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DUALITY BETWEEN STABLE STRONG SHAPE MORPHISMS AND STABLE HOMOTOPY CLASSES

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Abstract. Let SStrSh_n be the full subcategory of the stable strong shape category SStrSh of pointed compacta [H-N] whose objects are all pointed subcompacta of S^n and let SO_n be the full subcategory of the stable homotopy category S ([S-W] or [S]) whose objects are all open subsets of S^n. In this paper it is shown that there exists a contravariant additive functor $D_n : \mathrm{SStrSh}_n \to SO_n$ such that $D_n(X) = S^n \setminus X$ for every subcompactum X of S^n and $D_n : \mathrm{SStrSh}_n(X, Y) \to SO_n(S^n \setminus Y, S^n \setminus X)$ is an isomorphism of abelian groups for all compacta $X, Y \subset S^n$. Moreover, if $X \subset Y \subset S^n, j : S^n \setminus Y \to S^n \setminus X$ is an inclusion and $\boldsymbol{\alpha} \in \mathrm{SStrSh}_n(X, Y)$ is induced by the inclusion of X into Y then $D_n(\boldsymbol{\alpha}) = \{j\}$.

Introduction. Basic definitions.

In [N] it has been proved that the stable shape category of subcompacta of S^n is isomorphic to the stable weak homotopy category of their complements. This theorem generalizes the Spanier-Whitehead Duality Theorem and corresponds to the Chapman Complement Theorem [S_1].

In [H-N] the authors have constructed a stable strong shape category of pointed metric compacta (see also [M]).

The purpose of the present note is to prove that there exists a contravariant functor from the stable strong shape category of pointed subcompacta of

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S^n to the stable homotopy category of their complements which induces an isomorphism between the sets of morphisms.

The Hilbert space l^2 consists of all real sequences $(x_1, x_2, ...)$ with $\sum x_i^2 < \infty$ and $\mathbb{R}^n$ consists of all points $(x_1, x_2, ...)$ of l^2 such that $x_k = 0$ for $k > n$. It follows that $\mathbb{R}^n \subset \mathbb{R}^m$ for $n \leq m$. The point $(x_1, x_2, ..., x_n, 0, ...)$ is denoted by $(x_1, x_2, ..., x_n)$.

The n-sphere S^n consists of all points $(x_1, x_2, ..., x_{n+1})$ of $\mathbb{R}^{n+1}$ with $x_1^2 + x_2^2 + ... + x_{n+1}^2 = 1$. It follows that $S^0 \subset S^1 \subset ...$ and S^n is embedded as an equator in S^{n+1} (S^0 consists of two points -1 and 1).

By the (unreduced) suspension ΣX of a subset X of S^n we understand the union od all segments joining points of X with the poles $v = (0, 0, ..., 1)$ and $v' = (0, 0, ..., -1)$ of S^{n+1} and $\{v, v'\}$ ($\Sigma\emptyset = \{v, v'\}$). If (X, x_0) is a pointed subcompactum of S^n, then we will consider that $(\Sigma X, x_0)$ is a pointed spaces with the base point $x_0 \in X$. The k-fold suspension $\Sigma^k X$, and the suspensions $\Sigma f, \Sigma^k f$ of a map are defined as usual.

If $(z, t) \in Z \times I$, we denote by $[z, t]$ the corresponding point of the reduced suspension $\mathbf{S}Z$ under the quotient map $q_Z : Z \times I \to \mathbf{S}Z$. Then $[z, 0] = [z_0, t] = [z', 1]$ for $z, z' \in Z$ and $t \in I$, where z_0 is a base point of Z. The point $[z_0, 0] \in \mathbf{S}Z$ is also denoted by z_0. If z_0 is a base point of Z, then $\mathbf{S}Z$ is a pointed space with base point z_0. If $f : Z \to Z'$, the map $\mathbf{S}f : \mathbf{S}Z \to \mathbf{S}Z'$ is defined by $\mathbf{S}f([z, t]) = [f(x), t]$.

It is known that $\mathbf{S}X$ is a cogroup object in the homotopy category H. Thus, $H((\mathbf{S}Z, z_0), (Z', z_0'))$ is always a group, which is abelian when $Z = \mathbf{S}W$.

The reduced suspension $(\mathbf{S}X, x_0)$ is obtained from the unreduced suspension $(\Sigma X, x_0)$ by shrinking to a point (which is taken as a base point) the two segments $\overline{v, x_0}$ and $\overline{v', x_0}$. The quotient map $p_X : (\Sigma X, x_0) \to (\mathbf{S}X, x_0)$ is a homotopy equivalence if $X \in ANR(\mathfrak{M})$. It induces an addition in $H((\Sigma X, x_0), (\Sigma X', x_0'))$, which makes this set a group.

If $X, X' \in ANR(\mathfrak{M})$ are simply connected, then the forgetful functor obtained by suppressing base points induces an isomorphism of the set of all pointed homotopy classes $H((X, x_0), (X', x_0'))$ onto the set $H(X, X')$ of all free homotopy classes. We shall identify $H(\Sigma^k X, \Sigma^k X')$ with $H((\Sigma^k X, x_0), (\Sigma^k X', x_0'))$ for $k \geq 2$.

The operation Σ induces a function $\Sigma : H(\Sigma^k X, \Sigma^k X') \to H(\Sigma^{k+1} X, \Sigma^{k+1} X')$ between sets of homotopy classes which is a homomorphism.

The stable homotopy category S was introduced by Spanier and Whitehead (see [S-W] or [S]). We will consider the full subcategory SO_n of S whose objects are open subsets of S^n (the complements of compact subsets of S^n). The set of morphisms $SO_n(U, V) = \{U, V\}$ equals to the direct limit of the sequence

$$H(U, V) \xrightarrow{\Sigma} H(\Sigma U, \Sigma V) \xrightarrow{\Sigma} H(\Sigma^2 U, \Sigma^2 V) \xrightarrow{\Sigma} ....$$

If $f : \Sigma^k U \to \Sigma^k V$ is a map, then $\{f\} = \alpha$ will denote the corresponding element of $\{U, V\}$.

Since the inclusion of $S^n \setminus X$ into $S^{n+k} \setminus \Sigma^k X$ is a homotopy equivalence for $n, k \geq 1$ and $X \subset S^n$, we have a canonical bijection $\{S^{n+k} \setminus \Sigma^k X, S^{n+k} \setminus \Sigma^k Y\} \leftrightarrow \{S^n \setminus X, S^n \setminus Y\}$.

