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k-PROPER FAMILIES AND ALMOST APPROXIMATELY POLYNOMIAL FUNCTIONS

Jacek Tabor
Institute of Mathematics, Jagiellonian University, Kraków, Poland

Abstract. Using the notion of k-proper family we prove general results on the almost approximate stability of the polynomial functional equation. As one of the corollaries we obtain the generalization of the Theorem of R. Ger (cf. [4]):

Theorem *Let G be a uniquely $n!$ divisible commutative semigroup and let $\mathcal{I}$ be a translation invariant proper ideal in G such that $\frac{1}{i}U \in \mathcal{I}$ for every $U \in \mathcal{I}$, $i = 1, \dots, n$. Let E be a Banach space.*

Then there for every $\varepsilon > 0$ and every $f : G \to E$ satisfying

$$\|\Delta^n f(x,h)\| \leq \varepsilon \quad \text{for } \Omega(\mathcal{I})\text{-a.a. } (x,h) \in G \times G$$

there exists a unique up to a constant function polynomial $p : G \to E$ of order $n-1$ such that

$$\|f(x) - p(x)\| \leq 2(2^n - 1)\varepsilon \quad \text{for } \mathcal{I}\text{-a.a. } x \in G.$$

1. Introduction

Due to their importance polynomials have for a long time attracted mathematical attention. When one considers the stability of polynomials, which is the aim of the present paper, one has to mention the results of M. Albert and J. Baker who proved in [1] that the polynomial equation is stable in the Hyers-Ulam sense and that of R. Ger from [4] who proved that if a given function satisfies a polynomial equation almost everywhere then it is equal to a certain polynomial almost everywhere.

Using the notion of a k-proper family which we have introduced in [9] in Section 2 we join and generalize the just mentioned results of R. Ger and M. Albert, J. Baker and obtain the so called almost approximate stability

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of the polynomial functional equation. In other words it means that if a given function almost everywhere satisfies with certain small bounded from above error the polynomial functional equation then there exists a polynomial function which approximates the given one almost everywhere.

In Section 3 we obtain an analogue of results from [10] for polynomials. We prove that if the n-th difference of a given function is integrable then it is in fact near to a polynomial.

2. Basic definitions

Throughout this paper G will denote commutative semigroup and H will denote abelian group. For functions $f : G \to H$ we define the *zero difference* of f by the formula

$$\Delta^0 f(x,h) := f(x) \quad \text{for } x,h \in G.$$

For $n \in \mathbf{N}$ we define inductively the *n-th difference* of f

$$\Delta^n f(x,h) := \Delta^{n-1} f(x+h,h) - \Delta^{n-1} f(x,h).$$

One can easily check by induction that for $n \in \mathbf{N}$

$$\Delta^n f(x,h) = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} f(x+jh) \quad \text{for } x,h \in G. \tag{2.1}$$

or equivalently that

$$f(x) = (-1)^n \Delta^n f(x,h) - \sum_{j=1}^{n} (-1)^{j} \binom{n}{j} f(x+jh) \quad \text{for } x,h \in G. \tag{2.2}$$

Definition 1. *We say that f is a* polynomial function of order $n-1$ *if*

$$\Delta^n f(x,h) = 0 \quad \textit{for } x,h \in G.$$

It is well-known that if $f : \mathbf{R} \to \mathbf{R}$ is a continuous polynomial function (in the above sense) of degree n then it is a common polynomial, that is there exist $a_0, \ldots a_n$ such that $f(x) = \sum_{i=0}^{n} a_n x^n$. Thus we see that the above definition generalizes the idea of polynomials for commutative groups.

Let $\mathcal{I}$ be a family of subsets of G. $\mathcal{I}$ is a *translation invariant* family iff $\mathcal{I}$ is proper and

$$-a + A := \{x \in G \mid a + x \in A\} \in \mathcal{I} \quad \text{for } a \in G, A \in \mathcal{I}.$$

We say that family $\mathcal{I}$ is *proper* if $G \notin \mathcal{I}$.

Now we explain what we mean by small sets.

Definition 2. *For a family $\mathcal{I}$ of subsets of G and $k \in \mathbf{N}$ we define*

$$\mathcal{I}^k := \{A_1 \cup \ldots \cup A_k \mid A_1, \ldots, A_k \in \mathcal{I}\}.$$

We say that $\mathcal{I}$ is k-proper *iff $\mathcal{I}^k$ is proper.*

If $\mathcal{I}$ is a k-proper translation invariant family then for the shortness of notation we write that $\mathcal{I}$ is a k-p.t.i. family.

