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n-SHAPE EQUIVALENCE AND TRIADS

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Abstract. This paper concerns the shape theory for triads of spaces which was introduced by the author. More precisely, in the first part, the shape dimension for triads of spaces $(X; X_0, X_1)$ is introduced, and its upper and lower bounds are given in terms of the shape dimensions of X_0, X_1, $X_0 \cap X_1$ and X. In the second part, a Whitehead type theorem for triads of spaces and a Mayer-Vietoris type theorem concerning n-shape equivalence are obtained.

1. Introduction

Throughout the paper, spaces and maps mean topological spaces and continuous maps, respectively. A *triad of spaces* $(X; X_0, X_1)$ means a space X and subspaces X_0 and X_1 of X such that $X = X_0 \cup X_1$. A *map of triads* $f : (X; X_0, X_1) \to (Y; Y_0, Y_1)$ means a map $f : X \to Y$ such that $f(X_0) \subseteq Y_0$ and $f(X_1) \subseteq Y_1$. A *homotopy of triads* means a map of triads $h : (X \times I; X_0 \times I, X_1 \times I) \to (Y; Y_0, Y_1)$. Pointed versions of a triad, map and homotopy are also defined in the obvious way. Using this homotopy, the auther defined the shape theory for triads [9] and reproved the Blakers-Massey homotopy excision theorem for shape theory (see also [11]). The purpose of this paper is to investigate properties that concern the shape dimension for triads and n-shape equivalence between triads.

First recall that Günther gave an upper bound of the shape dimension $\mathrm{Sd}\, X$ of a single space X:

Theorem 1.1 (Günther [2]). *Let $(X; X_0, X_1)$ be a triad of spaces such that X_0 and X_1 are closed and $X_0 \cap X_1$ is normally embedded in X. Then we*

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have

$$\operatorname{Sd} X \le \max\{\operatorname{Sd} X_0, \operatorname{Sd} X_1, 1 + \operatorname{Sd}(X_0 \cap X_1)\}.$$

For each triad of spaces $(X; X_0, X_1)$ and $n \ge 0$, the *shape dimension* $\operatorname{Sd}(X; X_0, X_1)$ is said to be at most n, denoted $\operatorname{Sd}(X; X_0, X_1) \le n$, provided each map $f : (X; X_0, X_1) \to (P; P_0, P_1)$ into a polyhedral triad factors up to homotopy of triads through a polyhedral triad $(Q; Q_0, Q_1)$ such that $\dim Q \le n$. Let $\operatorname{Sd}(X; X_0, X_1) = \min\{n \ge 0 : \operatorname{Sd}(X; X_0, X_1) \le n\}$.

As the first result we show that this shape dimension of a triad is close to the upper bound given by Günther:

THEOREM A. *Let $(X; X_0, X_1)$ be a triad of metric spaces such that X_0 and X_1 are closed. Then we have*

$$\begin{aligned}\max\{\operatorname{Sd} X, \operatorname{Sd} X_0, \operatorname{Sd} X_1, \operatorname{Sd}(X_0 \cap X_1)\} &\le \operatorname{Sd}(X; X_0, X_1)\\ &\le \max\{\operatorname{Sd} X_0, \operatorname{Sd} X_1, 1 + \operatorname{Sd}(X_0 \cap X_1)\}.\end{aligned}$$

Let $(X; X_0, X_1, *)$ and $(Y; Y_0, Y_1, *)$ be pointed triads of spaces such that X is normal, X_0, X_1 and Y_0, Y_1 are connected closed subsets of X and Y, respectively, and that $X_0 \cap X_1$ and $Y_0 \cap Y_1$ are connected and normally embedded in X and Y, respectively. Let $\varphi : (X; X_0, X_1, *) \to (Y; Y_0, Y_1, *)$ be a shape morphism. Then φ induces the restricted shape morphisms $\varphi|_X : (X, *) \to (Y, *)$, $\varphi|_{X_0 \cap X_1} : (X_0 \cap X_1, *) \to (Y_0 \cap Y_1, *)$, $\varphi|_{X_0} : (X_0, *) \to (Y_0, *)$ and $\varphi|_{X_1} : (X_1, *) \to (Y_1, *)$. The second result is the following Whitehead type theorem:

THEOREM B. *Suppose that* $\operatorname{Sd}(X; X_0, X_1) \le n - 1$ *and* $\operatorname{Sd}(Y; Y_0, Y_1) \le n$, *where* $1 \le n < \infty$. *If a shape morphism $\varphi : (X; X_0, X_1, *) \to (Y; Y_0, Y_1, *)$ induces n-shape equivalences $\varphi|_{X_0 \cap X_1} : (X_0 \cap X_1, *) \to (Y_0 \cap Y_1, *)$, $\varphi|_{X_0} : (X_0, *) \to (Y_0, *)$ and $\varphi|_{X_1} : (X_1, *) \to (Y_1, *)$, then φ is an equivalence.*

The Whitehead theorem for ordinary shape is well-known, and the most general form can be found in [8, Theorem 7, p. 152].

As the third result we show the following statements of Mayer-Vietoris type concerning n-shape equivalence:

THEOREM C. *Let $\varphi : (X; X_0, X_1, *) \to (Y; Y_0, Y_1, *)$ be a shape morphism, and suppose it induces n-shape equivalences $\varphi|_{X_0 \cap X_1} : (X_0 \cap X_1, *) \to (Y_0 \cap Y_1, *)$, $\varphi|_{X_0} : (X_0, *) \to (Y_0, *)$ and $\varphi|_{X_1} : (X_1, *) \to (Y_1, *)$, where $0 \le n < \infty$. Then*

1. *if $n \ge 2$, the induced map* $\varphi_* : \text{pro-}\pi_q(X; X_0, X_1, *) \to \text{pro-}\pi_q(Y; Y_0, Y_1, *)$ *is an isomorphism for $2 \le q \le n - 1$ and an epimorphism for $q = n$; and*
2. *$\varphi|_X : (X, *) \to (Y, *)$ is an n-shape equivalence.*

For any functions $f, g : X \to Y$ between sets and for any covering $\mathcal{V}$ of Y, $(f, g) < \mathcal{V}$ means that f and g are $\mathcal{V}$-near. For any covering $\mathcal{U}$ of a set X and for any subset A of X, let $\mathcal{U}|A = \{U \cap A : U \in \mathcal{U}\}$ and $\mathrm{St}(A,\mathcal{U}) = \cup\{U \in \mathcal{U} : U \cap A \neq \emptyset\}$.

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2. Preliminaries

Shape of triads. Let $\mathbf{Top}^T$ denote the category of triads of spaces and maps of triads. Recall that a *resolution* of a triad $(X; X_0, X_1)$ is a morphism

$$\boldsymbol{p} = (p_\lambda) : (X; X_0, X_1) \to (\boldsymbol{X}; \boldsymbol{X}_0, \boldsymbol{X}_1) = ((X_\lambda; X_{0\lambda}, X_{1\lambda}), p_{\lambda\lambda'}, \Lambda)$$

in $\textbf{pro-Top}^T$ with the following two properties [7]:

(R1) Let $(P; P_0, P_1)$ be an ANR triad, and let $\mathcal{V}$ be an open covering of P. Then every map of triads $f : (X; X_0, X_1) \to (P; P_0, P_1)$ admits $\lambda \in \Lambda$ and a map of triads $g : (X_\lambda; X_{0\lambda}, X_{1\lambda}) \to (P; P_0, P_1)$ such that $(gp_\lambda, f) < \mathcal{V}$; and

(R2) Let $(P; P_0, P_1)$ be an ANR triad. Then for each open covering $\mathcal{V}$ of P there exists an open covering $\mathcal{V}'$ of P such that whenever $\lambda \in \Lambda$ and $g, g' : (X_\lambda; X_{0\lambda}, X_{1\lambda}) \to (P; P_0, P_1)$ are maps of triads such that $(gp_\lambda, g'p_\lambda) < \mathcal{V}'$, then $(gp_{\lambda\lambda'}, g'p_{\lambda\lambda'}) < \mathcal{V}$ for some $\lambda' \geq \lambda$.

