GLASNIK MATEMATIČKI
Vol. 36(56)(2001), 311 – 318

# APPROXIMATION BY KANTOROVICH TYPE GENERALIZATION OF MEYER- KÖNIG AND ZELLER OPERATORS

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Abstract. In this study, we define a Kantorovich type generalization of W. Meyer- König and K. Zeller operators and we will give the approximation properties of these operators with the help of Korovkin theorems. Then we compute the approximation order by modulus of continuity.

## 1. Introduction

A constructive approach to the characterization of functions is to define and approximate them in terms of some defined positive operators. A large reference list in the study of the subject is cited for the intersted readers ([1]-[20]). Specifically, in [8], we generalized the Meyer - König and Zeller operators. In this paper, we make a Kantorovich type generalization of the operators $L_n(f;x)$ defined in [8] . Then, we give some approximation properties of these operators.

## 2. Kantorovich Type Generalization of Meyer-König and Zeller Operators

Let $A$ be a real number in the interval $(0,1)$. Assume that a sequence of functions $\{\varphi_n\}$ satisfies the following conditions:

2000 *Mathematics Subject Classification.* 41A36.
*Key words and phrases.* Linear positive operators, Korovkin theorem, Modulus of continuity.
Supported by Ankara University Resarch Fund Grant Number 99.05.02.01.

*i*. Every element of the sequence $\{\varphi_n\}$ is analytic on a domain $D$ containing the disk $B = \{z \in \mathbf{C} : |z| \leq A\}$

*ii*. $\varphi_n^{(0)}(x) = \varphi_n(x) > 0$

*iii*. $\varphi_n^{(k)}(x) = \gamma_n(k+n)\,(1+\ell_{n,k})\,\varphi_n^{(k-1)}(x)$ , $\quad k = 1, 2, ...$
where, $\varphi_n^{(k)}(x) = \frac{d^k}{dx^k}\varphi_n(x)$, and $\ell_{n,k}$ and $\gamma_n$ are sequences of numbers satisfying the conditions;

$$\ell_{n,k} = O\left(\frac{1}{n}\right), \ell_{n,k} \geq 0, \gamma_n = 1 + O\left(\frac{1}{n}\right), \ \gamma_n \geq 1.$$

For

$$0 < \alpha_{n,k} \leq 1 \quad (n, k = 1, 2, ...),$$

consider the sequence of linear positive operators

$$M_n^*(f;x) = \frac{1}{\varphi_n(x)} \sum_{k=0}^{\infty} \frac{1}{\alpha_{n,k}} \int_k^{k+\alpha_{n,k}} f\left(\frac{\xi}{k+n}\right) d\xi \ \varphi_n^{(k)}(0) \frac{x^k}{k!} \tag{2.1}$$

where $f$ is an integrable function on $(0,1)$.

$M_n^*(f;x)$ is a Kantorovich type generalization of Meyer - König and Zeller operators.

## 3. Approximation Properties

In this section, we give approximation properties of the operator $M_n^*(f;x)$ with the help of Korovkin theorem.

Theorem 3.1. *The sequence of linear positive operators defined by (2.1) with conditions $i - iii$ converges uniformly to the function $f \in C[0,A]$ in $[0,A]$.*

Proof. It is enough to prove the conditions of Korovkin theorem [12] which are

$$M_n^*(t^k;x) \longrightarrow x^k, \ \ k = 0, 1, 2 \ ,$$

uniformly in $[0,A]$.

First, since

$$\frac{1}{\varphi_n(x)} \sum_{k=0}^{\infty} \varphi_n^{(k)}(0) \frac{x^k}{k!} = 1, \tag{3.1}$$

we get $M_n^*(1;x) = 1$.

Second, we can write

$$\begin{aligned} M_n^*(t;x) - x &= \tfrac{1}{\varphi_n(x)} \sum_{k=0}^{\infty} \tfrac{(k+\alpha_{n,k})^2-k^2}{2(k+n)} \varphi_n^{(k)}(0)\tfrac{x^k}{k!} - x \\ &= \tfrac{1}{\varphi_n(x)} \sum_{k=0}^{\infty} \tfrac{k}{k+n} \varphi_n^{(k)}(0)\tfrac{x^k}{k!} + \tfrac{1}{2}\tfrac{1}{\varphi_n(x)} \sum_{k=0}^{\infty} \tfrac{\alpha_{n,k}}{k+n} \varphi_n^{(k)}(0)\tfrac{x^k}{k!} - x \\ &= L_n(t;x) - x + \tfrac{1}{2}\tfrac{1}{\varphi_n(x)} \sum_{k=0}^{\infty} \tfrac{\alpha_{n,k}}{k+n} \varphi_n^{(k)}(0)\tfrac{x^k}{k!} \end{aligned}$$

where $L_n(t;x)$ is a generalization of Meyer- König and Zeller operators defined, in general, as

$$L_n(f;x) = \frac{1}{\varphi_n(x)} \sum_{k=0}^{\infty} f(\frac{k}{k+n})\varphi_n^{(k)}(0)\frac{x^k}{k!}$$

(see [8]).