Clearly a family $\mathcal{I}$ in G is proper if it is 1-proper. We write that a given condition is satisfied for $\mathcal{I}$-*almost all* (shortly $\mathcal{I}$-a.a.) $x \in G$ if there exists $U \in \mathcal{I}$ such that this condition holds for $x \in G \setminus U$. For $Q \in G \times G$, $x \in G$ we put $Q_{[x]} := \{y \in G \mid (x, y) \in Q\}$. For $U \in \mathcal{I}$ we define

$$\Omega(U, \mathcal{I}) := \{Q \subset G \times G \mid Q_{[x]} \in \mathcal{I} \quad \text{for } x \in G \setminus U\},$$
$$\Omega(\mathcal{I}) := \bigcup_{U \in \mathcal{I}} \Omega(U, \mathcal{I}).$$

$\Omega(\mathcal{I})$ is a product of the family $\mathcal{I}$ on $G \times G$. One can easily observe that if $\mathcal{I}$ is k-proper then also $\Omega(\mathcal{I})$ is a k-proper family in $G \times G$.

3. Almost approximately polynomial functions

For $B \subset H$ and $k \in \mathbf{N}$ we define

$$k \bullet B := \{x_1 + \ldots + x_k \mid x_1, \ldots, x_k \in B\}.$$

The following Theorem is a generalization of Theorems 1 and 2 from [4]. However, what shows the advantage of the notion of a k-proper family over the ideal, is that although we use nearly exactly the same idea as that of R. Ger in [4], making use of k-proper families we can obtain stability.

Theorem 1. *Let $n \in \mathbf{N}$, and let G be a uniquely $n!$ divisible commutative semigroup and let H be abelian group. We assume that $\mathcal{I}$ is a $(n^2 + n)$-p.t.i. family in G such that*

$$\frac{1}{i} I \in \mathcal{I} \quad \textit{for } I \in \mathcal{I},\ i = 1, \ldots, n.$$

Let $B \subset H$, $B = -B$. Let $U \in \mathcal{I}$ and let $f : G \to H$ be such that

$$\Delta^n f(x, h) \in B \quad \textit{for } \Omega(U, \mathcal{I})\textit{-a.a. } (x, h) \in G \times G. \tag{3.3}$$

Then there exists a function $p : G \to H$ such that

$$f(x) = p(x) \quad \textit{for } x \in G \setminus U \tag{3.4}$$

and

$$\Delta^n p(x, y) \in (2^{2n} - 1) \bullet B \quad \textit{for } x, y \in G. \tag{3.5}$$

Moreover, for every functions p_0, p_1 satisfying (3.4) and (3.5)

$$p_0(x) - p_1(x) \in 2(2^{2n} - 1) \bullet B \quad \textit{for } x \in G. \tag{3.6}$$

Proof. By $Q \in \Omega(U, I)$ we denote the set such that

$$\Delta^n f(x, h) \in B \quad \text{for } (x, h) \in (G \times G) \setminus Q. \tag{3.7}$$

If $a, b \in H$, $a - b \in k \bullet B$ then we write $a \overset{k}{\approx} b$. As $B = -B$, $a \overset{k}{\approx} b$ iff $b \overset{k}{\approx} a$. One can easily notice that if $a \overset{k}{\approx} b$ and $b \overset{l}{\approx} c$ then $a \overset{k+l}{\approx} c$. Also if $a \overset{k}{\approx} b$ and $c \overset{l}{\approx} d$ then $a + b \overset{k+l}{\approx} c + d$.

To each $x \in G$ we assign a set $A_x \in \mathcal{I}^n$ by the formula

$$A_x := \bigcup_{i=1}^{n} \frac{1}{i}(-x + U).$$

If G does not contain neutral element then by $G \cup \{0\}$ we denote the semigroup G with added neutral element. We fix a function $\phi : G \to G \cup \{0\}$ satisfying

(a) $\phi(x) = 0 \quad$ for $x \in G \setminus U$,
(b) $\phi(x) \in G \setminus A_x \quad$ for $x \in U$.

Then

$$x + i\phi(x) \in G \setminus U \quad \text{for } i = 1, \ldots, n.$$

We define

$$p(x) := \sum_{i=1}^{n} (-1)^{i-1} \binom{n}{i} f(x + i\phi(x)). \tag{3.8}$$

It is obvious that $f(x) = p(x)$ for $x \in G \setminus U$, so p satisfies (3.4).

In order to prove that p satisfies (3.5) we first show that

$$p(x) \overset{2^n-1}{\approx} \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} f(x + kh) \quad \text{for } x \in G, h \in G \setminus A_x. \tag{3.9}$$

In fact, let us fix arbitrarily an $x \in G$ and $h \in G \setminus A_x$. Then one can easily notice that the sets $Q_{[x+i\phi(x)]}$ and $Q_{[x+kh]}$ are elements of $\mathcal{I}$ for $i, k = 1, \ldots, n$. Then there exists

$$\psi_h \in G \setminus \Big(\bigcup_{k=1}^{n} \frac{1}{k}(-\phi(x) + Q_{[x+kh]}) \cup \bigcup_{i=1}^{n} \frac{1}{i}(-h + Q_{[x+i\phi(x)]})\Big) \in \mathcal{I}^{2n},$$

as $\mathcal{I}^{2n} \subset \mathcal{I}^{n^2+n}$ which is proper by assumptions. Thus we have

$$x + kh \in G \setminus U, \; \phi(x) + k\psi_h \in G \setminus Q_{[x+kh]}, \; h + i\psi_h \in G \setminus Q_{[x+i\phi(x)]} \tag{3.10}$$

and so, in particular

$$(x + kh, \phi(x) + k\psi_h) \in (G \times G) \setminus Q \quad \text{for } k = 1, \ldots, n. \tag{3.11}$$