$\boldsymbol{p}$ is an *ANR-resolution* (resp., *polyhedral resolution*) if $(X_\lambda; X_{0\lambda}, X_{1\lambda})$ are all ANR triads (resp., polyhedral triads). Then we have

Theorem 2.1 ([7, 9]). *Every triad $(X; X_0, X_1)$ of spaces admits*

1. *an ANR-resolution*

$$\boldsymbol{p} = (p_\lambda) : (X; X_0, X_1) \to (\boldsymbol{X}; \boldsymbol{X}_0, \boldsymbol{X}_1) = ((X_\lambda; X_{0\lambda}, X_{1\lambda}), p_{\lambda\lambda'}, \Lambda)$$

such that Λ is cofinite and $X_\lambda = \mathrm{Int}(X_{0\lambda}) \cup \mathrm{Int}(X_{1\lambda})$ for each $\lambda \in \Lambda$; and

2. *a polyhedral resolution*

$$\boldsymbol{p} = (p_\lambda) : (X; X_0, X_1) \to (\boldsymbol{X}; \boldsymbol{X}_0, \boldsymbol{X}_1) = ((X_\lambda; X_{0\lambda}, X_{1\lambda}), p_{\lambda\lambda'}, \Lambda)$$

such that Λ is cofinite.

Theorem 2.2 ([9]). *Every resolution of a triad of spaces $(X; X_0, X_1)$ induces an expansion of $(X; X_0, X_1)$.*

Let $\mathbf{HTop}^T$ be the category of triads of spaces and homotopy classes of maps of triads, and let $\mathbf{HPol}^T$ be the full subcategory of $\mathbf{HTop}^T$ whose objects are the triads of spaces which have the homotopy type of a polyhedral triad (equivalently, an ANR triad) (see [9, Theorem 4.5]). Combining Theorems 2.1 and 2.2, we define the shape category $\mathbf{Sh}^T$ for triads of spaces as the abstract shape category for the pair $(\mathbf{HTop}^T, \mathbf{HPol}^T)$ ([9, §5]).

Theorem 2.3. *Let $(X;X_0,X_1)$ be a triad of spaces such that X is normal, X_0 and X_1 are closed and $X_0\cap X_1$ is normally embedded in X. Then for each resolution*

$$\boldsymbol{p}=(p_\lambda):(X;X_0,X_1)\to(\boldsymbol{X};\boldsymbol{X}_0,\boldsymbol{X}_1)=((X_\lambda;X_{0\lambda},X_{1\lambda}),p_{\lambda\lambda'},\Lambda)$$

such that X and X_λ, $\lambda\in\Lambda$, are normal, the following induced morphisms are resolutions:

$$\begin{cases}\boldsymbol{p}|_X=(p_\lambda|_X):X\to\boldsymbol{X}\\ \boldsymbol{p}|_{X_0}=(p_\lambda|_{X_0}):X_0\to\boldsymbol{X}_0\\ \boldsymbol{p}|_{X_1}=(p_\lambda|_{X_1}):X_1\to\boldsymbol{X}_1\\ \boldsymbol{p}|_{X_0\cap X_1}=\\ \quad(p_\lambda|_{X_0\cap X_1}):X_0\cap X_1\to\boldsymbol{X}_0\cap\boldsymbol{X}_1=(X_{0\lambda}\cap X_{1\lambda},p_{\lambda\lambda'}|_{X_{0\lambda'}\cap X_{1\lambda'}},\Lambda).\end{cases}$$

Proof. First note that our assumption implies that X_0 and X_1 are normally embedded (see [2, 2.7 a)]). By [7, Remark 1 and Theorem 4] the induced morphisms $\boldsymbol{p}|_{(X,X_0)}$, $\boldsymbol{p}|_{(X,X_1)}$ and $\boldsymbol{p}|_{(X,X_0\cap X_1)}$ are resolutions of pairs, which implies by [6, Theorem 2] that $\boldsymbol{p}|_X$ is a resolution and by [6, Theorem 3] that $\boldsymbol{p}|_{X_0}$, $\boldsymbol{p}|_{X_1}$ and $\boldsymbol{p}|_{X_0\cap X_1}$ are resolutions. □

Let $\mathbf{HTop}_*^T$ and $\mathbf{HPol}_*^T$ denote the pointed versions of the categories $\mathbf{HTop}^T$ and $\mathbf{HPol}^T$, respectively. Analogously, we can define the shape category $\mathbf{Sh}_*^T$ for pointed triads of spaces as the abstract shape category for the pair $(\mathbf{HTop}_*^T,\mathbf{HPol}_*^T)$. The pointed version of Theorem 2.3 also holds, and we have

Lemma 2.4. *Every pointed triad of spaces $(X;X_0,X_1,*)$ admits an $\mathbf{HPol}_*^T$-expansion $\boldsymbol{p}=(p_\lambda):(X;X_0,X_1,*)\to(\boldsymbol{X};\boldsymbol{X}_0,\boldsymbol{X}_1,*)$ such that $\boldsymbol{p}=(p_\lambda):(X;X_0,X_1)\to(\boldsymbol{X};\boldsymbol{X}_0,\boldsymbol{X}_1)$ is an $\mathbf{HPol}^T$-expansion.*

Proof. This follows from the constructions in [9, Theorems 3.2, 3.6]. □

ANR triads.

Lemma 2.5. *Let $(P;P_0,P_1)$ be an ANR triad, let $(X;X_0,X_1)$ be a triad of metric spaces such that X_0,X_1 are closed subsets of X, and let A be a closed subset of X. Then every map of triads $f:(A;A\cap X_0,A\cap X_1)\to(P;P_0,P_1)$ admits an extension $\tilde{f}:(U;U\cap X_0,U\cap X_1)\to(P;P_0,P_1)$ for some open neighborhood U of A in X.*

Lemma 2.6. *Let $(P;P_0,P_1),(X;X_0,X_1)$ and A be as in Lemma 2.5, and suppose that $f,g:(X;X_0,X_1)\to(P;P_0,P_1)$ are maps of triads. If $f|A\simeq g|A$ as maps of triads from $(A;A\cap X_0,A\cap X_1)$ to $(P;P_0,P_1)$, then there exists an open neighborhood V of A in X such that $f|V\simeq g|V$ as maps of triads from $(V;V\cap X_0,V\cap X_1)$ to $(P;P_0,P_1)$.*

LEMMA 2.7. *(Homotopy extension lemma) Let* $(P; P_0, P_1)$, $(X; X_0, X_1)$ *and* $A \subseteq X$ *be as in Lemma 2.5, and let* $(P; P_0, P_1)$ *be an ANR triad. If* $f, g : (A; A \cap X_0, A \cap X_1) \to (P; P_0, P_1)$ *are homotopic maps of triads, and if* g *extends to a map of triads* $\tilde{g} : (X; X_0, X_1) \to (P; P_0, P_1)$*, then there is an extension* $\tilde{f} : (X; X_0, X_1) \to (P; P_0, P_1)$ *of* f *such that* $\tilde{f} \simeq \tilde{g}$ *as maps of triads.*

PROOF OF LEMMA 2.5-2.7. These are proved in [9]. But note that the condition that $X = \mathrm{Int}(X_0) \cup \mathrm{Int}(X_1)$ can be dropped from the hypothesis of [9, Lemma 2.3]. □

Polyhedral triads.

