GLASNIK MATEMATIČKI
Vol. 36(56)(2001), 287 – 295

NONUNIFORM EXPONENTIAL UNSTABILITY OF EVOLUTION OPERATORS IN BANACH SPACES

Mihail Megan, Adina Luminiţa Sasu and Bogdan Sasu
University of the West, Timişoara, Romania

Abstract. In this paper we consider a nonuniform unstability concept for evolution operators in Banach spaces. The relationship between this concept and the Perron condition is studied. Generalizations to the nonuniform case of some results of Van Minh, Räbiger and Schnaubelt are obtained. The theory we present here is applicable for general time - varying linear equations in Banach spaces.

1. Evolution operators with exponential growth

Let X be a real or complex Banach space. The norm on X and on the Banach algebra $\mathcal{B}(X)$ of all bounded linear operators from X into itself will be denoted by $\|\cdot\|$. We recall that if

$$\Delta = \{(t, t_0) \in \mathbf{R}_+^2 : t \geq t_0\}$$

then an application $\Phi : \Delta \to \mathcal{B}(X)$ is called *evolution operator* on X if it satisfies the following conditions:

$e_1)$ $\Phi(t,t) = I$ (the identity operator on X) for every $t \geq 0$;

$e_2)$ $\Phi(t,s)\Phi(s,t_0) = \Phi(t,t_0)$, for all $(t,s),(s,t_0) \in \Delta$;

$e_3)$ for every $t, t_0 \in \mathbf{R}_+$ and every $x \in X$ the mappings $s \mapsto \Phi(t,s)x$ and $s \mapsto \Phi(s,t_0)x$ are continuous on $[0,t]$ and on $[t_0,\infty)$, respectively.

If $\Phi : \Delta \to \mathcal{B}(X)$ is an evolution operator on X, then we introduce the mapping

---

2000 *Mathematics Subject Classification.* 34D05, 34D20.

Key words and phrases. evolution operator, exponential unstability, admissibility.

$$U : \mathbf{R}_+ \to \mathcal{B}(X), \quad U(t) = \Phi(t, 0).$$

REMARK 1.1. If $U(t)$ is invertible for every $t \in \mathbf{R}_+$, then $\Phi(t, t_0)$ is invertible for all $(t, t_0) \in \Delta$ and

$$\Phi(t, t_0) = U(t)V(t_0), \qquad \text{where } V(t) = U^{-1}(t).$$

In this case, for $0 \leq t < t_0$ we define

$$\Phi(t, t_0) = \Phi^{-1}(t_0, t) = U(t)V(t_0)$$

and the equality from e_2) holds for all $t, t_0 \geq 0$.

DEFINITION 1.1. The evolution operator $\Phi : \Delta \to \mathcal{B}(X)$ is said to be with

i) *nonuniform exponential growth* if there are $M, \omega : \mathbf{R}_+ \to \mathbf{R}_+^* = (0, \infty)$ such that

$$||\Phi(t, t_0)|| \leq M(t_0)e^{\omega(t_0)\,(t-t_0)}, \qquad \text{for all } (t, t_0) \in \Delta;$$

ii) *uniform exponential growth* if there exist $M, \omega > 0$ such that

$$||\Phi(t, t_0)|| \leq Me^{\omega(t-t_0)}, \quad \text{for all } (t, t_0) \in \Delta.$$

REMARK 1.2. The evolution operator $\Phi : \Delta \to \mathcal{B}(X)$ is with nonuniform exponential growth if and only if there are $M, \omega : \mathbf{R}_+ \to \mathbf{R}_+^*$ such that

$$||\Phi(t, t_0)x_0|| \leq M(s)e^{\omega(s)\,(t-s)}||\Phi(s, t_0)x_0||$$

for all $(t, s), (s, t_0) \in \Delta$ and all $x_0 \in X$.

Similar equivalence holds for the case of evolution operators with uniform exponential growth.

REMARK 1.3. If $U(t) = \Phi(t, 0)$ is invertible for all $t \geq 0$ then the following statements are equivalent:

i) Φ is with nonuniform exponential growth;

ii) there exist $M, \omega : \mathbf{R}_+ \to \mathbf{R}_+^*$ such that

$$||\Phi(t_0, t)x_0|| \geq M(t_0)e^{-\omega(t_0)\,(t-t_0)}||x_0||$$

for all $(t, t_0) \in \Delta$ and all $x_0 \in X$;

iii) there are $M, \omega : \mathbf{R}_+ \to \mathbf{R}_+^*$ such that

$$||U(t)x_0|| \leq M(t_0)e^{\omega(t_0)\,(t-t_0)}||U(t_0)x_0||$$

for all $(t, t_0) \in \Delta$ and all $x_0 \in X$.