Now let us note that for an arbitrary i, $1 \le i \le n$, and for every $h_i \in G \setminus Q_{[x+i\phi(x)]}$ by (2.1) and (3.3)

$$f(x + i\phi(x)) \overset{1}{\approx} \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} f(x + i\phi(x) + kh_i).$$

In particular, in virtue of (3.10), we may take $h_i = h + i\psi_h$ and thus we can write

$$\begin{aligned} f(x+i\phi(x)) &\overset{1}{\approx} \sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k} f(x+i\phi(x)+k(h+i\psi_h)) \\ &= \sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k} f(x+kh+i(\phi(x)+k\psi_h)) \end{aligned} \tag{3.12}$$

Finally, (3.8) and (3.12) give

$$\begin{aligned} p(x) &\overset{2^n-1}{\approx} \sum_{i=1}^{n}(-1)^{i-1}\binom{n}{i}\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k} f(x+kh+i(\phi(x)+k\psi_h)) \\ &= \sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}\sum_{i=1}^{n}(-1)^{i-1}\binom{n}{i} f(x+kh+i(\phi(x)+k\psi_h)), \end{aligned}$$

whence (3.9) results by (2.1), (3.11) and (3.3).

Now we will prove that p satisfies (3.5). Let us fix arbitrary $u, v \in G$. By (3.9) we can write

$$p(u+jv) \overset{2^n-1}{\approx} \sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k} f(u+jv+h_j), \tag{3.13}$$

for $h_j \in G \setminus A_{u+jv}$, $j = 0, 1, \ldots, n$.

Let us take a $h \in G \setminus A_u = G \setminus \bigcup_{k=1}^{n} \frac{1}{k}(-u+U) \in \mathcal{I}^n$, and choose $\xi(h) \in G$ such that

$$\begin{aligned} h + j\xi(h) &\in G \setminus A_{u+jv} && \text{for } j = 0, 1, \ldots, n, \\ v + k\xi(h) &\in G \setminus Q_{[u+kh]} && \text{for } k = 1, 2, \ldots, n. \end{aligned} \tag{3.14}$$

Such a choice is always possible, because

$$\bigcup_{j=1}^{n} \frac{1}{j}(-h + A_{u+jv}) \cup \bigcup_{k=1}^{n} \frac{1}{k}(Q_{-v+[u+kh]}) \in \mathcal{I}^{n^2+n},$$

and by the assumptions $\mathcal{I}$ is (n^2+n)-proper, so $\mathcal{I}^{n^2+n}$ is a proper family.

Since $h + j\xi(h) \in G \setminus A_{u+jv}$, we obtain from (3.13) that

$$\begin{aligned} p(u+jv) &\overset{2^n-1}{\approx} \sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k} f(u+jv+k(h+j\xi(h))) \\ &= \sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k} f(u+kh+j(v+k\xi(h))) \end{aligned} \tag{3.15}$$

for $j = 0, 1, \ldots, n$.

Finally, from (2.1) and (3.15) we have

$$\begin{aligned}
\Delta^n p(u,v) &= \sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}p(u+jv)\\
&\overset{2^n\cdot(2^n-1)}{\approx} \sum_{j=0}^{n}(-1)^{j-1}\binom{n}{j}\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}f(u+kh+j(v+k\xi(h)))\\
&= \sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}\sum_{j=0}^{n}(-1)^{j-1}\binom{n}{j}f(u+kh+j(v+k\xi(h)))\\
&\overset{2^n-1}{\approx} 0
\end{aligned}$$

in view of the fact that $(u+kh, v+k\xi(h)) \in (G\times G)\setminus Q$ for $k=1,\ldots,n$, by (3.14). Thus p satisfies (3.5).

In order to prove (3.6) let $p_0, p_1 : G \to H$ be two functions satisfying (3.4) and (3.5). Let $x \in G$ be arbitrary. We have, by (2.2), for every h

$$p_0(x) \overset{2^{2n}-1}{\approx} \sum_{i=1}^{n}(-1)^{i-1}\binom{n}{i}p_0(x+ih),$$

$$p_1(x) \overset{2^{2n}-1}{\approx} \sum_{i=1}^{n}(-1)^{i-1}\binom{n}{i}p_1(x+ih).$$

Let $h \in A_x$ be arbitrary. Then $p_0(x+ih) = p_1(x+ih)$ for $i=1,2,\ldots,n$, and consequently $p_0(x) \overset{2(2^{2n}-1)}{\approx} p_1(x)$, which makes the proof complete. ∎

Now we show a generalization of Theorem 1 from [4] by stating $B=\{0\}$ in Theorem 1.