LEMMA 2.8 (Homotopy extension lemma for polyhedral triads). *Let* $(P; P_0, P_1)$ *be a polyhedral triad, and let* Q *be a subpolyhedron of* P*. Then for any triad of spaces* $(Y; Y_0, Y_1)$ *any map of triads*

$$H : ((P \times 0) \cup (Q \times I); (P_0 \times 0) \cup ((P_0 \cap Q) \times I), (P_1 \times 0) \cup ((P_1 \cap Q) \times I)) \to (Y; Y_0, Y_1)$$

extends to a map of triads

$$\tilde{H} : (P \times I; P_0 \times I, P_1 \times I) \to (Y; Y_0, Y_1).$$

PROOF. The same argument as for [8, Theorem 3, p. 291] works for polyhedral triads. □

LEMMA 2.9 (Cellular approximation theorem for polyhedral triads). *For each map of triads* $f : (P; P_0, P_1) \to (Q; Q_0, Q_1)$ *between polyhedral triads, there exists a map of triads* $g : (P; P_0, P_1) \to (Q; Q_0, Q_1)$ *such that* $g(P^{(n)}) \subseteq Q^{(n)}$ *and* $f \simeq g$ *as maps of triads. Here for any polyhedron* R, $R^{(n)}$ *denotes the* n*-skeleton of* R*.*

PROOF. By the cellular approximation theorem, the restricted map $f|_{P_0 \cap P_1} : P_0 \cap P_1 \to Q_0 \cap Q_1$ admits a cellular map $g' : P_0 \cap P_1 \to Q_0 \cap Q_1$ such that $f|_{P_0 \cap P_1} \simeq g'$. By Lemma 2.8 (with $Q = P_0 \cap P_1$), g' extends to a map of triads $g' : (P; P_0, P_1) \to (Q; Q_0, Q_1)$ such that $f \simeq g'$ as maps of triads. By the cellular approximation theorem ([10, Theorem 17, p. 404]), there exist cellular maps of pairs $g_0 : (P_0, P_0 \cap P_1) \to (Q_0, Q_0 \cap Q_1)$ such that $g_0 \simeq g'|_{P_0}$ rel $(P_0 \cap P_1)$ and $g_1 : (P_1, P_0 \cap P_1) \to (Q_1, Q_0 \cap Q_1)$ such that $g_1 \simeq g'|_{P_1}$ rel $(P_0 \cap P_1)$. Since $g_0|_{P_0 \cap P_1} = g'|_{P_0 \cap P_1} = g_1|_{P_0 \cap P_1}$, g_0 and g_1 define a map of triads $g : (P; P_0, P_1) \to (Q; Q_0, Q_1)$ such that $g \simeq g'$ as maps of triads. □

LEMMA 2.10. *Let* $0 \leq n \leq \infty$*, and let* $(X; X_0, X_1, *)$ *and* $(Y; Y_0, Y_1, *)$ *be pointed polyhedral triads such that* X_0, X_1, $X_0 \cap X_1$, Y_0, Y_1, $Y_0 \cap Y_1$ *are connected, and let* $f : (X; X_0, X_1, *) \to (Y; Y_0, Y_1, *)$ *be a map of triads with the following property:*

(E)$_n$ *The restricted maps* $f|_{X_0\cap X_1} : (X_0 \cap X_1, *) \to (Y_0 \cap Y_1, *)$, $f|_{X_0} : (X_0, *) \to (Y_0, *)$ *and* $f|_{X_1} : (X_1, *) \to (Y_1, *)$ *are* n*-equivalences.*

and for each pointed polyhedral triad $(P; P_0, P_1, *)$ *consider the map*

$$f_* : \mathbf{HPol}_*^T((P; P_0, P_1, *), (X; X_0, X_1, *)) \longrightarrow \longrightarrow \mathbf{HPol}_*^T((P; P_0, P_1, *), (Y; Y_0, Y_1, *)).$$

Then if $\dim P \le n$, f_* *is an epimorphism, and if* $\dim P \le n - 1$, f_* *is a monomorphism.*

Proof. This is proved similarly to [10, Theorem 23]. ∎

Lemma 2.11. *Let* $1 \le n \le \infty$, *let* $(X; X_0, X_1, *)$ *and* $(Y; Y_0, Y_1, *)$ *be pointed polyhedral triads such that* X_0, X_1, $X_0 \cap X_1$, Y_0, Y_1, $Y_0 \cap Y_1$ *are connected, and let* $f : (X; X_0, X_1) \to (Y; Y_0, Y_1)$ *be a map of triads with property (E)*$_n$. *Then if* $\dim X \le n-1$ *and* $\dim Y \le n$, *then* $f : (X; X_0, X_1) \to (Y; Y_0, Y_1)$ *is a homotopy equivalence.*

Lemma 2.12. *Let* $0 \le n \le \infty$, *let* $(X; X_0, X_1, *)$ *and* $(Y; Y_0, Y_1, *)$ *be pointed polyhedral triads such that* X_0, X_1, $X_0 \cap X_1$, Y_0, Y_1, $Y_0 \cap Y_1$ *are connected, and let* $f : (X; X_0, X_1, *) \to (Y; Y_0, Y_1, *)$ *be a map of triads with property (E)*$_n$. *Then the restricted map* $f|_X : (X, *) \to (Y, *)$ *is an* n*-equivalence.*

Proof. This is essentially proved in [1, 16.24]. ∎

Lemma 2.13. *Let* $2 \le n \le \infty$, *let* $(X; X_0, X_1, *)$ *and* $(Y; Y_0, Y_1, *)$ *be pointed polyhedral triads such that* X_0, X_1, $X_0 \cap X_1$, Y_0, Y_1, $Y_0 \cap Y_1$ *are connected, and let* $f : (X; X_0, X_1, *) \to (Y; Y_0, Y_1, *)$ *be a map of triads with property (E)*$_n$. *Then the induced map* $f_* : \pi_q(X; X_0, X_1, *) \to \pi_q(Y; Y_0, Y_1, *)$ *is an isomorphism for* $2 \le q \le n - 1$ *and an epimorphism for* $q = n$.

Proof. This follows from the homotopy sequences for polyhedral pairs and triads (see [3, p. 160]) and the Five Lemma (see [5, p.201]). ∎

3. Shape dimension for triads of spaces

In this section we obtain fundamental properties of shape dimension and prove Theorem A. First, let us note that the properties analogous to [8, Theorem 2, p. 96] hold:

Proposition 3.1. *For each triad of spaces* $(X; X_0, X_1)$, *the following statements are equivalent:*

1. $\mathrm{Sd}(X; X_0, X_1) \le n$;
2. *There exists an* $\mathbf{HPol}^T$*-expansion*

$$\boldsymbol{p} = (p_\lambda) : (X; X_0, X_1) \to (\boldsymbol{X}; \boldsymbol{X}_0, \boldsymbol{X}_1) = ((X_\lambda; X_{0\lambda}, X_{1\lambda}), p_{\lambda\lambda'}, \Lambda)$$

such that each X_λ *is a polyhedral triad with* $\dim X_\lambda \le n$;

3. *For each* $\mathbf{HPol}^T$*-expansion*

$$\boldsymbol{p} = (p_\lambda) : (X; X_0, X_1) \to (\boldsymbol{X}; \boldsymbol{X}_0, \boldsymbol{X}_1) = ((X_\lambda; X_{0\lambda}, X_{1\lambda}), p_{\lambda\lambda'}, \Lambda),$$

each λ *admits* $\lambda' \geq \lambda$ *such that* $p_{\lambda\lambda'}$ *factors in* $\mathbf{HPol}^T$ *through a polyhedral triad* $(P; P_0, P_1)$ *such that* $\dim P \leq n$*; and*

4. *Each map* $f : (X; X_0, X_1) \to (P; P_0, P_1)$ *into a polyhedral triad* $(P; P_0, P_1)$ *factors in* $\mathbf{HTop}^T$ *through a map of triads* $g : (X; X_0, X_1) \to (P^{(n)}; P_0^{(n)}, P_1^{(n)})$ *and the inclusion map* $i : (P^{(n)}; P_0^{(n)}, P_1^{(n)}) \hookrightarrow (P; P_0, P_1)$.

Proof. Note that Lemma 2.9 is used in 1) $\Rightarrow$ 4), and the other cases are similar to the ordinary case. □

Shape dimension for pointed triads is similarly defined, and we have

Theorem 3.2. *For each triad of spaces* $(X; X_0, X_1)$ *with a base point* $*$, $\mathrm{Sd}(X; X_0, X_1, *) = \mathrm{Sd}(X; X_0, X_1)$.

Proof. The same argument as in the proof of [8, Theorem 7, p.104] applies to our case, using Lemma 2.8 and the following lemma in appropriate places. □

Lemma 3.3. *Let* $f, g : (P; P_0, P_1, *) \to (Q; Q_0, Q_1, *)$ *be maps of pointed polyhedral triads such that* $f \simeq g$ *as maps of unpointed triads and* $g(P) \subseteq Q^{(n)}$ *for some* $n \geq 0$. *Then there exists a map of pointed triads* $h : (P; P_0, P_1, *) \to (Q; Q_0, Q_1, *)$ *such that* $f \simeq h$ *as maps of pointed triads and* $h(P) \subseteq Q^{(n)}$.