The space $(\mathbf{S}Z, z_0)$ is a cogroup object in StrSh and thus $\mathrm{StrSh}((\mathbf{S}Z, z_0), (Z', z_0'))$ is a group, which is abelian when $Z = \mathbf{S}W$ (compare [H-N]).

By [D-S] (see Theorem 7.10 and Corollary 4.6) the quotient map $p_X : (\Sigma X, x_0) \to (\mathbf{S}X, x_0)$ induces a strong shape equivalence for every compactum $X \subset S^n$. The map p_X canonically induces on $(\Sigma X, x_0)$ the structure of a cogroup object in StrSh.

In [H-N] the authors defined the stable strong shape category SStrSh of pointed compacta.

We shall consider the full subcategory SStrSh_n of SStrSh, whose objects are pointed subcompacta of S^n. The set $\mathrm{SStrSh}_n((X, x_0), (Y, y_0))$ is the direct limit of the sequence

$$\mathrm{StrSh}((X, x_0), (Y, y_0)) \xrightarrow{\Sigma} \mathrm{StrSh}((\Sigma X, x_0), (\Sigma Y, y_0)) \xrightarrow{\Sigma} \ldots$$

$\mathrm{SStrSh}_n(X, Y)$ is an abelian group. Hereafter we shall omit references to base points.

The strong (stable) shape morphism represented by a map f is denoted by the bold letter $\boldsymbol{f}$.

1. The main theorem and schedule of its proof.

Theorem 1.1. *There exists a contravariant additive functor $D_n : \mathrm{SStrSh}_n \to SO_n$ such that*

$$D_n(X) = S^n \setminus X$$

for every compactum X of S^n and

$$D_n : \mathrm{SStrSh}_n(X, Y) \to SO_n(S^n \setminus Y, S^n \setminus X)$$

is an isomorphism of the abelian group $\mathrm{SStrSh}_n(X, Y)$ onto the abelian group $SO_n(S^n \setminus Y, S^n \setminus X)$ for all compacta $X, Y \subset S^n$. If $X \subset Y \subset S^n$, $j : S^n \setminus Y \to S^n \setminus X$ is an inclusion and $\boldsymbol{\alpha} \in \mathrm{SStrSh}_n(X, Y)$ is induced by the inclusion of X into Y, then

$$D_n(\boldsymbol{\alpha}) = \{j\}.$$

The notion of mapping cylinder of a strong shape morphism ([D-S], p.24) provides the starting place for the proof of Theorem 1.1.

Suppose that X and Y are pointed compacta. We say that a compactum $M \supset X \cup Y$ is a mapping cylinder of $\boldsymbol{f} \in \mathrm{StrSh}(X, Y)$ iff the inclusion $j : Y \to M$ induces a strong shape equivalence and $\boldsymbol{j}\boldsymbol{f} = \boldsymbol{i}$ as unpointed strong shape morphisms, where $i : X \to M$ denotes the inclusion of X into M.

We have (see [N]) the following.

Lemma 1.2. *Suppose that $A \subset B$ are subcompacta of S^n and the inclusion of A into B induces a shape equivalence. Then the inclusion of $S^n \setminus B$ into $S^n \setminus A$ is a stable homotopy equivalence.*

We need the following.

Lemma 1.3. *Suppose that $A \subset B$ are compacta and that $h : A \to S^n$ is an embedding. Then for sufficiently large $m > n$ there exists an embedding $\widetilde{h} : B \to S^m$ such that $\widetilde{h}(x) = h(x)$ for $x \in A$.*

J. Dydak and J. Segal have proved ([D-S], p.23) that if $X \cap Y = \emptyset$ then for every $\boldsymbol{f} \in \mathrm{StrSh}(X, Y)$ there exists a mapping cylinder M of $\boldsymbol{f}$ with $\dim M \leq \dim X + 1, \dim Y$.

Hence we may assume that $M \subset S^m$ for sufficiently large $m > n$ in the case when $X, Y \subset S^n$ (see Lemma 1.3). Moreover, it follows from Lemma 1.2 that the inclusion $j_0 : S^m \setminus M \to S^m \setminus Y$ is a homotopy equivalence for almost all m.

Since the inclusion of $\Sigma^k(S^n \setminus X)$ into $S^{n+k} \setminus X$ and the inclusion of $\Sigma^k(S^n \setminus Y)$ into $S^{n+k} \setminus Y$ are homotopy equivalences for $k = 1, 2, ...$, we infer that every map of $S^n \setminus Y$ into $S^n \setminus X$ determines uniquely an element of $\{S^n \setminus Y, S^n \setminus X\}$. We will use this convention in the whole paper.

Suppose that $X \cap Y = \emptyset$ and $\boldsymbol{f} \in \mathrm{StrSh}(X, Y)$, where X, Y are subcompacta of S^n. Let $r : S^m \setminus Y \to S^m \setminus M$ denote the homotopy inverse of the inclusion $j_0 : S^m \setminus M \to S^m \setminus Y$, where $M \subset S^m$ is a mapping cylinder of $\boldsymbol{f}$.

It will be proved that the stable homotopy class $\Delta_n(\boldsymbol{f}) : S^n \setminus Y \to S^n \setminus X$ determined by the composition $i_0 r : S^m \setminus Y \to S^m \setminus X$ does not depend on the choice of M, where $i_0 : S^m \setminus M \to S^m \setminus X$ denotes the inclusion of $S^m \setminus M$ into $S^m \setminus X$.

The next step is to define $\Delta_n(\boldsymbol{f})$ for every $\boldsymbol{f} \in \mathrm{StrSh}(X, Y)$.

Suppose that $\boldsymbol{f} \in \mathrm{StrSh}(X, Y)$, where X and Y are arbitrary subcompacta of S^n. We can find a homeomorphism $h : X \to X_1 \subset S^{n+1}$ such that $X_1 \cap (X \cup Y) = \emptyset$. We shall prove that $\Delta_{n+1}(\boldsymbol{h})\Delta_{n+1}(\boldsymbol{f}\boldsymbol{h}^{-1})$ does not depend on the choice of h. Let α denote the element of $\{S^n \setminus Y, S^n \setminus X\}$ determined by $\Delta_{n+1}(\boldsymbol{h})\Delta_{n+1}(\boldsymbol{f}\boldsymbol{h}^{-1})$.