Corollary 1. *Let $n\in\mathbf{N}$, and let G be a uniquely $n!$ divisible commutative semigroup and let H be abelian group. We assume that $\mathcal{I}$ is a (n^2+n)-p.t.i. family in G such that*

$$\frac{1}{i}I \in \mathcal{I} \quad \text{for } I\in\mathcal{I},\ i=1,\ldots,n.$$

Let $U\in\mathcal{I}$ and let $f : G\to H$ be such that

$$\Delta^n f(x,h) = 0 \quad \text{for } \Omega(U,\mathcal{I})\text{-a.a. } (x,h)\in G\times G. \tag{3.16}$$

Then there exists a unique polynomial $p : G\to H$ of degree $n-1$ such that

$$f(x) = p(x) \quad \text{for } x\in G\setminus U \tag{3.17}$$

Now we formulate a generalization of Theorem 3 from [4] (similair results for the Cauchy equation were earlier obtained by S. Hartman in [6] and N. G. de Bruijn in [2]). We skip the proof as it is an exact repetition of the proof of Theorem 3 from [4].

However, we were not able to obtain the same generality as in Theorem 1 – we assume also that G is a group and $\mathcal{I}$ is invariant with respect to invarsion. It seems interesting whether this "additional" assumptions are essential (to our opinion it is really the case).

Corollary 2. *Let $\mathcal{I}$ be a (n^2+n)-p.t.i. family in a group G, such that*

$$\frac{1}{i}I \in \mathcal{I} \quad \text{for } I \in \mathcal{I},\ i = 1,\dots,n. \tag{3.18}$$

We assume additionally that $-I \in \mathcal{I}$ for $I \in \mathcal{I}$.

Let $f : G \to H$ and let $S \in \mathcal{I}$ be such that

$$\Delta^n f(x,y) = 0 \quad \text{for } x, h \in G \setminus S.$$

Then f is a polynomial function of order $n-1$.

Let $B \subset H$ be such that $B = -B$. We say that the polynomial equation of order $n-1$ is *B-stable* for functions from G into H if there exists $K_B \in \mathbf{N}$ such that for every function $F : G \to H$ satisfying

$$\Delta^n F(x,y) \in B \quad \text{for } (x,y) \in G \times G$$

there exists a polynomial function $p : G \to E$ of order $n-1$ with

$$F(x) - p(x) \in K_B \bullet B \quad \text{for } x \in G.$$

Proposition 1. *Let $n \in \mathbf{N}$, let G be a uniquely $n!$ divisible commutative semigroup, let H be abelian group and let $B \subset H$. We assume that $B = -B$ and that the polynomial equation is $(2^{2n}-1) \bullet B$-stable with constant K.*

Let $\mathcal{I}$ be a (n^2+n)-p.t.i. family in G, and let $U \in \mathcal{I}$. Then for every function $f : G \to H$ satisfying

$$\Delta^n f(x,y) \in B \quad \text{for } \Omega(U,\mathcal{I})\text{-a.a. } (x,y) \in G \times G.$$

there exists a polynomial function $p : G \to H$ such that

$$f(x) - p(x) \in (2^{2n}-1)K \bullet B \quad \text{for } x \in G \setminus U. \tag{3.19}$$

Moreover, for every polynomial functions p_0, p_1 satisfying (3.19)

$$p_0(x) - p_1(x) \in 2(2^n-1)(2^{2n}-1)K \bullet B \quad \text{for } x \in G \tag{3.20}$$

Proof. By Theorem 1 we obtain that there exists a function $P : G \to H$ satisfying (3.4), (3.5). As the polynomial equation is $(2^{2n}-1) \bullet B$-stable (3.5) implies that there exists a polynomial p of order $n-1$ such that

$$P(x) - p(x) \in (2^{2n}-1)K \bullet B \quad \text{for } x \in G.$$

By (3.4) we obtain that

$$P(x) = f(x) \quad \text{for } x \in G \setminus U.$$

By joining the two above inequalities we obtain (3.19).

We show (3.20). Let p_0, p_1 be two polynomials satisfying (3.19). Take an $x \in G$ and an h such that $x + ih \notin U$ for $i = 1, 2, \ldots, n$, which is possible, since

$$\bigcup_{i=1}^{n} \frac{1}{i}(-x + U) \in \mathcal{I}^n.$$

Then

$$p_0(x + ih) \overset{K(2^{2n}-1)}{\approx} f(x + ih) \overset{K(2^{2n}-1)}{\approx} p_1(x + ih) \quad \text{for } i = 1, \ldots, n,$$

so by (2.2)

$$\begin{aligned} p_0(x) &= \sum_{i=1}^{n} (-1)^{i-1} \binom{n}{i} p_0(x + ih) \\ &\overset{(2^n-1)\cdot 2K(2^{2n}-1)}{\approx} \sum_{i=1}^{n} (-1)^{i-1} \binom{n}{i} p_1(x + ih) = p_1(x). \end{aligned}$$

□

The following result is a corollary of Theorem 4 from [1].