Proof. The same argument as in the proof of [8, Lemma 4, p. 104] applies to our case, using Lemmas 2.8 and 2.9. □

Before proving Theorem A, we prove

Lemma 3.4. *Let* $(X; X_0, X_1)$ *be a triad of metric spaces such that* X_0 *and* X_1 *are closed, and let*

$$\begin{cases} X' = (X_0 \times 0) \cup ((X_0 \cap X_1) \times I) \cup (X_1 \times 1) \\ X'_0 = X' \cap (X_0 \times [0, 2/3]) \\ X'_1 = X' \cap (X_1 \times [1/3, 1]) \end{cases}$$

and

$$\begin{cases} X'' = (X_0 \times [0, 2/3]) \cup (X_1 \times [1/3, 1]) \\ X''_0 = X_0 \times [0, 2/3] \\ X''_1 = X_1 \times [1/3, 1] \end{cases}.$$

Then the inclusion map $i : (X'; X'_0, X'_1) \hookrightarrow (X''; X''_0, X''_1)$ *is an equivalence in* $\mathbf{Sh}^T$.

PROOF. The proof follows the technique used in [2, Lemma 2.6]. It suffices to show that for each ANR triad $(P;P_0,P_1)$ the inclusion induced map

$$i^*:\mathbf{HTop}^T((X'';X_0'',X_1''),(P;P_0,P_1))\to\mathbf{HTop}^T((X';X_0',X_1'),(P;P_0,P_1))$$

is a bijection. By Lemma 2.5, each map $f:(X';X_0',X_1')\to(P;P_0,P_1)$ extends to a map $\overline{f}:(U;U\cap X_0'',U\cap X_1'')\to(P;P_0,P_1)$ for some open neighborhood U of X' in X''. By the compactness of I, there exists an open neighborhood V of $X_0\cap X_1$ in X such that $(V\times I)\cap X''\subseteq U$. By the Urysohn lemma, there exists a map $\phi:X\to I$ such that

$$\begin{cases}\phi|_{X_0\cap X_1}=1;\\ \phi|_{X\setminus V}=0.\end{cases}$$

Define a map $g:X\times I\to P$ by

$$g(x,t)=\begin{cases}\overline{f}(x,t\phi(x)) & \text{if } x\in X_0;\\ \overline{f}(x,1-(1-t)\phi(x)) & \text{if } x\in X_1.\end{cases}$$

Then $g|_{X'}=f$, and $g(X_0'')\subseteq P_0$ and $g(X_1'')\subseteq P_1$, so g defines a map of triads $g:(X'';X_0'',X_1'')\to(P;P_0,P_1)$ such that $g|_{(X';X_0',X_1')}=f$, showing that i^* is surjective. To see that i^* is injective, suppose that $g_1,g_2:(X'';X_0'',X_1'')\to(P;P_0,P_1)$ are maps of triads such that $g_1|_{X'}\simeq g_2|_{X'}$ as maps of triads from $(X';X_0',X_1')$ to $(P;P_0,P_1)$. Then by Lemma 2.6 there exists an open neighborhood W of X' in X'' such that $g_1|_W\simeq g_2|_W$ as maps of triads from $(W;W\cap X_0'',W\cap X_1'')$ to $(P;P_0,P_1)$. By the same argument as above, this homotopy of triads extends to a homotopy of triads $g_1\simeq g_2$ as required. □

PROOF OF THEOREM A. The first inequality follows from Theorem 2.3. To show the second inequality, let $n=\max\{\mathrm{Sd}\,X_0,\mathrm{Sd}\,X_1,1+\mathrm{Sd}(X_0\cap X_1)\}$, and let $f:(X;X_0,X_1)\to(P;P_0,P_1)$ be a map of triads. It suffices to verify the second inequality for each triad $(X;X_0,X_1)$ of spaces such that the inclusion maps $X_0\cap X_1\hookrightarrow X_0$ and $X_0\cap X_1\hookrightarrow X_1$ are cofibrations. Indeed, embed $(X;X_0,X_1)$ into $(X'';X_0'',X_1'')$ by $i(x)=(x,1/2)$ for $x\in X$, and let $r:(X'';X_0'',X_1'')\to(X;X_0,X_1)$ be the projection. Then $ri=1_X$, in particular, $(X;X_0,X_1)$ is dominated by $(X'';X_0'',X_1'')$ in $\mathbf{Sh}^T$, so that $\mathrm{Sd}(X;X_0,X_1)\le\mathrm{Sd}(X'';X_0'',X_1'')$. Since by Lemma 3.4 $\mathrm{Sd}(X'';X_0'',X_1'')=\mathrm{Sd}(X';X_0',X_1')$, then $\mathrm{Sd}(X;X_0,X_1)\le\mathrm{Sd}(X';X_0',X_1')$. On the other hand, by [2, Theorem 2.6], X_0 and X_1 are shape equivalent to X_0' and X_1', respectively, and clearly $X_0\cap X_1$ is shape equivalent to $X_0'\cap X_1'$. Hence we can replace $(X;X_0,X_1)$ by $(X';X_0',X_1')$ if necessary.

Now since $\mathrm{Sd}(X_0\cap X_1)\le n-1$, $f|_{X_0\cap X_1}\simeq f'$ for some map $f':X_0\cap X_1\to P_0\cap P_1$ such that $f'(X_0\cap X_1)\subseteq(P_0\cap P_1)^{(n-1)}$. Considering f' as a map of triads $f':(A;A\cap X_0,A\cap X_1)\to(P;P_0,P_1)$ where $A=X_0\cap X_1$, by Lemma 2.7, f' extends to a map of triads $f':(X;X_0,X_1)\to(P;P_0,P_1)$ such that $f'\simeq f$ as maps of triads. Now by [2, Lemma 2.8 b)], there exists a map $g_0:X_0\to P_0$ such that $g_0(X_0)\subseteq P_0^{(n)}$ and $g_0\simeq f'|_{X_0}$ rel $(X_0\cap X_1)$, and similarly there exists a map $g_1 : X_1 \to P_1$ such that $g_1(X_1) \subseteq P_1^{(n)}$ and $g_1 \simeq f'|_{X_1}$ rel $(X_0 \cap X_1)$. So the map of triads $g : (X; X_0, X_1) \to (P; P_0, P_1)$ defined by $g|_{X_0} = g_0$ and $g|_{X_1} = g_1$ satisfies $g(X) \subseteq P^{(n)}$ and $g \simeq f$ as maps of triads. By Proposition 3.1, we conclude $\mathrm{Sd}(X; X_0, X_1) \leq n$. ☐

Remark. Note that the difference between the upper and lower bounds in Theorem A is at most 1, i.e.,

$$\begin{aligned}&\max\{\mathrm{Sd}\, X_0, \mathrm{Sd}\, X_1, \mathrm{Sd}(X_0 \cap X_1) + 1\} \leq \\ &\qquad \leq \max\{\mathrm{Sd}\, X, \mathrm{Sd}\, X_0, \mathrm{Sd}\, X_1, \mathrm{Sd}(X_0 \cap X_1)\} + 1.\end{aligned}$$

Moreover, there is an example with each one of the inequalities being strict. Indeed, there exists a polyhedral triad $(X; X_0, X_1)$ such that

$$\begin{aligned}\max\{\mathrm{Sd}\, X, \mathrm{Sd}\, X_0, \mathrm{Sd}\, X_1, \mathrm{Sd}(X_0 \cap X_1)\} &< \mathrm{Sd}(X; X_0, X_1) \\ &= \max\{\mathrm{Sd}\, X_0, \mathrm{Sd}\, X_1, \mathrm{Sd}(X_0 \cap X_1) + 1\}\end{aligned}$$

(e.g., take $X = X_0 = D^2$, $X_1 = \partial D^2$), and also there exists a polyhedral triad $(X; X_0, X_1)$ such that

$$\begin{aligned}\max\{\mathrm{Sd}\, X, \mathrm{Sd}\, X_0, \mathrm{Sd}\, X_1, \mathrm{Sd}(X_0 \cap X_1)\} &= \mathrm{Sd}(X; X_0, X_1) \\ &< \max\{\mathrm{Sd}\, X_0, \mathrm{Sd}\, X_1, \mathrm{Sd}(X_0 \cap X_1) + 1\}\end{aligned}$$

(e.g., take $X = X_0 = X_1 = S^1$).

4. n-Shape equivalence

Throughout the rest of the paper, all triads are pointed, and maps and homotopies preserve the base point, so that the indication of the base point is omitted.

