GLASNIK MATEMATIČKI
Vol. 36(56)(2001), 241 – 245

UNIVALENCE CRITERIA FOR INTEGRAL OPERATORS

Dorin Blezu and Radu N. Pascu
Lucian Blaga University and Transilvania University, Romania

Abstract. The aim of this paper is to obtain new univalence criteria for integral operators which were introduced by N.N.Pascu [3, 4] and S. Moldoveanu [3]. The improvement consist in the fact, that the new hypothesis are more simple that is they do not contain $|z|$. Of this reason these can be easily applicated.

1. Introduction

Let A the set of analytic functions defined in the unit disk U normalised as:

$$A = \{f \mid f \in \mathcal{H}(u),\ f(0) = 0,\ f'(0) = 1\}$$

and

$$S = \{f \mid f \in A,\ \text{and } f \text{ is univalent}\}$$

We hereby remind of two Theorems given by N.N.Pascu and S.Moldoveanu and N.N.Pascu respectively:

Theorem A. [3] *Let $f \in S$, $\alpha \in C$. If $|\alpha - 1| \leq \frac{1}{4}$ then the function $F_\alpha(z)$ given by*

$$F_\alpha(z) = \left[\alpha \int_0^z f^{\alpha-1}(u)du\right]^{1/\alpha}$$

is analytic and univalent in U.

Theorem B. [4] *Let $f \in A$, $\alpha \in C$, $Re\ \alpha > 0$. If*

2000 *Mathematics Subject Classification.* Primary 30C45.

Presented at the International Conference on Complex Analysis and Related Topics and the VIII-th Romanian Finnish Seminar, August 23–27 1999, Iassy, Romania.

$$\text{(1.1)} \qquad \frac{1-|z|^{2Re\ \alpha}}{Re\ \alpha}\left|\frac{zf''(z)}{f'(z)}\right| \leq 1 \qquad (\forall)\ z \in U$$

then for all complex numbers β, $Re\ \beta \geq Re\ \alpha$ the function

$$F_\beta(z) = \left[\beta \int_0^z u^{\beta-1} f'(u)du\right]^{1/\beta}$$

is analytic and univalent in U.

In the following lemma we consider the Caratheodory inequality which is based on the Schwarz's lemma.

Thanks to this Lemma we can give a univalence criterion which does not contain $|z|$.

Lemma C. (Caratheodory) *Let be $g(z) \in \mathcal{H}(U)$, $g(0) = 0$, $M > 0$. If $Re\ g(z) < M$, $z \in U$ then*

$$(1-|z|)|g(z)| \leq 2M|z| \qquad \forall\ z \in U$$

Proof. Let be the function h

$$h(z) = \frac{g(z)}{2M - g(z)}$$

that is $h(0) = 0$, $h(z) \in \mathcal{H}(u)$ and $|h(z)| \leq 1$ because

$$|g(z)| \leq |2M - g(z)|$$

According to the Schwarz's lemma we can write

$$|h(z)| \leq |z| \qquad (\forall)\ z \in U$$

that is

$$|g(z)| \leq |z|\ |2M - g(z)| \leq |z|(2M + |g(z)|)$$

hence

$$(1-|z|)|g(z)| \leq 2M|z|$$

which completes the proof of the Lemma 1. □

Main results

By means of Lemma C we obtain a set of simple univalence criteria depending on parameter θ using also the well known Becker's criterion, [1].

THEOREM 1. [2] *Let be* $f \in A,\ \ \theta \in [0, 2\pi]$.
If

$$Re\ e^{i\theta} \frac{zf''(z)}{f'(z)} \le \frac{1}{4}$$

then $f \in S$.

PROOF. In Lemma 1 we take $g(z) = e^{i\theta} \frac{zf''(z)}{f'(z)}$, $M = \frac{1}{4}$ it follows that

$$(1 - |z|) \left| \frac{zf''(z)}{f'(z)} \right| \le 2\frac{1}{4}|z| = \frac{|z|}{2}$$

and hence

$$(1 - |z|^2) \left| \frac{zf''(z)}{f'(z)} \right| = (1 + |z|)(1 - |z|) \left| \frac{zf''(z)}{f'(z)} \right| \le (1 + |z|)\frac{|z|}{2} \le 1, \quad (\forall)\ z \in U$$

According to the Becker's univalence criterions it follows that $f \in S$. □

THEOREM 2. *Let* $f \in A$, $\alpha \in C$ *and* $|\alpha - 1| \le \frac{1}{4},\ \ \theta \in [0, 2\pi]$.
If

$$Re\ e^{i\theta} \frac{zf''(z)}{f'(z)} \le \frac{1}{4}$$

then the function

$$F_\alpha(z) = \left[\alpha \int_0^z f^{\alpha - 1}(u) du \right]^{1/\alpha}$$

is analytic and univalent in U.