Theorem A-B Let G be uniquely $n!$ divisible commutative semigroup, let E be a Banach space, let $\varepsilon \geq 0$ and let $f : G \to E$ be such that

$$\|\Delta^n f(x, y)\| \leq \varepsilon \quad \text{for } x, y \in G.$$

Then there exists a unique up to a constant function polynomial $p : G \to E$ of order $n - 1$ such that

$$\|f(x) - p(x)\| \leq 2\varepsilon \quad \text{for } x \in G.$$

Now we are able to prove the almost approximate stability of polynomial functional equation.

THEOREM 2. *Let G be uniquely $n!$ divisible commutative semigroup, let $\mathcal{I}$ be a $(n^2 + n)$-p.t.i. family in G such that*

$$\frac{1}{i} I \in \mathcal{I} \quad \textit{for } I \in \mathcal{I}, i = 1, \ldots, n.$$

Let E be a Banach space.

Then for every $\varepsilon > 0$, $U \in \mathcal{I}$ and every $f : G \to E$ satisfying

$$\|\Delta^n f(x, y)\| \leq \varepsilon \quad \textit{for } \Omega(U, \mathcal{I})\textit{-a.a. } (x, y) \in G \times G$$

there exists a unique up to a constant function polynomial $p : G \to E$ of order $(n - 1)$ such that

$$\|f(x) - a(x)\| \leq 2(2^{2n} - 1)\varepsilon \quad \textit{for } x \in G \setminus U.$$

PROOF. We put $B = B(0, \varepsilon)$, a ball with the center at zero and radius ε in E. Then the assertion of the Theorem follows immediately from Theorem A-B and Proposition 1. □

4. Generalized Stability

If $\mathcal{L}$ is a set of functions from $G \to H$ then by $k \bullet \mathcal{L}$ we denote $\overbrace{\mathcal{L} + \ldots + \mathcal{L}}^{k}$. We define the translation by $a \in G$ by the formula

$$T_a(x) := a + x \quad \text{for } x \in G.$$

We say that $\mathcal{L}$ is translation invariant if $f \circ T_a \in \mathcal{L}$ for every $a \in G$, $f \in \mathcal{L}$. We also define

$$M_i(x) := ix \quad \text{for } i = 1, \ldots, n,\ x \in G.$$

Proposition 2. *Let G be uniquely $n!$ divisible commutative semigroup, and let $\mathcal{I}$ be a p.t.i. family in G such that*

$$\frac{1}{i} I \in \mathcal{I} \quad \textit{for } I \in \mathcal{I},\ i = 1, \ldots, n.$$

Let $B \subset H$ be such that $B = -B$.

Let $\mathcal{L}$ be a translation invariant set of functions from G into H such that $\mathcal{L} = -\mathcal{L}$. We assume additionaly that if a constant function $c(x) = c$ belongs to $(2^{n+1} - 1) \bullet \mathcal{L}$ then $c \in B$, and that

$$f \circ M_i \in \mathcal{L} \quad \textit{for } f \in \mathcal{L},\ i = 1, \ldots, n.$$

Let $\mathcal{I}$ be a p.t.i. family in G and let $U \in \mathcal{I}$. We assume that $f : G \to H$ is a function such that there exists $V \subset G$, $nV \in \mathcal{I}$ with

$$\begin{aligned} &\Delta^n f(x, \cdot) \in \mathcal{L} \quad \textit{for } x \in G \setminus U, \\ &\Delta^n f(\cdot, h) \in \mathcal{L} \quad \textit{for } h \in G \setminus V. \end{aligned} \tag{4.21}$$

Then

$$\Delta^n f(x, h) \in B \quad \textit{for } \Omega(U, \mathcal{I}^{2n-1})\textit{-a.a. } (x, h) \in G \times G.$$

Proof. The main part of the proof consists of some trivial but tedious manipulations. For arbitrary $a, x, h \in G$ we have

$$\begin{aligned} &\sum_{k=0}^{n} \binom{n}{k} (-1)^k \Delta^n f(x + knh, a + (n-k)h) \\ &= \sum_{k=0}^{n} \binom{n}{k} (-1)^k \sum_{i=0}^{n} \binom{n}{i} (-1)^i f(x + knh + (n-i)(a + (n-k)h)) \\ &= \sum_{k=0}^{n} \sum_{i=0}^{n} \binom{n}{i} \binom{n}{k} (-1)^i (-1)^k f(x + knh + (n-i)(a + (n-k)h)) \\ &= \sum_{i=0}^{n} \binom{n}{i} (-1)^i \sum_{k=0}^{n} \binom{n}{k} (-1)^k f(x + (n-i)(a + nh) + k(ih)) \\ &= \sum_{i=0}^{n} \binom{n}{i} (-1)^i (-1)^n \Delta^n f(x + (n-i)(a + nh), ih). \end{aligned}$$