Using $L_n(t;x) - x \geq 0$, we obtain $M_n^*(t;x) - x \geq 0$.

On the other hand, since $\frac{\alpha_{n,k}}{k+n} \leq \frac{1}{n}$ we get

$$0 \leq M_n^*(t;x) - x \leq L_n(t;x) - x + \frac{1}{2n}.$$

In [8], it is shown that $\lim_{n\to\infty}(L_n(t;x) - x) = 0$. Therefore, we obtain $\lim_{n\to\infty}(M_n^*(t;x) - x) = 0$, and thus the second condition of Korovkin theorem is hold.

Finally, using *iii* in $M_n^*(t^2;x)$,we obtain

$$\begin{aligned} M_n^*(t^2;x) = \quad & L_n(t^2;x) + \tfrac{1}{\varphi_n(x)} \sum_{k=0}^{\infty} \tfrac{k\cdot\alpha_{n,k}}{(k+n)^2} \varphi_n^{(k)}(0)\tfrac{x^k}{k!} \\ & + \tfrac{1}{3\varphi_n(x)} \sum_{k=0}^{\infty} \tfrac{\alpha_{n,k}^2}{(k+n)^2} \varphi_n^{(k)}(0)\tfrac{x^k}{k!}. \end{aligned}$$

Thus we can write

$$\begin{aligned} M_n^*(t^2;x) - x^2 &= L_n(t^2;x) - x^2 + \tfrac{1}{\varphi_n(x)} \sum_{k=0}^{\infty} \left[\tfrac{k\cdot\alpha_{n,k}}{(k+n)^2} + \tfrac{\alpha_{n,k}^2}{3(k+n)^2}\right] \varphi_n^{(k)}(0)\tfrac{x^k}{k!} \\ &\geq 0. \end{aligned}$$

Now,

$$L_n(t^2;x) - x^2 \geq 0$$

( See [8] ).

In addition, since $\frac{\alpha_{n,k}}{k+n} \leq \frac{1}{n}$ and $\frac{\alpha_{n,k}^2}{(k+n)^2} \leq \frac{1}{n^2}$ , the following inequality holds:

$$M_n^*(t^2;x) - x^2 \leq (L_n(t^2;x) - x^2) + \frac{1}{n}L_n(t;x) + \frac{1}{3n^2}.$$

On the other hand

$$L_n(t;x) \longrightarrow x\,, \quad L_n(t^2;x) \longrightarrow x^2$$

(See [8] ). Thus $M_n^*(t^2;x) \longrightarrow x^2$. Because of Korovkin theorem, the proof is complete. ∎

## 4. The Order of Approximation

In this section, we compute the approximation order of $L_n(f;x)$ defined in [8] and $M_n^*(f;x)$ with the help of modulus of continuity which is

$$\omega(f;\delta) = \sup\{|f(t)-f(x)|\,;\ \ |t-x|\le\delta,\ \ t,x\in[0,A]\}\,. \tag{4.1}$$

The following theorems are analogous to one given in [7] for Gadjiev-Ibragimov operators.

Theorem 4.1. *The following inequality holds:*

$$\|L_n(f;x)-f(x)\|_{C[0,A]} \le C\ \omega(f;\frac{1}{\sqrt{n}})$$

*where $\omega(f;\frac{1}{\sqrt{n}})$ is a modulus of continuity defined by (4.1), and $C$ is a positive constant.*

Proof. From well-known properties of $\omega(f;\delta)$, we can write

$$|L_n(f;x)-f(x)| \le \omega(f;\delta)\left[\frac{1}{\delta}(L_n(t^2;x)-2xL_n(t;x)+x^2)^{1/2}+1\right] \tag{4.2}$$

If we use (3.1) and inequalities

$$L_n(t;x)-x \le (\gamma_n-1)x+\frac{dx\gamma_n}{n},$$

$$L_n(t^2;x)-x^2 \le x^2(\gamma_n^2-1)+\frac{2dx^2\gamma_n^2+x\gamma_n}{n}+\frac{x^2\gamma_n^2d^2+x\gamma_n d}{n^2}\,,$$