In this section we wish to prove Theorems B and C. First we prove the following lemmas.

Lemma 4.1. *Suppose that we are given a commutative diagram:*

$$\begin{array}{ccccc}(Z_{\lambda_0}; Z_{0\lambda_0}, Z_{1\lambda_0}) & \xrightarrow{p_0} & (Z_{\lambda_1}; Z_{0\lambda_1}, Z_{1\lambda_1}) & \xrightarrow{p_1} & \cdots \\ \subseteq\uparrow & & \subseteq\uparrow & & \\ (X_{\lambda_0}; X_{0\lambda_0}, X_{1\lambda_0}) & \longrightarrow & (X_{\lambda_1}; X_{0\lambda_1}, X_{1\lambda_1}) & \longrightarrow & \cdots\end{array}$$

$$\begin{array}{ccc}\cdots & \xrightarrow{p_{n-1}} & (Z_{\lambda_n}; Z_{0\lambda_n}, Z_{1\lambda_n}) \\ & & \subseteq\uparrow \\ \cdots & \longrightarrow & (X_{\lambda_n}; X_{0\lambda_n}, X_{1\lambda_n})\end{array}$$

where $(Z_{\lambda_i}; Z_{0\lambda_i}, Z_{1\lambda_i})$ *and* $(X_{\lambda_i}; X_{0\lambda_i}, X_{1\lambda_i})$, $i = 0, 1, \ldots, n$, *are polyhedral triads such that* $Z_{0\lambda_0}$, $Z_{1\lambda_0}$ *and* $Z_{0\lambda_0} \cap Z_{1\lambda_0}$ *are connected, and suppose that the induced maps*

$$\begin{cases} (p_i|_{(Z_{0\lambda_i}, X_{0\lambda_i})})_* : \pi_i(Z_{0\lambda_i}, X_{0\lambda_i}) \to \pi_i(Z_{0\lambda_{i+1}}, X_{0\lambda_{i+1}}) \\ (p_i|_{(Z_{1\lambda_i}, X_{1\lambda_i})})_* : \pi_i(Z_{1\lambda_i}, X_{1\lambda_i}) \to \pi_i(Z_{1\lambda_{i+1}}, X_{1\lambda_{i+1}}) \\ (p_i|_{(Z_{0\lambda_i} \cap Z_{1\lambda_i}, X_{0\lambda_i} \cap X_{1\lambda_i})})_* : \\ \quad \pi_i(Z_{0\lambda_i} \cap Z_{1\lambda_i}, X_{0\lambda_i} \cap X_{1\lambda_i}) \to \pi_i(Z_{0\lambda_{i+1}} \cap Z_{1\lambda_{i+1}}, X_{0\lambda_{i+1}} \cap X_{1\lambda_{i+1}}) \end{cases}$$

are trivial for $i = 0, 1, \ldots, n-1$. *Then there exist polyhedral triads* $(P; P_0, P_1)$ *and* $(Q; Q_0, Q_1)$ *and a map of triads* $g : (P; P_0, P_1) \to (Z_{\lambda_n}, Z_{0\lambda_n}, Z_{1\lambda_n})$ *with the following properties:*

1. P_0, P_1, $P_0 \cap P_1$, Q_0, Q_1, $Q_0 \cap Q_1$ *are connected;*
2. $(Q; Q_0, Q_1) \subseteq (P; P_0, P_1)$, *and the inclusion map* $k : (Q; Q_0, Q_1) \hookrightarrow (P; P_0, P_1)$ *satisfies condition (E)*$_{n-1}$*;*
3. $(Z_{\lambda_0}; Z_{0\lambda_0}, Z_{1\lambda_0}) \subseteq (P; P_0, P_1)$ *and* $(X_{\lambda_0}; X_{0\lambda_0}, X_{1\lambda_0}) \subseteq (Q; Q_0, Q_1)$*;*
4. $g|_{(Z_{\lambda_0}; Z_{0\lambda_0}, Z_{1\lambda_0})} = p_{n-1} \cdots p_1 p_0$*; and*
5. *The restriction of* g *to* $(Q; Q_0, Q_1)$ *defines a map of triads*

$$g|_{(Q; Q_0, Q_1)} : (Q; Q_0, Q_1) \to (X_{\lambda_n}; X_{0\lambda_n}, X_{1\lambda_n}).$$

Proof. Let $\begin{Bmatrix} (K; K_0, K_1) \\ (L; L_0, L_1) \end{Bmatrix}$ be triangulations of $\begin{Bmatrix} (Z_{\lambda_0}; Z_{0\lambda_0}, Z_{1\lambda_0}) \\ (X_{\lambda_0}; X_{0\lambda_0}, X_{1\lambda_0}) \end{Bmatrix}$ such that $(L; L_0, L_1)$ is a subcomplex of $(K; K_0, K_1)$ and that L is a full subcomplex of K. For each $i = 0, 1, \ldots, n-1$, let

$$\begin{cases} Q_i = (X_{\lambda_0} \times I) \cup (|K^i| \times I) \\ Q_{0i} = (X_{0\lambda_0} \times I) \cup (|K_0^i| \times I) \\ Q_{1i} = (X_{1\lambda_0} \times I) \cup (|K_1^i| \times I) \end{cases}$$

and

$$\begin{cases} P_i = Q_i \cup (Z_{\lambda_0} \times 0) \\ P_{0i} = Q_{0i} \cup (Z_{0\lambda_0} \times 0) \\ P_{1i} = Q_{1i} \cup (Z_{1\lambda_0} \times 0) \end{cases}.$$

Then for each $i = 0, 1, \ldots, n-1$, the inclusion map $k_i : (Q_i; Q_{0i}, Q_{1i}) \hookrightarrow (P_i; P_{0i}, P_{1i})$ satisfies condition (E)$_i$. We wish to define a map of triads

$$g_i : (P_i; P_{0i}, P_{1i}) \to (Z_{\lambda_{i+1}}; Z_{0\lambda_{i+1}}, Z_{1\lambda_{i+1}}) \ (i = 0, 1, \ldots, n-1)$$

with the following properties:

1. The restriction of g_i to $(Q_i; Q_{0i}, Q_{1i})$ defines a map of triads

$$g_i|_{(Q_i; Q_{0i}, Q_{1i})} : (Q_i; Q_{0i}, Q_{1i}) \to (X_{\lambda_{i+1}}; X_{0\lambda_{i+1}}, X_{1\lambda_{i+1}});$$

and

2. The following diagram commutes:

$$\begin{array}{ccccc}
(Z_{\lambda_0}; Z_{0\lambda_0}, Z_{1\lambda_0}) & \xrightarrow{\subseteq} & (P_0; P_{00}, P_{10}) & \xrightarrow{\subseteq} & \cdots \\
\Big\| & & \Big\downarrow{\scriptstyle g_0} & & \\
(Z_{\lambda_0}; Z_{0\lambda_0}, Z_{1\lambda_0}) & \xrightarrow{p_0} & (Z_{\lambda_1}; Z_{0\lambda_1}, Z_{1\lambda_1}) & \xrightarrow{p_1} & \cdots
\end{array}$$

$$\begin{array}{ccc}
\cdots & \xrightarrow{\subseteq} & (P_{n-1}; P_{0n-1}, P_{1n-1}) \\
 & & \Big\downarrow{\scriptstyle g_{n-1}} \\
\cdots & \xrightarrow{p_{n-1}} & (Z_{\lambda_n}; Z_{0\lambda_n}, Z_{1\lambda_n}).
\end{array}$$

For $i = 0$, first let

$$\begin{cases} g_0 | Z_{\lambda_0} \times 0 = p_0; \\ g_0(x \times I) = p_0(x) \quad \text{for } x \in X_{\lambda_0}, \end{cases}$$

and for each vertex v of $K \setminus L$, let $g_0|v \times I$ be a path from $p_0(v)$ to the base point $*$ in

$$\left\{ \begin{array}{cl} Z_{0\lambda_1} & \text{if } v \in K_0 \setminus K_1 \\ Z_{1\lambda_1} & \text{if } v \in K_1 \setminus K_0 \\ Z_{0\lambda_1} \cap Z_{1\lambda_1} & \text{if } v \in K_0 \cap K_1 \end{array} \right\}.$$