2. Exponential unstable evolution operators

Let $\Phi : \Delta \to \mathcal{B}(X)$ be an evolution operator on a Banach space X.

Definition 2.1. Φ is said to be

i) *nonuniformly exponentially unstable* if there exist a constant $\nu > 0$ and a function $N : \mathbf{R}_+ \to \mathbf{R}_+^*$ such that

$$N(t)||\Phi(t, t_0)x_0|| \geq e^{\nu(t-t_0)}||x_0||$$

for all $(t, t_0) \in \Delta$ and all $x_0 \in X$;

ii) *uniformly exponentially unstable* if there are $N, \nu > 0$ such that

$$N||\Phi(t, t_0)x_0|| \geq e^{\nu(t-t_0)}||x_0||$$

for all $(t, t_0) \in \Delta$ and all $x_0 \in X$.

Remark 2.1. The evolution operator Φ is nonuniformly exponentially unstable if and only if there are $\nu > 0$ and a function $N : \mathbf{R}_+ \to \mathbf{R}_+^*$ such that

$$N(t)||\Phi(t, t_0)x_0|| \geq e^{\nu(t-s)}||\Phi(s, t_0)x_0||$$

for all $(t, s), (s, t_0) \in \Delta$ and all $x_0 \in X$.

Remark 2.2. If $U(t)$ is surjective for all $t \geq 0$ then the following assertions are equivalent:

i) Φ is nonuniformly exponentially unstable;

ii) for all $(t, t_0) \in \Delta$, $\Phi(t, t_0)$ is invertible and there exist $\nu > 0$ and a function $N : \mathbf{R}_+ \to \mathbf{R}_+^*$ such that

$$||\Phi(t_0, t)x_0|| \leq N(t)e^{-\nu(t-t_0)}||x_0||$$

for all $(t, t_0) \in \Delta$ and all $x_0 \in X$;

iii) there are $\nu > 0$ and a function $N : \mathbf{R}_+ \to \mathbf{R}_+^*$ such that

$$N(t)||U(t)x_0|| \geq e^{\nu(t-t_0)}||U(t_0)x_0||$$

for all $(t, t_0) \in \Delta$ and all $x_0 \in X$.

Similar equivalences hold for the case of uniform exponential unstability.

3. Auxiliary results

We start with the following

Lemma 3.1. *Let $f : \mathbf{R}_+ \to \mathbf{R}_+^*$ be a continuous function such that there are $c > 0$ and $M, \omega : \mathbf{R}_+ \to \mathbf{R}_+^*$ with the properties:*

i) $\int_s^\infty f(t)\,dt \le c f(s)$, *for all* $s \ge 0$;

ii) $f(s) \le M(s) e^{\omega(s)\,(t-s)} f(t)$, *for all* $(t,s) \in \Delta$.
Then

$$f(t) \le c\,\omega(t)\,M(t)\,e^{-\frac{t-s}{c}} f(s), \quad \text{for all } (t,s) \in \Delta.$$

Proof. The function $F : \mathbf{R}_+ \to \mathbf{R}_+^*$ defined by

$$F(s) = \int_s^\infty f(t)\,dt$$

is differentiable with

$$F(s) \le c f(s) = -cF^{'}(s), \quad \text{for all } s \ge 0.$$

This inequality implies

$$F(t) \le F(s) e^{-\frac{t-s}{c}}, \quad \text{for all } (t,s) \in \Delta.$$

We observe that

$$\frac{F(s)}{f(s)} = \int_s^\infty \frac{f(t)}{f(s)}\,dt \ge \int_s^\infty \frac{e^{-\omega(s)\,(t-s)}}{M(s)}\,dt = \frac{1}{\omega(s)\,M(s)}$$

for all $s \ge 0$. Hence

$$f(t) \le \omega(t) M(t) F(t) \le \omega(t) M(t) F(s) e^{-\frac{t-s}{c}} \le c\,\omega(t) M(t) e^{-\frac{t-s}{c}} f(s),$$

which concludes the proof. □

Lemma 3.2. *Let $\Phi : \Delta \to \mathcal{B}(X)$ be an evolution operator with nonuniform exponential growth and with the property that $U(t) = \Phi(t,0)$ is invertible, for all $t \ge 0$.*