(see[8]) in (4.2) and making the simplifications, we obtain,

$$\begin{aligned}|L_n(f;x)-f(x)| &\le \omega(f;\delta)\left[\tfrac{\sqrt{5}}{\delta}O(\tfrac{1}{\sqrt{n}})+1\right]\\ &\le \omega(f;\delta)\left[\tfrac{1}{\delta}\tfrac{C_1}{\sqrt{n}}+1\right].\end{aligned}$$

By choosing $\delta=\frac{1}{\sqrt{n}}$, we obtain the desired result. ∎

Theorem 4.2. *Let $f$ be uniform continuous on $[0,1]$. Then, the sequence of linear positive operators defined by (2.1) under the conditions $i-iii$, satisfies the inequality*

$$|M_n^*(f;x)-f(x)| \le C\omega(f;\frac{1}{\sqrt{n}}).$$

PROOF. By using the triangle inequality, we can write

(4.3)
$$|M_n^*(f;x)-f(x)| \le \frac{1}{\varphi_n(x)} \sum_{k=0}^{\infty} \frac{1}{\alpha_{n,k}} \int_k^{k+\alpha_{n,k}} \left| f\left(\frac{\xi}{k+n}\right) - f(x)\right| d\xi \; \varphi_n^{(k)}(0)\frac{x^k}{k!}$$

On the other hand, since

$$\left| f\left(\frac{\xi}{k+n}\right) - f(x)\right| \le \left( \frac{\left|\frac{\xi}{k+n}-x\right|}{\delta} + 1 \right) \omega(f;\delta)$$

we get

$$\int_k^{k+\alpha_{n,k}} \left| f\left(\frac{\xi}{k+n}\right) - f(x)\right| d\xi \le \left( \frac{\int_k^{k+\alpha_{n,k}} \left|\frac{\xi}{k+n}-x\right| d\xi}{\delta} + \alpha_{n,k} \right) \omega(f;\delta).$$

Using the last inequality and (3.1) in (4.3), we obtain

$$|M_n^*(f;x)-f(x)| \le$$

$$\le \left[ \tfrac{1}{\delta} \tfrac{1}{\varphi_n(x)} \sum_{k=0}^{\infty} \left\{ \tfrac{1}{\alpha_{n,k}} \int_k^{k+\alpha_{n,k}} \left|\tfrac{\xi}{k+n}-x\right| d\xi \; \varphi_n^{(k)}(0)\tfrac{x^k}{k!} \right\} + 1 \right] \omega(f;\delta)$$

$$\le \left[ \tfrac{1}{\delta} \left( \tfrac{1}{\varphi_n(x)} \sum_{k=0}^{\infty} \left( \tfrac{1}{\alpha_{n,k}} \int_k^{k+\alpha_{n,k}} \left|\tfrac{\xi}{k+n}-x\right| d\xi \right)^2 \varphi_n^{(k)}(0)\tfrac{x^k}{k!} \right)^{\frac{1}{2}} + 1 \right] \omega(f;\delta)$$

$$\le \left[ \tfrac{1}{\delta} \left( L_n(t^2;x) + \tfrac{2}{n}L_n(t;x) + \tfrac{1}{2n^2} + 2xL_n(t;x) + \tfrac{x}{n} - 3x^2 \right)^{\frac{1}{2}} + 1 \right] \omega(f;\delta)$$

$$\le \left[ \tfrac{1}{\delta} C_1 O(\tfrac{1}{\sqrt{n}}) + 1 \right] \omega(f;\delta).$$

If we choose $\delta = \frac{1}{\sqrt{n}}$, then the proof is completed. □

## 5. A Differential Equation

We refer to some papers, in which equations analogous to the following theorem are given: May [14], Volkov [20], Alkemade [2] and [8].

Let $g$ satisfy the equality

$$g\left(\frac{k}{k+n}\right) = \frac{ak}{bn}$$

where $a$ and $b$ are arbitrary nonzero constant. Hence letting $n = \dfrac{k}{t} - k$, we obtain $g(t) = \dfrac{at}{b(1-t)}$ .

Now we give the following theorem.