Thus we have obtained that for $a, h, x \in G$,

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k \Delta^n f(x + knh, a + (n-k)h)$$

$$= \sum_{i=0}^{n} \binom{n}{i} (-1)^{n+i} \Delta^n f(x + (n+i)(a+nh), ih),$$

or equivalently, replacing h by $\frac{1}{n}h$, that for $a, h, x \in G$

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k \Delta^n f(x + kh, a + (1 - \frac{k}{n})h)$$

$$= \sum_{i=0}^{n} \binom{n}{k} (-1)^{n+i} \Delta^n f(x + (n-i)(a+h), \frac{i}{n}h).$$

This means that for $a, x, h \in G$

$$\begin{aligned} \Delta^n f(x,h) \;&=\; (-1)^n \sum_{k=0}^{n} \binom{n}{k} (-1)^k \Delta^n f(x + kh, a + (1 - \frac{k}{n})h) \\ &\quad - \sum_{i=0}^{n-1} \binom{n}{i} (-1)^i \Delta^n f(x + (n-i)(a+h), \frac{i}{n}h). \end{aligned} \tag{4.22}$$

Now we are ready to prove the assertion of the Proposition. Suppose that $x, h \in G$ are such that

$$\frac{i}{n}h \notin V, x + kh \notin U \quad \text{for } k = 1, \ldots, n, i = 1, \ldots, n-1.$$

Then by (4.22)

$$\begin{aligned} \Delta^n f(x,h) \;&=\; (-1)^n \sum_{k=0}^{n} \binom{n}{k} (-1)^k \Delta^n f(x + kh, a + (1 - \frac{k}{n})h) \\ &\quad - \sum_{i=0}^{n-1} \binom{n}{i} (-1)^i \Delta^n f(x + (n-i)h + (n-i)a, ih) \\ &=\; (-1)^n \sum_{k=0}^{n} \binom{n}{k} (-1)^k \Delta^n f(x + kh, \cdot) \circ T_{(1-\frac{k}{n})h}(a) \\ &\quad - \sum_{i=0}^{n-1} \binom{n}{i} (-1)^i \Delta^n f(\cdot, ih) \circ M_{n-i} \circ T_{x+(n-i)h}(a). \end{aligned}$$

Thus we have obtained that $\Delta^n f(x,h)$ as a function of variable a is an element of $(2^{n+1} - 1) \bullet \mathcal{L}$. Trivially $\Delta^n f(x,h)$ as a function of a is constant, so by the assumptions we obtain that $\Delta^n f(x,h) \in B$. This means that

$$\Delta^n f(x,h) \in B \quad \text{for } (x,h) \in G \times G \setminus Q,$$

where

$$Q := (U \times G) \cup \bigcup_{x \in G \setminus U} (x, \bigcup_{i=1}^{n} \frac{1}{i}(nV) \cup \bigcup_{k=1}^{n-1} \frac{1}{k}(-x+U)).$$

One can easily notice that $Q \in \Omega(U, \mathcal{I}^{2n-1})$. □

Theorem 3. *Let G be uniquely $n!$ divisible commutative semigroup, let $\mathcal{I}$ be a $(2n-1)(n^2+n)$-p.t.i. family in G such that*

$$\frac{1}{i} I \in \mathcal{I} \quad \text{for } i \in \mathcal{I},\ i = 1, \dots, n.$$

Let H be abelian group and let $B \subset H$ be such that $B = -B$. Let $\mathcal{L}$ be a translation invariant set of functions from G into H such that $\mathcal{L} = -\mathcal{L}$ and that $f \circ M_i \in \mathcal{L}$ for $i = 1, \dots, n$. We assume additionaly that if a constant function $c(x) = c$ belongs to $(2^{n+1}-1) \bullet \mathcal{L}$ then $c \in B$.

Let $U \in \mathcal{I}$. Suppose that $f : G \to H$ is a function such that

$$\begin{aligned} &\Delta^n f(x, \cdot) \in \mathcal{L} \quad \text{for } x \in G \setminus U, \\ &\Delta^n f(\cdot, h) \in \mathcal{L} \quad \text{for } \mathcal{I}\text{-a.a. } x \in G. \end{aligned} \tag{4.23}$$

Then there exists a function $F : G \to H$ such that

$$\Delta^n F(x, y) \in (2^{2n} - 1) \bullet B \quad \text{for } x, y \in G$$

and

$$f(x) = F(x) \quad \text{for } x \in G \setminus U.$$

Proof. Making use of Proposition 2 with the family $(n\mathcal{I})$ we obtain that

$$\Delta^n f(x, h) \in B \quad \text{for } \Omega(U, (n\mathcal{I})^{2n-1})\text{-a.a. } (x, h) \in G \times G.$$

As G is uniquely n divisible as $\mathcal{I}$ is a $(2n-1)(n^2+n)$-p.t.i., so is $n\mathcal{I}$, and therefore $(n\mathcal{I})^{2n-1}$ is a (n^2+n)-p.t.i. family. Theorem 1 makes the proof complete. □

Let $p \in [1, \infty)$. For the definition of the Banach space $\mathcal{L}_p(\mathbf{R}^n, E)$ of p-integrable functions on $\mathbf{R}^n$ with respect to Lebesgue measure λ_n and values in a Banach space E we refer the reader to [5].