If there exists $c > 0$ such that

$$\int_s^\infty \frac{||U(s)x_0||}{||U(t)x_0||}\,dt \le c, \text{ for all } s \ge 0 \text{ and all } x_0 \in X \setminus \{0\}$$

then there is $N : \mathbf{R}_+ \to \mathbf{R}_+^*$ such that

$$N(t)||U(t)x_0|| \ge e^{\frac{t-t_0}{c}} ||U(t_0)x_0||, \text{ for all } (t,t_0) \in \Delta \text{ and all } x_0 \in X.$$

PROOF. We apply Lemma 3.1. to the function

$$f : \mathbf{R}_+ \to \mathbf{R}_+^*, \quad f(t) = \frac{1}{||U(t)x_0||},$$

where $x_0 \in X \setminus \{0\}$.

Indeed, from hypothesis we have that

$$\int_s^\infty f(t)\, dt \le c f(s), \qquad \text{for all } s \ge 0.$$

and by Remark 1.3. it follows

$$f(s) \le M(s)\, e^{\omega(s)\,(t-s)}\, f(t), \qquad \text{for all } (t,s) \in \Delta.$$

Then by Lemma 3.1. we have

$$\frac{1}{||U(t)x_0||} = f(t) \le c\,\omega(t) M(t) e^{-\frac{t-s}{c}} f(s) = c\,\omega(t) M(t) e^{-\frac{t-s}{c}} \frac{1}{||U(s)x_0||},$$

for all $(t,s) \in \Delta$. Setting $N(t) = c\omega(t)M(t)$, for all $t \ge 0$, we obtain the conclusion. ∎

4. Complete admissibility and nonuniform exponential unstability

In what follows we shall denote by C the Banach space of all continuous functions $u : \mathbf{R}_+ \to \mathbf{X}$ with the property that $\lim_{t\to\infty} u(t) = 0$. The norm on C is defined by

$$|||u||| = \sup_{t \ge 0} ||u(t)||, \quad u \in C.$$

Simillary, if $t_0 \ge 0$ then $C(t_0)$ denotes the set of all continuous functions $u : [t_0, \infty) \to \mathbf{X}$ with $\lim_{t\to\infty} u(t) = 0$.

DEFINITION 4.1. We say that the evolution operator $\Phi : \Delta \to \mathcal{B}(X)$ satisfies *the Perron condition* or *the pair* (C, C) *is completely admissible for* Φ if for every $t_0 \ge 0$ and for every $u \in C(t_0)$ there exists an unique $\varphi_u \in C(t_0)$ such that

$$\varphi_u(t) = \Phi(t,s)\varphi_u(s) + \int_s^t \Phi(t,\tau) u(\tau)\, d\tau \tag{1}$$

for all $(t,s), (s,t_0) \in \Delta$.

REMARK 4.1. If the pair (C, C) is completely admissible for Φ then there exists $c > 0$ such that

$$|||\varphi_u||| \leq c|||u|||, \qquad \text{for all } u \in C,$$

where φ_u is given by (1).

To prove this, by using the closed graph theorem, it is sufficient to show that the linear mapping

$$P : C \to C, \qquad P(u) = \varphi_u$$

is closed.

Indeed, if $u_n \to u$ in C and $Pu_n \to v$ in C then $u_n(t) \to u(t)$ and $Pu_n(t) \to v(t)$ uniformly on $\mathbf{R}_+$. By dominated convergence theorem, we have

$$v(t) = \lim_{n\to\infty} Pu_n(t) = \lim_{n\to\infty} \Phi(t,s)\, Pu_n(s) + \lim_{n\to\infty} \int_s^t \Phi(t,\tau)u_n(\tau)\, d\tau =$$

$$= \Phi(t,s)\, v(s) + \int_s^t \lim_{n\to\infty} \Phi(t,\tau)u_n(\tau)\, d\tau = \Phi(t,s)\, v(s) + \int_s^t \Phi(t,\tau)u(\tau)d\tau$$

for all $(t, s) \in \Delta$. This shows that $Pu = v$, so P is closed. □

THEOREM 1. *If the pair* (C, C) *is completely admissible for* Φ *then* $U(t) = \Phi(t, 0)$ *is bijective, for all* $t \geq 0$.

PROOF. *Injectivity.* Suppose the contrary, i.e. there exists $t_0 > 0$ and $x_0 \neq 0$ such that $U(t_0)x_0 = 0$.