Setting

$$\Delta_n(\boldsymbol{f}) = \alpha$$

one can define $\Delta_n(\boldsymbol{f})$ for every $\boldsymbol{f} \in \mathrm{StrSh}(X, Y)$ and extend Δ_n to a function $\Delta_n : \mathrm{StrSh}(X, Y) \to SO_n(S^n \setminus Y, S^n \setminus X)$.

Suppose that $\boldsymbol{\alpha} \in \mathrm{SStrSh}_n(X, Y)$ is represented by $\boldsymbol{f} \in \mathrm{SStrSh}(\Sigma^k(X), \Sigma^k(Y))$ and that $\beta \in \{S^n \setminus Y, S^n \setminus X\}$ corresponds to

$\Delta_{n+k}(\boldsymbol{f}) \in \{S^{n+k} \setminus \Sigma^k(Y), S^{n+k} \setminus \Sigma^k(X)\}$ under the canonical isomorphism of $\{S^n \setminus Y, S^n \setminus X\}$ onto $\{S^{n+k} \setminus \Sigma^k(Y), S^{n+k} \setminus \Sigma^k(X)\}$.

We define $D_n : \mathrm{SStrSh}_n(X,Y) \to SO_n(S^n \setminus Y, S^n \setminus X)$ by the formula

$$D_n(\boldsymbol{\alpha}) = \beta.$$

The last sections are devoted to the proof that $D_n : \mathrm{SStrSh}_n(X,Y) \to \{S^n \setminus Y, S^n \setminus X\}$ is an isomorphism.

2. Auxiliary facts on mapping cylinders.

First we present the Dydak-Segal construction of mapping cylinder (see [D-S]). Suppose that $\boldsymbol{f} \in \mathrm{StrSh}(X,Y)$ is represented by a proper map $f : \mathrm{Tel}\,\boldsymbol{X} \to \mathrm{Tel}\,\boldsymbol{Y}$, where $\boldsymbol{X}$ and $\boldsymbol{Y}$ are nets (see [H-N]) associated with X and Y such that $X_0 = Y_0 = A \in AR$.

It is clear that $f(x,t)$ can be written in the form

$$f(x,t) = (f'(x,t), f''(x,t))$$

where $f' : \mathrm{Tel}\,\boldsymbol{X} \to Y_0 = A$ is a map and $f'' : \mathrm{Tel}\,\boldsymbol{X} \to [0,\infty)$ is a proper map.

Let $A' = A \times [0,\infty) \cup \{w\}$ be a one-point compactification of $A \times [0,\infty)$ and $M(\boldsymbol{f})$ be defined by the formula

$$M(\boldsymbol{f}) = \{((x,t), f'(x,t)) \in A' \times A : (x,t) \in X \times [0,\infty)\} \cup \{w\} \times Y.$$

We will identify (respectively) X and Y with the set consisting of all points $((x,0), f'(x,0)) \in M(\boldsymbol{f})$ such that $x \in X$ and with the set consisting of all points $(w,y) \in M(\boldsymbol{f})$ such that $y \in Y$.

In [D-S] it has been proved that $M(\boldsymbol{f})$ is a mapping cylinder of $\boldsymbol{f}$.

PROPOSITION 2.1. *Suppose that X, Y, M' and M'' are compacta lying in S^n, $\boldsymbol{f} \in \mathrm{StrSh}(X,Y)$, $M' \cap M'' = X \cup Y$ and M' and M'' are mapping cylinders of $\boldsymbol{f}$. Then for sufficiently large m there exists a mapping cylinder $S^m \supset M \supset M' \cup M''$ of $\boldsymbol{f}$.*

PROOF. Let $\boldsymbol{M}', \boldsymbol{M}'', \boldsymbol{X}$ and $\boldsymbol{Y}$ denote nets associated respectively with M', M'', X and Y such that

$$M'_0 = M''_0 = X_0 = Y_0 \text{ and } M'_n \cap M''_n \supset X_n \cup Y_n \text{ for } n = 1, 2, \ldots.$$

We can find a proper map $f : \mathrm{Tel}\,\boldsymbol{X} \to \mathrm{Tel}\,\boldsymbol{Y}$ which represents $\boldsymbol{f}$.

Consider also proper maps $r_1 : \mathrm{Tel}\,\boldsymbol{M}' \to \mathrm{Tel}\,\boldsymbol{Y}$ and $r_2 : \mathrm{Tel}\,\boldsymbol{M}'' \to \mathrm{Tel}\,\boldsymbol{Y}$ which represent (respectively) the inverses of $\boldsymbol{j}_1$ and $\boldsymbol{j}_2$, where j_1 and j_2 denote the inclusions of Y into M' and M''.

By the twofold application of the Proper Homotopy Extension Theorem ([B-S] and [D-S]) we obtain that one can assume that

$$r_1(x,t) = (x,t) = r_2(x,t) \text{ for } (x,t) \in Y \times [0,\infty)$$

and that there exists a proper map $g : \operatorname{Tel} \boldsymbol{M}' \to \operatorname{Tel} \boldsymbol{M}''$ such that

$$g(x,t) = (x,t) \text{ for } (x,t) \in (X \cup Y) \times [0,\infty).$$

For every $x \in (X \cup Y)$ we denote by $L_x \subset M(g)$ the closure of the set consisting of all points $((x,t),x) \in M(g)$ with $x \in X \cup Y$. Then L_x is an arc for every $x \in X \cup Y$.

Consider the decomposition space M of $M(g)$ of the upper semicontinuos decomposition of $M(g)$ into individual points $M(g) \setminus \bigcup_{x \in (X \cup Y)} L_x$ and the arcs L_x for $x \in (X \cup Y)$.

Then the quotient map of $M(g)$ into M is the strong shape equivalence ([D-S], p.32).

Identifying M' and M'' with suitable subsets of M one can easily check that M is a mapping cylinder of $\boldsymbol{f}$ and $M \supset M' \cup M''$.