Such a path exists since $Z_{0\lambda_0}, Z_{1\lambda_0}, Z_{0\lambda_0} \cap Z_{1\lambda_0}$ are path-connected. Then $g_0(P_{00}) \subseteq Z_{0\lambda_1}$, $g_0(P_{10}) \subseteq Z_{1\lambda_1}$, $g_0(Q_{00}) \subseteq X_{0\lambda_1}$, $g_0(Q_{10}) \subseteq X_{1\lambda_1}$, and thus g_0 defines a map of triads

$$g_0 : (P_0; P_{00}, P_{10}) \to (Z_{\lambda_1}; Z_{0\lambda_1}, Z_{1\lambda_1})$$

so that the restriction to $(Q_0; Q_{00}, Q_{01})$ defines a map of triads

$$g_0|_{(Q_0;Q_{00},Q_{01})} : (Q_0; Q_{00}, Q_{10}) \to (X_{\lambda_1}; X_{0\lambda_1}, X_{1\lambda_1}).$$

Suppose g_{i-1} has beed defined for some i such that $1 \leq i \leq n-2$. To define g_i, first let $p_i|P_{i-1} = p_i g_{i-1}$. Let σ be an i-simplex of $K \setminus L$, and let $v \in K \setminus L$ be a vertex of σ. Then either one of the following occurs: $\sigma \in K_0 \setminus K_1$, $\sigma \in K_1 \setminus K_0$, $\sigma \in K_0 \cap K_1$. So $((\partial\sigma \times I) \cup (\sigma \times 0), \partial\sigma \times 1, v \times 1)$ forms a cell in $\left\{ \begin{array}{c} P_{0i-1} \\ P_{1i-1} \\ P_{0i-1} \cap P_{1i-1} \end{array} \right\}$ with its boundary in $\left\{ \begin{array}{c} Q_{0i-1} \\ Q_{1i-1} \\ Q_{0i-1} \cap Q_{1i-1} \end{array} \right\}$ if $\left\{ \begin{array}{c} \sigma \in K_0 \setminus K_1 \\ \sigma \in K_1 \setminus K_0 \\ \sigma \in K_0 \cap K_1 \end{array} \right\}$, and the map $g_{i-1}|_{((\partial\sigma \times I) \cup (\sigma \times 0), \partial\sigma \times 1, v \times 1)}$ defines an element of $\left\{ \begin{array}{c} \pi_i(Z_{0\lambda_i}, X_{0\lambda_i}) \\ \pi_i(Z_{1\lambda_i}, X_{1\lambda_i}) \\ \pi_i(Z_{0\lambda_i} \cap Z_{1\lambda_i}, X_{0\lambda_i} \cap X_{1\lambda_i}) \end{array} \right\}$.

By assumption, $p_i g_{i-1}|_{((\partial\sigma\times I)\cup(\sigma\times 0),\partial\sigma\times 1,v\times 1)}$ extends to a map

$$g_i|_{(\sigma\times I,\sigma\times 1,v\times 1)} : (\sigma\times I,\sigma\times 1,v\times 1) \to \left\{\begin{matrix} (Z_{0\lambda_{i+1}}, X_{0\lambda_{i+1}}) \\ (Z_{1\lambda_{i+1}}, X_{1\lambda_{i+1}}) \\ (Z_{0\lambda_{i+1}} \cap Z_{1\lambda_{i+1}}, X_{0\lambda_{i+1}} \cap X_{1\lambda_{i+1}}) \end{matrix}\right\}.$$

Repeating the same process for each i-simplex of $K \setminus L$, we obtain a map g_i with the desired property. Now it suffices to set $(P;P_0,P_1) = (P_{n-1};P_{0n-1},P_{1n-1})$, $(Q;Q_0,Q_1) = (Q_{n-1};Q_{0n-1},Q_{1n-1})$ and $g = g_{n-1}$. □

LEMMA 4.2. *Let* $(\boldsymbol{Z};\boldsymbol{Z}_0,\boldsymbol{Z}_1) = ((Z_\lambda;Z_{0\lambda},Z_{1\lambda}),r_{\lambda\lambda'},\Lambda)$ *and* $(\boldsymbol{X};\boldsymbol{X}_0,\boldsymbol{X}_1) = ((X_\lambda;X_{0\lambda},X_{1\lambda}),p_{\lambda\lambda'},\Lambda)$ *be inverse systems of polyhedral triads such that for each* $\lambda \in \Lambda$, $(X_\lambda;X_{0\lambda},X_{1\lambda}) \subseteq (Z_\lambda;Z_{0\lambda},Z_{1\lambda})$ *and* $Z_{0\lambda}$, $Z_{1\lambda}$ *and* $Z_{0\lambda}\cap Z_{1\lambda}$ *are connected. Suppose that the inclusion induced morphism* $\boldsymbol{j} = (j_\lambda) : (\boldsymbol{X};\boldsymbol{X}_0,\boldsymbol{X}_1) \to (\boldsymbol{Z};\boldsymbol{Z}_0,\boldsymbol{Z}_1)$ *satisfies the following condition:*

(EE)$_n$ *The induced morphisms* $\boldsymbol{j}|_{\boldsymbol{X}_0\cap\boldsymbol{X}_1} : \boldsymbol{X}_0\cap\boldsymbol{X}_1 \to \boldsymbol{Z}_0\cap\boldsymbol{Z}_1$, $\boldsymbol{j}|_{\boldsymbol{X}_0} : \boldsymbol{X}_0 \to \boldsymbol{Z}_0$ *and* $\boldsymbol{j}|_{\boldsymbol{X}_1} : \boldsymbol{X}_1 \to \boldsymbol{Z}_1$ *are* n*-equivalences.*

Then for each $\lambda \in \Lambda$ *there exist* $\lambda' \geq \lambda$, *polyhedral triads* $(P;P_0,P_1)$ *and* $(Q;Q_0,Q_1)$ *and a map of triads* $g : (P;P_0,P_1) \to (Z_\lambda;Z_{0\lambda},Z_{1\lambda})$ *with the following properties:*

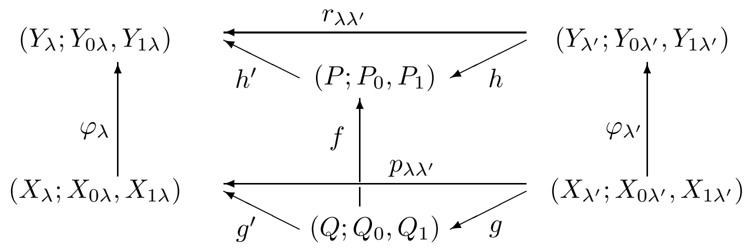
1. P_0, P_1, $P_0\cap P_1$, Q_0, Q_1, $Q_0\cap Q_1$ *are connected;*
2. $(Q;Q_0,Q_1) \subseteq (P;P_0,P_1)$, *and the inclusion map* $k : (Q;Q_0,Q_1) \hookrightarrow (P;P_0,P_1)$ *satisfies condition (E)*$_n$*;*
3. $(Z_{\lambda'};Z_{0\lambda'},Z_{1\lambda'}) \subseteq (P;P_0,P_1)$ *and* $(X_{\lambda'};X_{0\lambda'},X_{1\lambda'}) \subseteq (Q;Q_0,Q_1)$*;*
4. $g|_{(Z_{\lambda'};Z_{0\lambda'},Z_{1\lambda'})} = r_{\lambda\lambda'}$*; and*
5. *The restriction of* g *to* $(Q;Q_0,Q_1)$ *defines a map of triads*

$$g|_{(Q;Q_0,Q_1)} : (Q;Q_0,Q_1) \to (X_\lambda;X_{0\lambda},X_{1\lambda}).$$

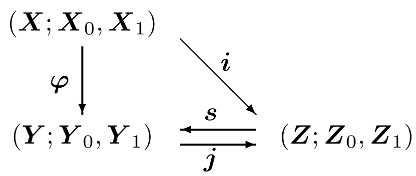
PROOF. By assumption, the inverse systems of pairs $(\boldsymbol{Z}_0,\boldsymbol{X}_0)$, $(\boldsymbol{Z}_1,\boldsymbol{X}_1)$ and $(\boldsymbol{Z}_0\cap\boldsymbol{Z}_1,\boldsymbol{X}_0\cap\boldsymbol{X}_1)$ are n-connected. This implies that each $\lambda\in\Lambda$ admits $\lambda = \lambda_{n+1} \leq \lambda_n \leq \cdots \leq \lambda_1 \leq \lambda_0 = \lambda'$ in Λ so that the hypothesis of Lemma 4.1 is satisfied with $p_i = p_{\lambda_{i+1}\lambda_i}$. Our assertion follows from Lemma 4.1. □