By complete admissibility of (C, C) for Φ it follows that for $u_0(t) = 0$ there is an unique function $\varphi \in C$ such that

$$\varphi(t) = \Phi(t,s)\varphi(s), \qquad \text{for all } (t,s) \in \Delta.$$

Because $\varphi_1(t) = 0$ and $\varphi_2(t) = U(t)x_0$ satisfy this equation, it follows that

$$U(t)x_0 = 0. \qquad \text{for all } t \geq 0.$$

This shows that $x_0 = U(0)x_0 = 0$. which is a contradiction.

Surjectivity. Let $t_0 \geq 0$ and $y_0 \in X$. We shall prove that there exists $x_0 \in X$ such that $y_0 = U(t_0)x_0$.

Let $\alpha : \mathbf{R}_+ \to [-1, 1]$ be a continuous function with compact support such that

$$supp\, \alpha \subset (t_0, \infty) \quad \text{and} \quad \int_{t_0}^{\infty} \alpha(s)\, ds = 1,$$

and let $\psi : [t_0, \infty) \to X$ be defined by

$$\psi(t) = \int_t^\infty \alpha(s)ds\ \Phi(t,t_0)y_0.$$

Then

$$u : \mathbf{R}_+ \to X, \quad u(t) = \begin{cases} -\alpha(t)\Phi(t,t_0)y_0\,, & \text{if} \quad t > t_0 \\ 0\,, & \text{if} \quad t \in [0.t_0] \end{cases}$$

belongs to C and using the complete admissibility of (C, C) for Φ, it follows that there exists an unique $\varphi_u \in C$ such that (1) holds.

On the other hand, we observe that if $(t,s),(s,t_0) \in \Delta$ then

$$\Psi(t) = \Phi(t,s)\psi(s) + \int_s^t \Phi(t,\tau)u(\tau)\,d\tau$$

i.e. ψ verifies the relation (1) for $t \geq s \geq t_0$.

By Definition 1. it follows that

$$\psi(t) = \varphi_u(t), \qquad \text{for all } t \geq t_0,$$

which implies that

$$y_0 = \psi(t_0) = \varphi_u(t_0) = \Phi(t_0,0)\varphi_u(0) + \int_0^{t_0} \Phi(t_0,\tau)u(\tau)\,d\tau = U(t_0)x_0$$

where $x_0 = \varphi_u(0)$. ∎

Theorem 4.2. *Let $\Phi : \Delta \to \mathcal{B}(X)$ be an evolution operator with nonuniform exponential growth. If the pair (C, C) is completely admissible for Φ, then Φ is nonuniformly exponentially unstable.*

Proof. For $n \in \mathbf{N}^*$ we consider a continuous function $\alpha_n : \mathbf{R}_+ \to [0.1]$ such that

$$\alpha_n(t) = \begin{cases} 1, & \text{if} \quad t \leq n \\ 0. & \text{if} \quad t > n+1 \end{cases}.$$

If the pair (C, C) is completely admissible for Φ then by the previous theorem we have that

$$U(t)x_0 \neq 0. \qquad \text{for all } t \geq 0 \quad \text{and all } x_0 \neq 0.$$

Let $x_0 \in X \setminus \{0\}$. For every $n \in \mathbf{N}^*$ we define:

$$u_n : \mathbf{R}_+ \to X, \quad u_n(t) = -\frac{\alpha_n(t)U(t)x_0}{||U(t)x_0||}.$$

If we denote by

$$\psi_n(t) = \int_t^\infty \frac{\alpha_n(\tau)U(t)x_0}{||U(\tau)x_0||}\, d\tau$$

then we observe that for every $(t,s) \in \Delta$

$$\Phi(t,s)\psi_n(s) + \int_s^t \Phi(t,\tau)u_n(\tau)\, d\tau = \int_s^\infty \frac{\alpha_n(\tau)U(t)x_0}{||U(\tau)x_0||}\, d\tau -$$

$$-\int_s^t \frac{\Phi(t,\tau)\alpha_n(\tau)U(\tau)x_0}{||U(\tau)x_0||}\, d\tau = \int_t^\infty \frac{\alpha_n(\tau)U(t)x_0}{||U(\tau)x_0||}\, d\tau = \psi_n(t),$$

i.e. ψ_n is a solution of (1) for $u = u_n$.