Corollary 3. *Let E be a Banach space, let $p \in [1, \infty)$ and let $f : \mathbf{R}^n \to E$ be a function such that*

$$\Delta^n f \in \mathcal{L}_p(\mathbf{R}^n \times \mathbf{R}^n, E).$$

Then there exists a unique polynomial $p : \mathbf{R}^n \to E$ of degree $n-1$ such that

$$f(x) = p(x) \quad \text{for } \lambda\text{-a.a. } x \in \mathbf{R}^n$$

Proof. We put $B = \{0\}$ and $\mathcal{I}$ define as the family of all sets with measure zero in $\mathbf{R}^n$. The reader can now easily check that all the assumptions of Theorem 3 are satisfied, so its assertion makes the proof complete. □

Theorem 4. *Suppose that all the assumptions of Theorem 3 hold. We assume additionally that H is a Banach space and that $B = B(0,\varepsilon)$, where $B(0,\varepsilon)$ denotes the ball in H with the center at zero and radius ε.*

Then there exists a unique up to a constant function polynomial p of order $n-1$ such that

$$\|f(x) - p(x)\| \le 2(2^{2n} - 1)\varepsilon \quad \text{for } x \in G \setminus U. \tag{4.24}$$

Proof. Making use of Theorem 3 we obtain that there exists a function $F : G \to H$ such that

$$\|\Delta^n F(x,y)\| \le (2^{2n} - 1)\varepsilon \quad \text{for } x, y \in G.$$

and that

$$f(x) = F(x) \quad \text{for } x \in G \setminus U.$$

Now making use of Theorem A-B we obtain the existence of a polynomial $p : G \to H$ such that

$$\|p(x) - F(x)\| \le 2(2^{2n} - 1)\varepsilon \quad \text{for } x \in G,$$

which implies that

$$\|p(x) - f(x)\| \le 2(2^{2n} - 1)\varepsilon \quad \text{for } x \in G \setminus U.$$

Now suppose that there are two polynomials p_0, p_1 satisfying (4.24). Then p_0, p_1 satisfy (3.19), so by (3.20) we obtain that

$$\|p_0(x) - p_1(x)\| \le 2(2^n - 1) \cdot 2(2^{2n} - 1)\varepsilon \quad \text{for } x \in G,$$

which clearly implies that $p_0 - p_1$ is a constant function. □

The following result shows that Theorem 4 enables us to "join" the classical almost approximate type of error with integral and still obtain almost approximate stability. Roughly speaking the reason for this is that if the n-th difference of a given function belongs to the space $\mathcal{L}^p$ then in fact it enforces the function to be almost everywhere equal to some polynomial (see Corollary 3).

However, first we have to introduce some notation. Let E be a Banach space and let $\mathcal{K}$ be a vector space of functions from G to E. Let $U \in \mathcal{I}$. In analogy to the definition $\Omega(\mathcal{I})$ we define

$$\Omega_{\mathcal{I}}(\mathcal{K}) := \{F : G \times G \to E \mid F(x,\cdot) \in \mathcal{K} \quad \text{for } \mathcal{I}\text{-a.a. } x \in G\},$$

For $f : G \to E$ we put

$$\|f\|_{\mathcal{K},\sup} := \inf\{\|f - f_{\mathcal{K}}\|_{\sup} \mid f_{\mathcal{K}} \in \mathcal{K}\}.$$

Theorem 5. *Let G be uniquely $n!$ divisible commutative semigroup. Let E be a Banach space and let $\mathcal{K}$ be a translation invariant vector space of* *functions from G into E such that $K = -\mathcal{K}$ and that $f \circ M_i \in \mathcal{K}$ for $f \in \mathcal{K}$, $i = 1, \dots, n$. We additionally assume that*

$$\inf_{x \in G} \|f(x)\| = 0 \quad \text{for } f \in \mathcal{K}.$$

Let $\mathcal{I}$ be a family in G such that

$$\frac{1}{i} I \in \mathcal{I} \quad \text{for } I \in \mathcal{I},\ i = 1, \dots, n.$$

We also assume that $(n\mathcal{I})$ is a $(2n-1)(n^2+n)$-p.t.i. family.

Let $\varepsilon > 0$ and $f : G \to E$ be such that

$$\|\Delta^n f\|_{\Omega_U(\mathcal{K}),\sup} \le \varepsilon$$

and that

$$\|\Delta^n f(\cdot, h)\|_{\mathcal{K},\sup} \le \varepsilon \quad \text{for } h \in G \setminus U.$$

Then there exists a unique up to a constant function polynomial $p : G \to E$ of order $n-1$ such that

$$\|f(x) - p(x)\| \le 2(2^{2n} - 1)(2^{n+1} - 1)\varepsilon \quad \text{for } x \in G \setminus U.$$

Proof. We put

$$\mathcal{L} := \{f : G \to E \mid \|f\|_{\mathcal{K},\sup} \le \varepsilon\}.$$

One can easily notice that $\mathcal{L}$ is a translation invariant set such that $\mathcal{L} = -\mathcal{L}$. We show that if a constant function $c : G \to E$ belongs to $(2^{n+1} - 1) \bullet \mathcal{L}$ then $\|c\| \le (2^{n+1} - 1)\varepsilon$.