Since $\dim M < \infty$ we may assume that $M \subset S^m$ for sufficiently large m. $\square$

Proposition 2.2. *Suppose that X, Y, Z are subcompacta of S^n, $\boldsymbol{f} \in \operatorname{StrSh}(x, Y), \boldsymbol{g} \in \operatorname{StrSh}(Y, Z)$ and $X \cap Y = Y \cap Z = X \cap Z = \emptyset$. Then for sufficiently large m there are compacta $N \supset X \cup Y$ and $S^m \supset M \supset N \cup Z$ such that*

(a) *N is a cylinder of $\boldsymbol{f}$*
(b) *M is a cylinder of $\boldsymbol{gf}$*
(c) *M is a cylinder of $\boldsymbol{g}$*

Proof. For sufficiently large m there are subcompacta N and V of S^m such that $N \cap V = Y$ and

$$V \text{ is a mapping cylinder of } \boldsymbol{f}$$

and

$$N \text{ is a mapping cylinder of } \boldsymbol{g}.$$

The compacta $M = N \cup V$ and N satisfy the required conditions. $\square$

Lemma 2.3. *If M is a mapping cylinder of $\boldsymbol{f} \in \operatorname{StrSh}(X,Y)$, then ΣM is a mapping cylinder of $\Sigma \boldsymbol{f} \in \operatorname{StrSh}(\Sigma X, \Sigma Y)$.*

3. The function Δ_n and its properties.

Lemma 3.1. *$\Delta_n(\boldsymbol{f})$ does not depend on the choice M, if $X \cap Y = \emptyset$.*

Proof. Let $M_1, M_2 \subset S^m$ be mapping cylinders of $\boldsymbol{f} \in \operatorname{StrSh}(X,Y)$ such that the inclusion $j_{m,k} : S^m \setminus M_k \to S^m \setminus Y$ is a stable homotopy equivalence for $k = 1,2$ and let $\alpha_k \in \{S^m \setminus Y, S^m \setminus M_k\}$ denote the inverse of $\{j_{m,k}\}$ for $k = 1,2$.

Since for every pair of compacta $A \subset B, B \subset S^m$ there exists a compactum $B' \subset S^{n+1}$ and a homeomorphism $h : B \to B'$ such that $h(x) = x$ for $x \in A$ and $B' \cap S^m = A$, we may assume that $M_1 \cap M_2 = X \cup Y$.

¿From Proposition 2.1 it follows that there exists a mapping cylinder $S^m \supset P \supset M_1 \cup M_2$ of $\boldsymbol{f}$. Let $\beta \in \{S^m \setminus Y, S^m \setminus P\}$ denote the inverse of stable homotopy class of the inclusion of $S^m \setminus P$ into $S^m \setminus Y$. Let $\alpha'_k \in \{S^m \setminus M_k, S^m \setminus P\}$ denote the inverse of the stable homotopy class of the inclusion of $S^m \setminus P$ into $S^m \setminus M_k$ for $k = 1, 2$.

We have $\beta = \alpha'_k \alpha_k$ and $\gamma\beta = \gamma\alpha'_k\alpha_k = \gamma_k\gamma'_k\alpha'_k\alpha_k = \gamma_k\alpha_k$, where $\gamma, \gamma_k, \gamma'_k$ denote respectively the stable homotopy classes of the inclusions of $S^m \setminus P$ into $S^m \setminus X$, of $S^m \setminus P$ into $S^m \setminus M_k$ and of $S^m \setminus M_k$ into $S^m \setminus X$. ☐

Lemma 3.2. *Suppose that $\boldsymbol{f} \in \mathrm{StrSh}(X, Y)$ and $\boldsymbol{g} \in \mathrm{StrSh}(Y, Z)$, where $X, Y, Z \subset S^n$ are compacta and $X \cap Y = Y \cap Z = X \cap Z = \emptyset$. Then*

$$\Delta_n(\boldsymbol{gf}) = \Delta_n(\boldsymbol{f})\Delta_n(\boldsymbol{g}).$$

Proof. Let N and $M \supset N \cup Z$ be respectively mapping cylinders of $\boldsymbol{f}$ and $\boldsymbol{g}$ such that M is also a mapping cylinder of $\boldsymbol{gf}$ (see Proposition 2.2).

Let $i_1 : S^m \setminus M \to S^m \setminus Y$, $i_2 : S^m \setminus M \to S^m \setminus N$, $i_3 : S^m \setminus N \to S^m \setminus Y$, $i_4 : S^m \setminus N \to S^m \setminus X$, $i_5 : S^m \setminus M \to S^m \setminus X$ and $i_6 : S^m \setminus M \to S^m \setminus Z$ be inclusions.

We have $\Delta_n(\boldsymbol{gf}) = \{i_5\}\{i_6\}^{-1}$, $\{i_5\} = \{i_4\}\{i_2\} = \{i_4 i_2\}$ and $\{i_2\} = \{i_3\}^{-1}\{i_1\}$. Hence

$$\Delta_n(\boldsymbol{gf}) = \{i_4\}\{i_2\}\{i_6\}^{-1} = \{i_4\}\{i_3\}^{-1}\{i_1\}\{i_6\}^{-1} = \Delta_n(\boldsymbol{f})\Delta_n(\boldsymbol{g}).$$

☐

Lemma 3.3. *Suppose that $h : X \to Y$ is a homeomorphism and $X, Y \subset S^n$ are compacta such that $X \cap Y = \emptyset$. Then $\Delta_n(\boldsymbol{h})$ is a stable homotopy equivalence and $[\Delta_n(\boldsymbol{h})]^{-1} = \Delta_n(\boldsymbol{h}^{-1})$.*

Proof. There exists an embedding $\widetilde{h} : X \times I \to S^m$ such that $\widetilde{h}(x, 0) = x$ and $\widetilde{h}(x, 1) = h(x)$ for every $x \in X$. Using the fact that $M = \widetilde{h}(X \times I)$ is a mapping cylinder of $\boldsymbol{h}$ and $\boldsymbol{h}^{-1}$ one can easily prove that Lemma 3.3 holds true. ☐

Lemma 3.4. *Suppose that $h_1 : X \to X_1$ and $h_2 : X \to X_2$ are homeomorphisms such that $X_k \cap (X \cup Y) = \emptyset$ for $k = 1, 2$, where $X, Y \subset S^{n-1}$ and $X_1, X_2 \subset S^n$ are compacta. Then*

$$\Delta_n(\boldsymbol{h}_1)\Delta_n(\boldsymbol{f}\boldsymbol{h}_1^{-1}) = \Delta_n(\boldsymbol{h}_2)\Delta_n(\boldsymbol{f}\boldsymbol{h}_2^{-1})$$

for every $\boldsymbol{f} \in \mathrm{StrSh}(X, Y)$.