THEOREM 5.1. *Let*

$$g(t) = \frac{at}{b(1-t)} \quad (t \in [0, A]\,,\ a, b \neq 0).$$

*For each* $x \in [0, A]$ , $f \in C\,[0, A]$ and $\alpha_{n,k} = 1$; $M_n^*(f;x)$ *satisfies the following functional differential equation for* $n = 2, 3, ...$ :

$$(5.1)\qquad \begin{aligned} x\tfrac{d}{dx}M_n^*(f;x) = &\left[-\gamma_n(1+n)(1+\ell_{n,1})x + \tfrac{1}{\ln(\frac{n-1}{n})} - n\right] M_n^*(f;x) \\ &-\tfrac{b}{a\ln(\frac{n-1}{n})} M_n^*((f,\, g);x). \end{aligned}$$

where

$$M_n^*((f,\, g);x) = \frac{1}{\varphi_n(x)} \sum_{k=0}^{\infty} \int_k^{k+1} f\left(\frac{\xi}{k+n}\right) d\xi \int_k^{k+1} g\left(\frac{\xi}{k+n}\right) d\xi\ \varphi_n^{(k)}(0) \frac{x^k}{k!}.$$

PROOF. Since $M_n^*(f;x)$ is uniform convergent on $[0, A]$, we can differentiate this series term by term in this interval.

Hence,

$$x\frac{d}{dx}M_n^*(f;x) = \frac{-\gamma_n(1+n)(1+\ell_{n,1})x}{\varphi_n(x)} \sum_{k=0}^{\infty} \int_k^{k+1} f\left(\frac{\xi}{k+n}\right) d\xi\ \varphi_n^{(k)}(0) \frac{x^k}{k!}$$

$$(5.2)\qquad \begin{aligned} x\tfrac{d}{dx}M_n^*(f;x) = &\tfrac{-\gamma_n(1+n)(1+\ell_{n,1})x}{\varphi_n(x)} \sum_{k=0}^{\infty} \int_k^{k+1} f\left(\tfrac{\xi}{k+n}\right) d\xi\ \varphi_n^{(k)}(0) \tfrac{x^k}{k!} \\ &+\tfrac{1}{\varphi_n(x)} \sum_{k=0}^{\infty} \int_k^{k+1} f\left(\tfrac{\xi}{k+n}\right) d\xi\ \varphi_n^{(k)}(0) k \tfrac{x^k}{k!}. \end{aligned}$$

It can be shown that

$$k = \frac{\frac{a}{b} - \int_k^{k+1} g\left(\frac{\xi}{k+n}\right) d\xi}{\frac{a}{b}\ln(\frac{n-1}{n})} - n.$$

By using this equality in (5.2), after simplification, we get

$$x\tfrac{d}{dx}M_n^*(f;x) = \left[-\gamma_n(1+n)(1+\ell_{n,1})x + \tfrac{1}{\ln(\frac{n-1}{n})} - n\right]\tfrac{1}{\varphi_n(x)}$$

$$\times \sum_{k=0}^{\infty}\int_k^{k+1} f\left(\tfrac{\xi}{k+n}\right)d\xi\ \varphi_n^{(k)}(0)\tfrac{x^k}{k!} - \tfrac{1}{\frac{a}{b}\ln(\frac{n-1}{n})}\tfrac{1}{\varphi_n(x)}$$

$$\times \sum_{k=0}^{\infty}\int_k^{k+1} f\left(\tfrac{\xi}{k+n}\right)d\xi \int_k^{k+1} g\left(\tfrac{\xi}{k+n}\right)d\xi\ \varphi_n^{(k)}(0)\tfrac{x^k}{k!}.$$

Thus the theorem is proved. □

Proposition 5.1. For $\alpha_{n,k} = 1$, $M_n^*(t;x)$ *gives a solution of the differential equation*

$$xy' - \big[-\gamma_n(1+n)(1+\ell_{n,1})x + \frac{a+b}{a\ln(\frac{n-1}{n})} - n\big]y = \gamma_n(1+n)(1+\ell_{n,1})x - \frac{1}{\ln(\frac{n-1}{n})} + n. \tag{5.3}$$

Proof. Setting, in (5.1), $f = 1-t$, it follows that

$$x\frac{d}{dx}M_n^*(1-t;x) = \left[-\gamma_n(1+n)(1+\ell_{n,1})x + \frac{1}{\ln(\frac{n-1}{n})} - n\right]M_n^*(1-t;x)$$

$$-\frac{b}{a\ln(\frac{n-1}{n})}M_n^*(t;x). \tag{5.4}$$

Using the linearity of $M_n^*$ and $M_n^*(1;x) = 1$ in (5.4) we get (5.3). □

Acknowledgements.

Authors would like to thank to the referee for the valuable suggestions and remarks.

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*Received*: 12.06.2000.
*Revised*: 03.11.2000.