Suppose, for an indirect proof, that there exists a constant function $c \in (2^{n+1} - 1) \bullet \mathcal{L}$ such that $\|c\| > (2^{n+1} - 1)\varepsilon$. By the definition of $\| \ \|_{\mathcal{K},\sup}$ and the fact that $c \in (2^{n+1} - 1) \bullet \mathcal{L}$, $\|c\|_{\mathcal{K},\sup} \le (2^{n+1} - 1)\varepsilon$ there exists $c_{\mathcal{K}} \in \mathcal{K}$ such that

$$\|c - c_{\mathcal{K}}\|_{\sup} \le \frac{\|c\| + (2^{n+1} - 1)\varepsilon}{2}.$$

As $c_{\mathcal{K}} \in \mathcal{K}$, there exists $x \in G$ such that

$$\|c_{\mathcal{K}}(x)\| < \frac{\|c\| - (2^{n+1} - 1)\varepsilon}{2}.$$

Joining the two above inequalities and making use of the fact that $c = (c - c_{\mathcal{K}}) + c_{\mathcal{K}}$ we obtain

$$\begin{aligned} \|c\| &= \|c(x)\| = \|c_0(x) + c_1(x)\| \le \|c_0(x)\| + \|c_1(x)\| \\ &< \tfrac{\|c\| + (2^{n+1}-1)\varepsilon}{2} + \tfrac{\|c\| - (2^{n+1}-1)\varepsilon}{2} = \|c\|, \end{aligned}$$

which yields a contradiction.

Theorem 4 completes the proof. ∎

Corollary 4. *Let E be a Banach space and let $p \in [1,\infty)$. Then for every $f : \mathbf{R}^n \to E$ satisfying*

$$\varepsilon := \|\Delta^n f\|_{\mathcal{L}_p(\mathbf{R}^n \times \mathbf{R}^n, E),\sup} < \infty$$

there exists a unique up to a constant function polynomial $p : \mathbf{R}^n \to E$ of degree $n-1$ such that

$$\|f(x) - p(x)\| \le 2(2^{2n}-1)(2^{n+1}-1)\varepsilon \quad \text{for } \lambda_n\text{-a.a. } x \in G.$$

Proof. We put

$$\begin{aligned} \mathcal{I} &:= \{A \subset \mathbf{R}^n \mid \lambda(A) = 0\}, \\ \mathcal{K} &:= \mathcal{L}_p(\mathbf{R}^n, E). \end{aligned}$$

By the Fubini Theorem we obtain that

$$\mathcal{L}_p(\mathbf{R}^n \times \mathbf{R}^n, E) \subset \Omega_{\mathcal{I}}(\mathcal{L}_p(\mathbf{R}^n, E).$$

This explains why

$$\|\Delta^n f\|_{\Omega_{\mathcal{I}}(\mathcal{L}_p(\mathbf{R}^n,E)),\sup} \le \|\Delta^n f\|_{\mathcal{L}_p(\mathbf{R}^n \times \mathbf{R}^n, E),\sup} = \varepsilon.$$

Analogously we obtain that

$$\|\Delta^n f(\cdot, h)\|_{\mathcal{L}_p(\mathbf{R}^n,E),\sup} \le \varepsilon \quad \text{for } \mathcal{I}\text{-a.a. } h \in G.$$

Theorem 5 makes the proof complete. □

Corollary 5. *Let E, F be Banach spaces, let U be a bounded subset of E.*

Let $\varepsilon \ge 0$. We assume that $f : E \to F$ is such that there exists a bounded $V \subset E$ with

$$\limsup_{\|y\| \to \infty} \|\Delta^n f(x,y)\| \le \varepsilon \quad \text{for } x \in E \setminus U.$$

$$\limsup_{\|y\| \to \infty} \|\Delta^n f(y,h)\| \le \varepsilon \quad \text{for } h \in E \setminus V.$$

Then there exists a unique up to a constant function polynomial $p : E \to F$ such that

$$\|f(x) - p(x)\| \le 2(2^{2n}-1)(2^{n+1}-1)\varepsilon \quad \text{for } x \in E \setminus U.$$

Proof. We define

$$\begin{aligned} \mathcal{I} &:= \{W \subset E \mid W \text{ is bounded }\} \\ \mathcal{K} &:= \{g : E \to F \mid \lim_{\|x\| \to \infty} \|g(x)\| = 0\}. \end{aligned}$$

Theorem 5 makes the proof complete. □

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Jacek Tabor
Institute of Mathematics
Jagiellonian University
Reymonta 4 st.
30-059 Kraków
Poland
E-mail: tabor@im.uj.edu.pl