LEMMA 4.3. *Let* $\boldsymbol{\varphi} = (\varphi_\lambda) : (\boldsymbol{X};\boldsymbol{X}_0,\boldsymbol{X}_1) \to (\boldsymbol{Y};\boldsymbol{Y}_0,\boldsymbol{Y}_1)$ *be a level morphism of inverse systems of polyhedral triads such that for each* $\lambda\in\Lambda$, $X_{0\lambda}, X_{1\lambda}, X_{0\lambda}\cap X_{1\lambda}, Y_{0\lambda}, Y_{1\lambda}, Y_{0\lambda}\cap Y_{1\lambda}$ *are all connected. If* $\boldsymbol{\varphi}$ *satisfies condition (EE)*$_n$, *then for each* $\lambda\in\Lambda$ *there exist* $\lambda' \geq \lambda$, *polyhedral triads* $(P;P_0,P_1)$ *and* $(Q;Q_0,Q_1)$ *and a map of triads* $f : (Q;Q_0,Q_1) \to (P;P_0,P_1)$ *with the following properties:*

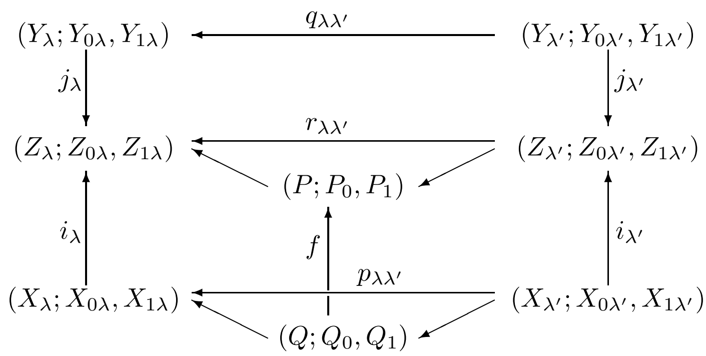
1. P_0, P_1, $P_0\cap P_1$, Q_0, Q_1, $Q_0\cap Q_1$ *are connected;*
2. f *satisfies condition* $(E)_n$*; and*
3. *The following diagram commutes for some maps of triads* h, h', g, g'*:*



Proof. Let $(\boldsymbol{Z}; \boldsymbol{Z}_0, \boldsymbol{Z}_1)$ be the inverse system of polyhedral triads of mapping cylinders $((M(\varphi_\lambda|_{X_\lambda}); M(\varphi_\lambda|_{X_{0\lambda}}), M(\varphi_\lambda|_{X_{1\lambda}})), r_{\lambda\lambda'}, \Lambda)$. Then there is a commutative diagram:



where $\boldsymbol{i}$ and $\boldsymbol{j}$ are the inclusion induced morphisms, and $\boldsymbol{s}$ is induced by the retractions. Then that $\boldsymbol{\varphi}$ satisfies condition $(\mathrm{EE})_n$ is equivalent to that $\boldsymbol{i}$ satisfies condition $(\mathrm{EE})_n$. Lemma 4.2 implies that each $\lambda \in \Lambda$ admits $\lambda' \geq \lambda$ and polyhedral triads $(P; P_0, P_1)$ and $(Q; Q_0, Q_1)$ and a map $f : (Q; Q_0, Q_1) \to (P; P_0, P_1)$ with properties 1) - 5) of Lemma 4.2, so that there is the following commutative diagram:



It is easy to see that such polyhedral triads $(P; P_0, P_1)$ and $(Q; Q_0, Q_1)$ and map of triads $f : (Q; Q_0, Q_1) \to (P; P_0, P_1)$ have the desired properties. ∎

Lemma 4.4. *Let* $\boldsymbol{\varphi} = (\varphi_\lambda) : (\boldsymbol{X}; \boldsymbol{X}_0, \boldsymbol{X}_1) \to (\boldsymbol{Y}; \boldsymbol{Y}_0, \boldsymbol{Y}_1)$ *be a level morphism of inverse systems of polyhedral triads such that for each* $\lambda \in \Lambda$,

$X_{0\lambda}, X_{1\lambda}, X_{0\lambda}\cap X_{1\lambda}, Y_{0\lambda}, Y_{1\lambda}, Y_{0\lambda}\cap Y_{1\lambda}$ are all connected. If $\boldsymbol{\varphi}$ satisfies condition (EE)$_n$, then each $\lambda\in\Lambda$ admits $\lambda'\geq\lambda$ with the following properties:

1. *For each map of triads $h:(R;R_0,R_1)\to(Y_{\lambda'};Y_{0\lambda'},Y_{1\lambda'})$ of a polyhedral triad $(R;R_0,R_1)$ such that $R_0, R_1, R_0\cap R_1$ are connected and $\dim R\leq n$, there exists a map of triads $k:(R;R_0,R_1)\to(X_\lambda;X_{0\lambda},X_{1\lambda})$ such that $\varphi_\lambda k\simeq q_{\lambda\lambda'}h$ as maps of triads; and*
2. *For each polyhedral triad $(R;R_0,R_1)$ such that $R_0,R_1,R_0\cap R_1$ are connected and $\dim R\leq n-1$ and for each pair of maps of triads $k_1,k_2:(R;R_0,R_1)\to(X_{\lambda'};X_{0\lambda'},X_{1\lambda'})$ such that $\varphi_{\lambda'}k_1\simeq\varphi_{\lambda'}k_2$ as maps of triads, we have $p_{\lambda\lambda'}k_1\simeq p_{\lambda\lambda'}k_2$ as maps of triads.*

Proof. For each $\lambda\in\Lambda$, take $\lambda'\geq\lambda$ as in Lemma 4.3. Then Lemmas 4.3 and 2.10 imply our assertion. □

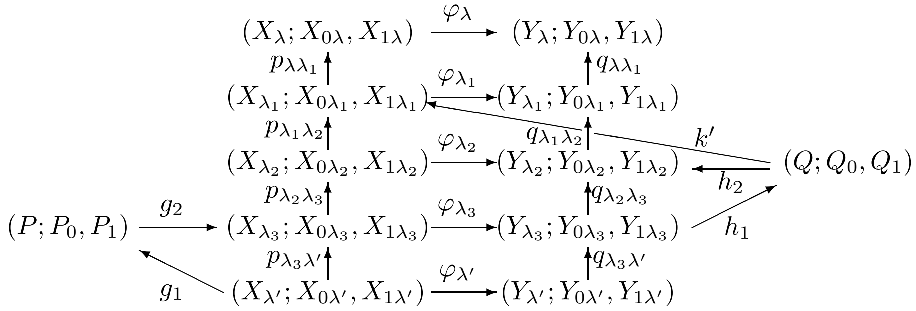
Proof of Theorem B. Let $\varphi:(X;X_0,X_1)\to(Y;Y_0,Y_1)$ be a shape morphism as in the hypothesis. Let φ be represented by a level morphism $\boldsymbol{\varphi}=(\varphi_\lambda):(\boldsymbol{X};\boldsymbol{X}_0,\boldsymbol{X}_1)\to(\boldsymbol{Y};\boldsymbol{Y}_0,\boldsymbol{Y}_1)$ where $\boldsymbol{p}=(p_\lambda):(X;X_0,X_1)\to(\boldsymbol{X};\boldsymbol{X}_0,\boldsymbol{X}_1)=((X_\lambda;X_{0\lambda},X_{1\lambda}),p_{\lambda\lambda'},\Lambda)$ and $\boldsymbol{q}=(q_\lambda):(Y;Y_0,Y_1)\to(\boldsymbol{Y};\boldsymbol{Y}_0,\boldsymbol{Y}_1)=((Y_\lambda;Y_{0\lambda},Y_{1\lambda}),q_{\lambda\lambda'},\Lambda)$ are $\mathbf{HTop}^T$-expansions of $(X;X_0,X_1)$ and $(Y;Y_0,Y_1)$, respectively, such that Λ is cofinite, $X_{0\lambda}$, $X_{1\lambda}$, $X_{0\lambda}\cap X_{1\lambda}$, $Y_{0\lambda}$, $Y_{1\lambda}$, $Y_{0\lambda}\cap Y_{1\lambda}$ are connected, and the following induced morphisms are expansions (see Theorem 2.3):

$$\begin{cases}\boldsymbol{p}|_{X_0}:X_0\to\boldsymbol{X}_0\\ \boldsymbol{p}|_{X_1}:X_1\to\boldsymbol{X}_1\\ \boldsymbol{p}|_{X_0\cap X_1}:X_0\cap X_1\to\boldsymbol{X}_0\cap\boldsymbol{X}_1\end{cases}$$

and

$$\begin{cases}\boldsymbol{q}|_{Y_0}:Y_0\to\boldsymbol{Y}_0\\ \boldsymbol{q}|_{Y_1}:Y_1\to\boldsymbol{Y}_1\\ \boldsymbol{q}|_{Y_0\cap Y_1}:Y_0\cap Y_1\to\boldsymbol{Y}_0\cap\boldsymbol{Y}_1\end{cases}.$$