Proof. Without loss of generality we may assume that $X_1 \cap X_2 = \emptyset$. Using Lemmas 3.2 and 3.3 we obtain that

$$\Delta_n(\boldsymbol{h}_2)\Delta_n(\boldsymbol{f}\boldsymbol{h}_2^{-1}) = \Delta_n(\boldsymbol{h}_1)\Delta_n(\boldsymbol{h}_1^{-1})\Delta_n(\boldsymbol{h}_2)\Delta_n(\boldsymbol{f}\boldsymbol{h}_2^{-1}) = \Delta_n(\boldsymbol{h}_1)\Delta_n(\boldsymbol{h}_2\boldsymbol{h}_1^{-1})\Delta_n(\boldsymbol{f}\boldsymbol{h}_2^{-1}) = \Delta_n(\boldsymbol{h}_1)\Delta_n(\boldsymbol{f}\boldsymbol{h}_1^{-1}).$$

□

Lemma 3.4 allows us to define $\Delta_n(\boldsymbol{f})$ for every $\boldsymbol{f} \in \mathrm{StrSh}(X, Y)$, where X and Y are arbitrary subcompacta of S^n (see the Section 1).

Lemma 3.5. *Suppose that $\boldsymbol{f} \in \mathrm{StrSh}(X, Y)$ and $\boldsymbol{g} \in \mathrm{StrSh}(Y, Z)$, where X, Y, Z are subcompacta of S^n. Then $\Delta_n(\boldsymbol{g}\boldsymbol{f}) = \Delta_n(\boldsymbol{f})\Delta_n(\boldsymbol{g})$.*

Proof. Without loss of generality we may assume that there are homeomorphisms $h : X \to X_1 \subset S^{n+1}$ and $k : Y \to Y_1 \subset S^{n+1}$ such that $(X \cup Y \cup Z) \cap X_1 = \emptyset = (X \cup Y \cup Z) \cap Y_1$ and $X_1 \cap Y_1 = \emptyset$.

Then it follows from Lemmas 3.2 and 3.3 that

$$\begin{aligned}\Delta_n(\boldsymbol{h})\Delta_n(\boldsymbol{g}\boldsymbol{f}\boldsymbol{h}^{-1}) &= \Delta_n(\boldsymbol{h})\Delta_n(\boldsymbol{g}\boldsymbol{k}^{-1}\boldsymbol{k}\boldsymbol{f}\boldsymbol{h}^{-1})\\ &= \Delta_n(\boldsymbol{h})\Delta_n(\boldsymbol{k}\boldsymbol{f}\boldsymbol{h}^{-1})\Delta_n(\boldsymbol{g}\boldsymbol{k}^{-1}) = \Delta_n(\boldsymbol{h})\Delta_n(\boldsymbol{f}\boldsymbol{h}^{-1})\Delta_n(\boldsymbol{k})\Delta_n(\boldsymbol{g}\boldsymbol{k}^{-1}).\end{aligned}$$

□

Theorem 3.6. *For every of compacta $X, Y \subset S^n$ there exists a function $\Delta_n : \mathrm{StrSh}(X, Y) \to \{S^n \setminus Y, S^n \setminus X\}$ satisfying the following conditions:*

(a) *if $X \subset Y$ then $\Delta_n(\boldsymbol{i}) = j$, where $i : X \to Y$ and $j : S^n \setminus Y \to S^n \setminus X$ are inclusion*

(b) *$\Delta_n(\boldsymbol{g}\boldsymbol{f}) = \Delta_n(\boldsymbol{f})\Delta_n(\boldsymbol{g})$ for all $\boldsymbol{f} \in \mathrm{StrSh}(X, Y)$ and $\boldsymbol{g} \in \mathrm{StrSh}(Y, Z)$*

(c) *for every $\boldsymbol{f} \in \mathrm{StrSh}(X, Y)$ the stable homotopy class $\Delta_n(\boldsymbol{f}) \in \{S^n \setminus Y, S^n \setminus X\}$ corresponds to the stable homotopy classes $\Delta_{n+1}(\Sigma\boldsymbol{f}) \in \{S^{n+1} \setminus \Sigma(Y), S^{n+1} \setminus \Sigma(X)\}$ and $\Delta_{n+1}(\boldsymbol{f}) \in \{S^{n+1} \setminus Y, S^{n+1} \setminus X\}$ under the canonical one-to-one correspondences $\{S^n \setminus Y, S^n \setminus X\} \leftrightarrow \{S^{n+1} \setminus \Sigma(Y), S^{n+1} \setminus \Sigma(X)\}$ and $\{S^n \setminus Y, S^n \setminus X\} \leftrightarrow \{S^{n+1} \setminus Y, S^{n+1} \setminus X\}$.*

Proof. The condition (c) is a consequence of Lemma 2.3. □

Corollary 3.7. *There exists a contravariant functor $D_n : \mathrm{SStrSh}_n \to SO_n$ such that*

(a) *$D_n(X) = S^n \setminus X$ for every $X \subset S^n$;*

(b) *if $X \subset Y$ then $D_n(\boldsymbol{\alpha}) = \{j\}$, where $\boldsymbol{\alpha} \in \mathrm{StrSh}(X, Y)$ is induced by the inclusion and $j : S^n \setminus Y \to S^n \setminus X$ is the inclusion;*

(c) *for every $\boldsymbol{\alpha} \in \mathrm{SStrSh}(X, Y)$ the stable homotopy class $D_n(\boldsymbol{\alpha})$ corresponds $D_{n+1}(\boldsymbol{\alpha})$ under the canonical one-one correspondence $\{S^n \setminus Y, S^n \setminus X\} \leftrightarrow \{S^{n+1} \setminus Y, S^{n+1} \setminus X\}$.*

4. Algebraic properties of D_n.

Theorem 4.1. *$D_n(\boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2) = D_n(\boldsymbol{\alpha}_1) + D_n(\boldsymbol{\alpha}_2)$ for all $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2 \in \mathrm{SStrSh}(X,Y)$.*