PROOF. According to Theorem 1, the hypothesis of Theorem 2 implies the univalence of the function f and using Theorem A we obtain the conclusion of Theorem 2.□

THEOREM 3. *Let* $f \in A$, $\alpha \in C$, $Re\ \alpha > 0,\ \ \theta \in [0, 2\pi]$
If

$$Re\ e^{i\theta} \frac{zf''(z)}{f'(z)} \le \begin{cases} \dfrac{Re\ \alpha}{4} & \text{for} \quad 0 < Re\ \alpha < 1 \\[2ex] \dfrac{1}{4} & \text{for} \quad Re\ \alpha \ge 1 \end{cases} \tag{1.2}$$

then for all complex numbers β, $Re\ \beta \geq Re\ \alpha$ the function

$$F_\beta(z) = \left[\beta \int_0^z u^{\beta-1} f'(u)du\right]^{1/\beta}$$

is analytic and univalent in U.

PROOF. In Lemma 1 we choose

$$g(z) = e^{i\theta}\frac{zf''(z)}{f'(z)}, \text{ and } M = \frac{Re\ \alpha}{4}$$

for $0 < Re\ \alpha < 1$ respectively $M = \frac{1}{4}$ for $Re\ \alpha \geq 1$ so that in the hypothesis (1.2)

a) It is easy to observe that the function $h : (0, \infty) \to R$

$$h(x) = \frac{1 - a^{2x}}{x} \quad \text{with} \quad 0 < a < 1$$

is a decreasing function.
If $Re\ \alpha \geq 1$, $z \in U$, $a = |z|$ then

$$\frac{1 - |z|^{2Re\ \alpha}}{Re\ \alpha} \leq 1 - |z|^2. \quad (\forall)\ z \in U$$

Using the conclusion of Lemma 1 with $M = \frac{1}{4}$, and a similar calculus as in Theorem 1 we obtain

$$\frac{(1 - |z|^{2Re\ \alpha})}{Re\ \alpha}\left|\frac{zf''(z)}{f'(z)}\right| \leq (1 - |z|^2)\left|\frac{zf''(z)}{f'(z)}\right| \leq 1$$

b) Now we consider the function $q : (0, \infty) \to R$, $q(x) = 1 - a^{2x}$, $0 < a < 1$ which is a increasing function. Then for $0 < Re\ \alpha \leq 1$ we have

$$1 - |z|^{2Re\ \alpha} \leq 1 - |z|^2, \qquad (\forall)\ z \in U$$

Hence by using the conclusion of Lemma 1 with $M = \frac{Re\ \alpha}{4}$ we obtain the inegality

$$(1 - |z|^{2Re\ \alpha})\left|\frac{zf''(z)}{f'(z)}\right| \leq (1 - |z|^2)\left|\frac{zf''(z)}{f'(z)}\right| \leq Re\ \alpha$$

which is (1.1).

Because the condition (1.2) implies inequality (1.1) for all $\alpha \in C$, $Re\ \alpha > 0$ from Theorem B, it follows that the conclusion of Theorem 3 is true. □

REMARK. The new forms of this hypothesis in the Theorems 1, 2, 3 are more simple, than others which contain $|z|$, that implies an additional scale of applications.

References

[1] Becker J.,*Löwnersche Differentialgleichung und Quasikonform fortsetzbare schlichte Functionen*, J.Reine Angen.Math.255(1972) 23–42.

[2] Blezu D., Pascu N.N.,, *Some univalence citerion*, Demonstratio Mathematica (to appear).

[3] Moldoveanu S, Pascu N.N., *Integral operators which preserve the univalence*, Mathematica (Cluj) 32(55), 2 (1990) 159–166.

[4] Pascu N.N., *An improvement of Becker's univalence criterion*, University of Braşov, Department of Mathematics, Procedings of the commemorative session Simion Stoilov (1987), 43–48.

Department of Mathematics
Lucian Blaga University
2400 Sibiu, str.Dr.I.Raţiu 5–7
Romania
E-mail: blezu@roger-univ.ro

Department of Mathematics
Transilvania University
Str. Iuliu Maniu 50
2200 Braşov
Romania
E-mail: pascu@info.unitbv.ro

Received: 06.09.2000.