By the uniqueness of ψ_n and by Remark 1. it follows that there exists $c > 0$ such that

$$|||\psi_n||| \le c\, |||u_n|||.$$

Using the fact that $|||u_n||| = 1$ we obtain that

$$\int_t^\infty \frac{\alpha_n(\tau)||U(t)x_0||}{||U(\tau)x_0||} d\tau \le c, \qquad \text{for all } t \ge 0 \text{ and all } n \in \mathbf{N}^*.$$

For $n \to \infty$ we have :

$$\int_t^\infty \frac{||U(t)x_0||}{||U(\tau)x_0||}\, d\tau \le c, \qquad \text{for all } t \ge 0.$$

By Lemma 3.2. and Remark 2.2. it results that Φ is nonuniformly exponentially unstable. ☐

Remark 4.2. In [6] N. van Minh, F. Räbiger and R. Schnaubelt have proved that for an evolution operator Φ with uniform exponential growth the pair (C, C) is completely admissible for Φ if and only if Φ is uniformly exponentially unstable and $\Phi(t, t_0)$ is invertible, for all $(t, t_0) \in \Delta$. For the case of evolution operators with nonuniform exponential growth this equivalence is not valid, as the following example shows.

Example 4.1. Let $X = \mathbf{R}, \gamma(t) = e^{t-4tsint}$ and

$$\Phi(t,s)x = \frac{\gamma(t)}{\gamma(s)}\, x$$

for all $(t,s) \in \Delta$ and all $x \in \mathbf{R}$. Then Φ is an evolution operator on $\mathbf{R}$ with

$$|\Phi(t,s)x| = \; e^{t-4t\sin t}\, e^{4\sin s - s}|x| \ge e^{-3t}\, e^{-5s}|x|.$$

It follows that Φ is nonuniformly exponentially unstable with $N(t) = e^{8t}$ and $\nu = 5$. We prove that the pair (C, C) is not completely admissible for Φ. For $u(t) = e^{-t}$ suppose that there exists $\varphi_u \in C$ such that (1) holds. Then:

$$\varphi_u(t) = \gamma(t)\,(\varphi_u(0) + \int_0^t e^{4ssins-2s} ds), \qquad \text{for all } t \geq 0.$$

For $t_n = 2n\pi + \frac{3\pi}{2}$ we have that $\gamma(t_n) = e^{5t_n}$ and:

$$\int_0^{t_n} e^{4sins-2s} ds > \int_{2n\pi+\frac{\pi}{4}}^{2n\pi+\frac{\pi}{2}} e^{4ssins-2s} ds > \frac{\pi}{4}\, e^{n\pi}.$$

It follows that

$$\varphi_u(t_n) > e^{10n\pi}\,(\varphi_u(0) + \frac{\pi}{4}\, e^{n\pi}), \qquad \text{for all } n \in \mathbf{N}^*$$

which contradicts the assumption that $\varphi_u \in C$. □

Acknowledgements.
The authors would like to thank the referee for helpful suggestions.

References

[1] Y. L. Dalecki, M. G. Krein - Stability of Differential Equations in Banach spaces, Amer. Math. Soc. Providence, R.I., 1974

[2] D. Hinrichsen, A. J. Pritchard - Robust stability of linear evolution operators on Banach spaces, SIAM J. Control and Optimizations 32 (1994), 1503-1541

[3] B. Jacob - Time-Varying Infinite Dimensional State-Space Systems, PhD thesis, Bremen

[4] J. L. Massera, J. J. Schäfer - Linear Differential Equations and Function Spaces, Academic Press, New York, 1966

[5] M. Megan - On (h, k) - dichotomy of evolution operators in Banach spaces, Dynamic Systems and Applications, vol.5, nr.2 (1996), 189-196

[6] M. Megan, D. R. Laţcu - On uniform exponential N-dichotomy, Ann. Math. Blaise Pascal, vol 1, no. 2 (1994), 33 - 41

[7] N. V. Minh, F. Räbiger, R. Schnaubelt - Exponential stability, exponential expansiveness and exponential dichotomy of evolution equations on the half - line, Integral Equations Operator Theory 32 (1998), 332-353

[8] P. Preda, M. Megan - Nonuniform dichotomy of evolutionary processes in Banach spaces, Bull. Austral. Math. Soc., vol. 27 (1983), 31-52

Department of Mathematics
University of the West
Timişoara
Bul. V.Pârvan 4
1900 - Timişoara
România
E-mail: megan@hilbert.math.uvt.ro
E-mail: sasu@hilbert.math.uvt.ro,lbsasu@yahoo.com