For the proof of Theorem 4.1 we need the following fact:

Proposition 4.2. *If $X \subset S^n \subset S^m$, $m > n$, then*

$$S^m \setminus (X \vee X) \simeq (S^m \setminus X) \vee (S^m \setminus X).$$

Proof. Let $a \in X$ be a base point of X and $c \in S^n \setminus X \subset S^m \setminus X$ be a base point of $S^m \setminus X$. Let $\sigma : S^n \to [0,1)$ be a map such that $\sigma^{-1}(0) = \{a\}$. If we regard $S^m = S^{m-1} \times [-1,1]/S^{m-1} \times \{-1\}, S^{m-1} \times \{1\}$, we define embeddings $h_i : S^n \to S^m$, $i = 1,2$, by putting $h_1(x) = [x, \sigma(x)]$ and $h_2(x) = [x, -\sigma(x)]$. Then $h_i(a) = [a,0] = a$, h_i are isotopic to the inclusion $S^n \hookrightarrow S^{m-1} \subset S^m$ and $W = X_1 \cup X_2$, where $X_i = h_i(X)$, $i = 1,2$, is a copy of $(X,a) \vee (X,a)$ in S^m. We shall show that

$$(S^m \setminus W, c) \simeq (S^m \setminus X, c) \vee (S^m \setminus X, c).$$

The homotopy equivalence will be a pointed equivalence.

There is a homeomorphism $h : S^m \setminus \{a\} \to S^m \setminus L$, where L is a compact arc on the great circle through a and the poles of S^m, $a \in \operatorname{int} L, c \notin L$. This exists for geometric reasons, simply deform all the great circles through the poles. Hence,

$$(S^m \setminus W, c) \simeq (S^m \setminus (X_1' \cup L \cup X_2'), c) \tag{4.1}$$

where $X_1' = h(X_1)$ is the upper hemisphere S^m_+ and $X_2' = h(X_2)$ is the lower hemisphere S^m_-.

Contracting $S^{m-1} \setminus \{a\} \subset S^m \setminus (X_1' \cup L \cup X_2')$ to a point yields a pointed homotopy equivalence

$$(S^m \setminus (X_1' \cup L \cup X_2'), c) \simeq (V_1, *) \vee (V_2, *) \tag{4.2}$$

where $V_i = (S^m_\pm \setminus X_i') \cup \mathrm{Con}(S^{n-1} \setminus \{a\})$, $i = 1,2$. ("Con" denotes the cone on a space and * denotes the top of the cone). This is true because $S^{m-1} \setminus \{a\}$ is contractible and the inclusion $S^{m-1} \setminus \{a\} \subset S^{m-1} \setminus (X_1' \cup L \cup X_2')$ is a cofibration (as an inclusion of a subcomplex).

There is an injection $i : \mathrm{Con}(S^{m-1} \setminus \{a\}) \to \mathrm{Con}(S^{m-1}) \setminus \{a\}$ which has a homotopy inverse r such that $ri \simeq \mathrm{id}\ \mathrm{rel}((S^{m-1} \setminus \{a\} \cup \{*\})$ and $ir \simeq \mathrm{id}\ \mathrm{rel}((S^{m-1} \setminus \{a\} \cup \{*\})$. This fact is obtained from [Mr], Lemma 2, replacing $J = [-1,1]$ by $I = [0,1]$ and A by $\{a\}$.

Thus,

$$(V_i, *) \simeq ((S^m_\pm \setminus X_i') \cup (\mathrm{Con}\, S^{m-1}) \setminus \{a\}, *) \simeq (S^m_i \setminus X_i', *),$$

where S^m_i is a homeomorphic copy of S^m, $i = 1,2$.

With this in mind, from 4.1 and 4.2 it follows that

$$(S^m \setminus W, c) \simeq (S^m \setminus (X_1 \cup X_2), c) \simeq (S_1^m \setminus X_1', *) \vee (S_2^m \setminus X_2', *).$$

Of course, we can isotope X_1', X_2' back to its position, so that

$$(S^m \setminus W, c) \simeq (S^m \setminus X, c) \vee (S^m \setminus X, c).$$

$\square$

Proof of Theorem 4.1. It suffices to show that $\Delta_n : \mathrm{StrSh}(X, Y) \to \{S^n \setminus Y, S^n \setminus X\}$ is a homomorphism if $X = \Sigma^{n-k}(A)$ and $Y = \Sigma^{n-k}(B)$, where A and B are subcompacta of S^k.

Suppose $\boldsymbol{f}_1, \boldsymbol{f}_2 \in \mathrm{StrSh}(X, Y)$. Let a be a base point of X and c a base point of $S^n \setminus X$. Let $m > 2n$ and for $i = 1, 2$ let $h_i : S^n \to S^m$ be a topological embedding as in the proof of Proposition 4.2. Then

$$(S^m \setminus (X_1 \cup X_2), c) \simeq (S_1^m \setminus X_1', *) \vee (S_2^m \setminus X_2', *),$$

where S_i^m are homeomorphic copies of S^m, X_i' are homeomorphic copies of X and $X_i = h_i(X)$, $i = 1, 2$. Using this fact we can identify $S^m \setminus (X_1 \cup X_2)$ with $U_1' \cup U_2'$, where U_i' is the $(m-n)$-fold suspension of $U_i = h_i(S^n \setminus X)$ in S^m, $i = 1, 2$.

Moreover, one can observe that the stable homotopy class of the inclusion $S^m \setminus (X_1 \cup X_2)$ into $S^m \setminus X_q$ equals to the stable homotopy class of the retraction $r_q' : U_1 \cup U_2 \to U_q$ with $r_q'(U_p) = \{c_q\}$, where $c_q = h_q(c)$ for $q \neq p$, $p, q = 1, 2$.

Similarly, the stable homotopy class r_q' of the inclusion U_q into $U_1 \cup U_2$ equals to the stable homotopy class $D_m(\boldsymbol{r}_q)$, where $r_q : X_1 \cup X_2 \to X_q$ is a retraction with $r_q(X_p) = \{a\}$ for $p, q = 1, 2$, $p \neq q$. Indeed, the one-point union of X_q and the cone over X_p is a mapping cylinder for r_q, where $q \neq p$, $p, q = 1, 2$.

