < \omega_1\}$. The condition (e) implies that the family $\mathcal{W} = \{W_\gamma : \gamma < \omega_1\}$ has the finite intersection property. Observe that the family $\{T_S(s_\alpha) : \alpha < \omega_1\}$ is discrete and consists of clopen subsets of Y. The condition (c) shows that $T_S(f_\gamma(\alpha)) \subset T_S(s_\alpha)$ for each $\alpha < \omega_1$. Thus, the family $\{T_S(f_\gamma(\alpha)) : \alpha < \omega_1\}$ is discrete for each $\gamma < \omega_2$. This proves (i). The property (ii) is an immediate consequence of (d) and our theorem is proved. $\square$

5. Unsolved problems

In this section we list some open problems, which indicate a natural line of further investigation of discretely and weakly discretely generated spaces.

Problem 5.1. *Let X be a discretely generated compact space. Is it true that $X \times X$ is discretely generated?*

Problem 5.2. *Let X be a discretely generated compact space. Is it true that any continuous image of X is discretely generated?*

Problem 5.3. *Let X be a (weakly) discretely generated space. Is it true that any perfect image of X is (weakly) discretely generated?*

Problem 5.4. *Let X be a weakly discretely generated space. Is it true that $X \times X$ is weakly discretely generated?*

Problem 5.5. *Is there in ZFC a countably compact Tychonoff (or regular) space which is not weakly discretely generated?*

Since scattered compact spaces as well as compact spaces of countable tightness have a point-countable π-base, it is natural to ask whether the same is true for discretely generated compact spaces.

Problem 5.6. *Let X be a discretely generated compact space. Must it have a point-countable π-base?*

The final problem must be very difficult, because it is answered positively in all known models of ZFC, while a negative answer would imply the existence of a model of ZFC without L-spaces.

Problem 5.7. *Let X be a discretely generated dyadic space. Is X metrizable in ZFC?*

References

[Ar] A. V. Arhangel'skii, *Structure and classification of topological spaces and cardinal invariants* (in Russian), Uspehi Mat. Nauk, **33** (1978), 29–84.

[vD] E. K. van Douwen, *Applications of maximal topologies,* Topology and Its Applications, **51** (1993), 125–139.

[Ef] B. A. Efimov, *Mapping and embeddings of dyadic spaces,* (in Russian), Matematicheskii Sbornik, **103** (1977), 52–68.

[En] R. Engelking, General Topology, PWN, Warszawa, 1977.

[Gr] G. Gruenhage, *Generalized metric spaces,* In: Handbook of Set-Theoretic Topology, Edited by K. Kunen and J. E. Vaughan, Elsevier Science Publishers B. V., 1984, 423–502.

[No] N. Noble, *Countably compact and pseudocompact products,* Czechoslovak Mathematical Journal, **19** (1969), 390–397.

[Tk] V. V. Tkachuk, *Spaces that are projective with respect to classes of mappings*, Transactions of Moscow Mathematical Society, **50** (1988), 139–156.

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