Now let $\lambda\in\Lambda$. Then take $\lambda_1\geq\lambda$ as in Lemma 4.4, and for this λ_1 repeatedly take $\lambda_2\geq\lambda_1$ as in Lemma 4.4. Since $\mathrm{Sd}(Y;Y_0,Y_1)\leq n$, by Proposition 3.1, there exists $\lambda_3\geq\lambda_2$ so that $q_{\lambda_2\lambda_3}$ factors through a polyhedral triad $(Q;Q_0,Q_1)$ with $\dim Q\leq n$. Similarly, by $\mathrm{Sd}(X;X_0,X_1)\leq n-1$, there exists $\lambda'\geq\lambda_3$ so that $p_{\lambda_3\lambda'}$ factors through a polyhedral triad $(P;P_0,P_1)$ with $\dim P\leq n-1$. Say $g_1:(X_{\lambda'};X_{0\lambda'},X_{1\lambda'})\to(P;P_0,P_1)$, $g_2:(P;P_0,P_1)\to(X_{\lambda_3};X_{0\lambda_3},X_{1\lambda_3})$ and $h_1:(Y_{\lambda_3};Y_{0\lambda_3},Y_{1\lambda_3})\to(Q;Q_0,Q_1)$, $h_2:(Q;Q_0,Q_1)\to(Y_{\lambda_2};Y_{0\lambda_2},Y_{1\lambda_2})$ are homotopy classes such that $p_{\lambda_3\lambda'}=g_2g_1$ and $q_{\lambda_2\lambda_3}=h_2h_1$. By Lemma 4.4, there exists a homotopy class $k':(Q;Q_0,Q_1)\to(X_{\lambda_1};X_{0\lambda_1},X_{1\lambda_1})$ so that $\varphi_{\lambda_1}k'=q_{\lambda_1\lambda_2}h_2$. Thus we have the following commutative diagram:



Now let $k = p_{\lambda\lambda_1}k'h_1q_{\lambda_3\lambda'} : (Y_{\lambda'}; Y_{0\lambda'}, Y_{1\lambda'}) \to (X_\lambda; X_{0\lambda}, X_{1\lambda})$. By tracing around the diagram we get $k\varphi_\lambda = q_{\lambda\lambda'}$. Also by the diagram $\varphi_{\lambda_1}k'h_1\varphi_{\lambda_3}g_2 = \varphi_{\lambda_1}p_{\lambda_1\lambda_3}g_2$. Since $\dim P \le n-1$, by the choice of λ_1, $p_{\lambda\lambda_1}k'h_1\varphi_{\lambda_3}g_2 = p_{\lambda\lambda_3}g_2$, which implies $p_{\lambda\lambda'} = k\varphi_{\lambda'}$. Now by Morita's lemma ([8, Theorem 5, p. 113]) we conclude that $\boldsymbol{\varphi}$ is an isomorphism. □

[8, Theorem 3, p. 109] partially holds for the case of pro-sets. More precisely, we have

Lemma 4.5. *Let $\boldsymbol{A} = (A_\lambda, a_{\lambda\lambda'}, \Lambda)$ and $\boldsymbol{B} = (B_\lambda, b_{\lambda\lambda'}, \Lambda)$ be pro-sets over the same index set Λ, and let $\boldsymbol{f} : \boldsymbol{A} \to \boldsymbol{B}$ be a morphism of pro-sets given by a level morphism of systems $(f_\lambda) : \boldsymbol{A} \to \boldsymbol{B}$. If the condition*

(EP) *For each $\lambda \in \Lambda$ there is $\lambda' \ge \lambda$ such that* $\operatorname{Im}(q_{\lambda\lambda'}) \subseteq \operatorname{Im}(f_\lambda)$.
holds, then the morphism $\boldsymbol{f}$ is an epimorphism.

Proof. The same proof as for the corresponding part of [8, Theorem 3, p. 109] applies to this case. □

Proof of Theorem C. Let $\boldsymbol{\varphi} = (\varphi_\lambda) : (\boldsymbol{X}; \boldsymbol{X}_0, \boldsymbol{X}_1) \to (\boldsymbol{Y}; \boldsymbol{Y}_0, \boldsymbol{Y}_1)$ be a level morphism representing the shape morphism φ as in the proof of Theorem B. Let $\lambda \in \Lambda$. Then take $\lambda' \ge \lambda$ and a map $f : (Q; Q_0, Q_1) \to (P; P_0, P_1)$ between polyhedral triads as in Lemma 4.3. By Lemma 2.13, $f_* : \pi_q(Q; Q_0, Q_1) \to \pi_q(P; P_0, P_1)$ is an isomorphism for $2 \le q \le n-1$ and an epimorphism for $q = n$. Thus by Morita's lemma $\boldsymbol{\varphi}_* : \pi_q(\boldsymbol{X}; \boldsymbol{X}_0, \boldsymbol{X}_1) \to \pi_q(\boldsymbol{Y}; \boldsymbol{Y}_0, \boldsymbol{Y}_1)$ is an isomorphism for $q \le n-1$. For $q = n$, $\operatorname{Im}\{(q_{\lambda\lambda'})_* : \pi_q(Y_{\lambda'}; Y_{0\lambda'}, Y_{1\lambda'}) \to \pi_q(Y_\lambda; Y_{0\lambda}, Y_{1\lambda})\} \subseteq \operatorname{Im}\{(\varphi_\lambda)_* : \pi_q(X_\lambda; X_{0\lambda}, X_{1\lambda}) \to \pi_q(Y_\lambda; Y_{0\lambda}, Y_{1\lambda})\}$, and so by Lemma 4.5 $\boldsymbol{\varphi}_* : \pi_q(\boldsymbol{X}; \boldsymbol{X}_0, \boldsymbol{X}_1) \to \pi_q(\boldsymbol{Y}; \boldsymbol{Y}_0, \boldsymbol{Y}_1)$ is an epimorphism. This proves the first assertion.

By Lemma 2.12, $f|_Q : Q \to P$ is an n-equivalence. By an argument similar to the above, we see that $(\boldsymbol{\varphi}|_{\boldsymbol{X}})_* : \pi_q(\boldsymbol{X}) \to \pi_q(\boldsymbol{Y})$ is an isomorphism for $q \le n-1$ and an epimorphism for $q = n$. This proves the second assertion. □

As an easy corollary to Theorems B and C, we have

Corollary 4.6. *Suppose that* $\varphi : (X; X_0, X_1) \to (Y; Y_0, Y_1)$ *is a shape morphism whose restrictions* $\varphi|_{X_0 \cap X_1} : X_0 \cap X_1 \to Y_0 \cap Y_1$, $\varphi|_{X_0} : X_0 \to Y_0$ *and* $\varphi|_{X_1} : X_1 \to Y_1$ *are isomorphisms. Then*

1. *if* $\mathrm{Sd}(X; X_0, X_1) < \infty$ *and* $\mathrm{Sd}(Y; Y_0, Y_1) < \infty$, *then* $\varphi : (X; X_0, X_1) \to (Y; Y_0, Y_1)$ *is an isomorphism; and*
2. *if* $\mathrm{Sd}\, X < \infty$ *and* $\mathrm{Sd}\, Y < \infty$, *then the restricted shape morphism* $\varphi|_X : X \to Y$ *is an isomorphism.*

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