Let $h_q' : X \to X_q$ be the map given by the formula

$$h_q'(x) = h_q(x) \text{ for } x \in X \text{ and } q = 1, 2,$$

and let

$$\boldsymbol{h} = \boldsymbol{i}_1 \boldsymbol{h}_1' + \boldsymbol{i}_2 \boldsymbol{h}_2'$$

where $i_q : X_q \to X_1 \cup X_2$ denotes the inclusion of X_q into $X_1 \cup X_2$ for $q = 1, 2$.

It is clear that $\boldsymbol{i}_1 \boldsymbol{r}_1 + \boldsymbol{i}_2 \boldsymbol{r}_2 : X_1 \cup X_2 \to X_1 \cup X_2$ equals to the strong shape morphism induced by the identity map $X_1 \cup X_2$.

We also known that $\Delta_m(\boldsymbol{i}_1 \boldsymbol{r}_1) + \Delta_m(\boldsymbol{i}_2 \boldsymbol{r}_2)$ equals to the stable homotopy class which is induced by the identity map of $U_1 \cup U_2$.

Let $\boldsymbol{v} : X_1 \cup X_2 \to Y$ be the strong shape morphism satisfying the condition

$$\boldsymbol{v} \boldsymbol{i}_q = \boldsymbol{f}_q \boldsymbol{h}_q',$$

where $q = 1, 2$.

Then $\boldsymbol{f}_q = \boldsymbol{v}\boldsymbol{i}_q\boldsymbol{h}'_q$ and $\boldsymbol{f}_1 + \boldsymbol{f}_2 = \boldsymbol{v}(\boldsymbol{i}_1\boldsymbol{h}'_1 + \boldsymbol{i}_2\boldsymbol{h}'_2) = \boldsymbol{v}\boldsymbol{h}$. Since $\boldsymbol{h}'_q = \boldsymbol{r}_q\boldsymbol{h}$, we have $\boldsymbol{f}_q = \boldsymbol{v}\boldsymbol{i}_q\boldsymbol{r}_q\boldsymbol{h}$ for $q = 1, 2$. Hence

$$\begin{aligned}\Delta_m(\boldsymbol{f}_1 + \boldsymbol{f}_2) &= \Delta_m(\boldsymbol{h})(\Delta_m(\boldsymbol{i}_1\boldsymbol{r}_1) + \Delta_m(\boldsymbol{i}_2\boldsymbol{r}_2))\Delta_m(\boldsymbol{v}) \\ &= \Delta_m(\boldsymbol{v}\boldsymbol{i}_1\boldsymbol{r}_1\boldsymbol{h}) + \Delta_m(\boldsymbol{v}\boldsymbol{i}_2\boldsymbol{r}_2\boldsymbol{h}) = \Delta_m(\boldsymbol{f}_1) + \Delta_m(\boldsymbol{f}_2).\end{aligned}$$

□

5. The proof that $D_n : \mathrm{SStrSh}_n(X, Y) \to \{S^n \setminus Y, S^n \setminus X\}$ is an isomorphism.

Theorem 5.1. *$D_n : \mathrm{SStrSh}_n(X, Y) \to \{S^n\setminus Y, S^n\setminus X\}$ is an isomorphism for all compacta $X, Y \subset S^n$.*

Proof. Suppose that $Q \subset S^n$ is a polyhedral cube which contains X and $\bigcap_{k=1}^{\infty} Y_k \subset ... \subset Y_2 \subset Y_1 \subset Y_0 = Q$, where Y_k is a subpolyhedron of Q for $k = 1, 2, ...$

We may assume that $X \cap Y = \emptyset$ and that the sets of all stable strong (pointed) shape morphisms $\mathrm{SStrSh}(X, Y)$ and $\mathrm{SStrSh}(\Sigma X, Y_k)$ is one-to-one correspondence with the sets $\mathrm{StrSh}(X, Y)$ and $\mathrm{StrSh}(\Sigma X, Y_k) \simeq H(\Sigma X, Y_k)$ for $k = 0, 1, 2, ...$

Suppose that $e = \{e_k\} \in \lim^1 H(\mathbf{S}X, Y_k)$ and $e_k \in H(\mathbf{S}X, Y_k)$ is represented by a map $f_k : \mathbf{S}(X) \to Y_k$ for $k = 0, 1, 2, ...$

We can construct a sequence $X_0 = Q \supset X_1 \supset X_2 \supset ...$ of subpolyhedra of Q such that $\cap_{k=1}^{\infty} X_k$ and for $k = 0, 1, 2, ...$ there exists $\widetilde{f}_k : \mathbf{S}X_k \to Y_k$ such that $\widetilde{f}_k(x) = f_k(x)$ for $x \in \mathbf{S}X \subset \mathbf{S}X_k$.

Setting

$$f'(x, r) = \widetilde{f}_k([x, r - k]) \text{ for } x \in X_k, r \in [k, k + 1] \text{ and } k = 0, 1, 2, ...$$

and

$$f''(x, r) = r \text{ for } x \in X_k, r \in [k, k + 1] \text{ and } k = 0, 1, 2, ...$$

and

$$f(x, r) = (f'(x, r), f''(x, r)) \text{ for } (x, r) \in \mathrm{Tel}\,\boldsymbol{X}$$

we get a map $f' : \mathrm{Tel}\,\boldsymbol{X} \to Y_0$ and proper maps $f'' : \mathrm{Tel}\,\boldsymbol{X} \to [0, \infty)$ and $f : \mathrm{Tel}\,\boldsymbol{X} \to \mathrm{Tel}\,\boldsymbol{Y}$, where $\boldsymbol{X}$ and $\boldsymbol{Y}$ denote respectively nets $X_0 \supset X_1 \supset X_2 \supset ...$ and $Y_0 \supset Y_1 \supset Y_2 \supset ...$ associated with X and Y.

The proper homotopy class of f determines uniquely an element $\alpha(e)$ of $\mathrm{StrSh}(X, Y) \cong \mathrm{SStrSh}(X, Y)$. It is known that α is a homorphism ([H-N]) and that the sequence

$$0 \to \lim{}^1\{\mathbf{S}(X), Y_k\} \xrightarrow{\alpha} \mathrm{SStrSh}(X, Y) \xrightarrow{\beta} \mathrm{SSh}(X, Y) \to 0$$

is exact, where β is induced by the natural functor from the strong shape category to the shape category.

Let $Q' = Q \times [0,\infty) \cup \{w\}$ be one-point compactification of $Q \times [0,\infty)$ and $Z = M(f)$ be defined as follows

$$Z = \{((x,r), f'(x,r)) \in Q' \times Q : (x,r) \in X \times [0,\infty)\} \cup \{w\} \times Y.$$

Let $s \in [k, k+1] \subset [0,\infty)$ and

$$Z_s = \{((x,r), f'(x,r)) \in Q' \times Q : (x,r) \in X_k \times [0,s)\} \cup Q'_s \times Y_k,$$

where $Q'_s \subset Q'$ is one-point compactification of $Q \times [s,\infty) \subset Q' = Q \times [0,\infty) \cup \{w\}$.

We identify X and Y with the subset of Z consisting of all points $((x,0), f'(x,0)) \in Z$ such that $x \in X$ and with the subset of Z consisting of all $(w,y) \in Z$ such that $y \in Y$.

We may assume that Z_k and $\bigcup_{s\in[k,k+1]} Z_k \times \{s\} = W_k$ are polyhedra for $k = 0,1,2,...$

Let us observe that $Z = \bigcap_{k=0}^{\infty} Z_k$.

We may assume that $Z \subset S^m$ for sufficiently large m.

Let $U = S^m \setminus X$ and $V = S^m \setminus Y$ and let $X_0^* \subset X_1^* \subset X_2^* \subset ...$ and $Y_0^* \subset Y_1^* \subset Y_2^* \subset ...$ be sequences of subpolyhedra of S^m such that X_k^* and Y_k^* are (respectively) Spanier-Whitehead duals of X_k and Y_k in S^m for $k = 0,1,2,...$, $\bigcup_{k=0}^{\infty} Y_k^* = V$ and $\bigcup_{k=0}^{\infty} X_k^* = U$.

Then we have a short exact sequence (see Theorem (5.1) of [N])

$$0 \to \lim{}^1\{\mathbf{S}Y_k^*, U\} \to \{V, U\} \to \{V, U\}_w \to 0.$$

Let $V^* = Y_0^* \times [0,1] \cup Y_1^* \times [1,2] \cup Y_2^* \times [2,3] \cup ...$ and let $g^* : V \to V^*$ be the natural homotopy equivalence.

We may also assume that the set of all stable homotopy classes $\{Y_k^*, X_t^*\}$ and $\{\mathbf{S}Y_k^*, X_t^*\}$ are in one-to-one correspondence with the sets $H(Y_k^*, X_t^*)$ and $H(\mathbf{S}Y_k^*, X_t^*)$ for $k, t = 0,1,2,....$

Suppose that $e^* \in \lim{}^1\{\mathbf{S}Y_k^*, U\}$ is represented by the sequence of maps $f_k^* : \mathbf{S}Y_k^* \to X_{\alpha(k)}^* \subset U$ for $k = 0,1,2,....$

Let $f^* : V^* \to U$ be defined by the formulas

$$f^*(x,r) = f_k^*([x, r-k]) \text{ for } x \in Y_k^*, r \in [k,k+1] \text{ and } k = 0,1,2,....$$

Then $\alpha'(e) = \{f^*g^*\}$.

Suppose that $\gamma : \lim{}^1\{\mathbf{S}X, Y_k\} \to \lim{}^1\{\mathbf{S}Y_k^*, U\}$ is induced by the Spanier-Whitehead Duality Theorem ([S-W])i.e. if $e \in \lim{}^1\{\mathbf{S}X, Y_k\}$ is represented by the sequence of the homotopy classes of maps $f_k : \mathbf{S}X_k \to Y_k$ then $\gamma(e)$ is represented by the sequence of homotopy classes of maps $f_k^* : \mathbf{S}Y_k^* \to X_k^* \subset \bigcup_{k=0}^{\infty} X_k^* = U$ such that $\{f_k\}$ and $\{f_k^*\}$ are dual (in the sense of the Spanier-Whitehead Duality Theorem) in S^{m+1}.

Consider the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \lim{}^1\{\mathbf{S}X, Y_j\} & \longrightarrow & \mathrm{SStrSh}(X,Y) & \longrightarrow & \mathrm{SSh}(X,Y) & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle\lambda} & & \Big\downarrow{\scriptstyle D_m} & & \Big\downarrow{\scriptstyle D'_m} & & \\
0 & \longrightarrow & \lim{}^1\{\mathbf{S}Y^*_j, U\} & \longrightarrow & \{V,U\} & \longrightarrow & \{V,U\}_w & \longrightarrow & 0
\end{array}$$

where $D'_m : \mathrm{SSh}(X,Y) \to \{V,U\}_w$ is the isomorphism which is induced by by the functor described in [N] (see [N], Theorem (4.1)).

Using the properties of the inclusions of $X_k \times [k, k+1]$ and $Y_k \times [k, k+1]$ into W_k we can check that the above diagram is commutative.

Since γ is an isomorphism we infer that D_m is an isomorphism. $\square$

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Final remarks.

The first version of the present note was written in 1987, but we believe some its aspects are still up-to-date and are not covered by newer results.

In the meantime Theorem 6.1 was obtained [B_1] (correction in [B_2]) by F. W. Bauer as a consequence of a very general version of the Alexander Duality Theorem. Next, he generalized it to arbitrary subsets of the n-sphere (see [B_2] and [B_3]).

Bauer's investigations are concentrated on the categories which contain the stable strong shape category of compacta.

The approach of the paper allows to compare various generalizations of the Spanier-Whitehead Duality (see [S_2] p. 217), i.e. the Lima Duality [L] defined on the stable shape category and the duality being the topic of the present paper.

In fact, (keeping the notations of the section 6) we show that the short exact sequence

$$0 \to \lim{}^1\{SX, Y_j\} \to \mathrm{SStrSh}(X,Y) \to \mathrm{SSh}(X,Y) \to 0$$

is mapped isomorphically onto the exact sequence

$$0 \to \lim{}^1\{SY^*_j, U\} \to \{V,U\} \to \{V,U\}_W \to 0$$

by these dualities. The crucial argument of the proof is to deduce that the middle homorphism $D_n : \mathrm{SStrSh}(X,Y) \to \{V,U\}$ is an isomorphism because the other vertical arrows (i.e. γ and D'_n) are isomorphisms.

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