GLASNIK MATEMATIČKI
Vol. 36(56)(2001), 1 – 9

ON FUNCTIONS WHICH ARE ALMOST ADDITIVE MODULO A SUBGROUP

Janusz Brzdęk
Pedagogical Academy of Cracow, Poland

Abstract. Let $(X, +)$ be a commutative semigroup, uniquely divisible by 2, $(G, +)$ be a topological group, and K be a discrete, normal and countable subgroup of G. We show that if X is endowed with a topology and the topologies in X and G satisfy some additional conditions, then for every measurable function f mapping X into G such that $f(x+y)-f(x)-f(y) \in K$ almost everywhere in X^2, with respect to some ideal in X^2, there is an additive function $A : X \to G$ with $f(x)-A(x) \in K$ almost everywhere in X.

1. Introduction

In connection with the problem of stability of the Cauchy equation (see [12]) several authors (see e.g. [1] – [4], [6], and [12]) have considered the following question:

Suppose that X is a real linear space,

(H_1) $(G, +)$ is a topological group (not necessarily commutative),

(H_2) K is a discrete and normal subgroup of G (discrete means that there is a neighbourhood $U \subset G$ of 0 such that $U \cap K = \{0\}$),

and $f : X \to G$ is a function satisfying

$$f(x + y) - f(x) - f(y) \in K \text{ for every } x, y \in X. \tag{1.1}$$

When does there exist an additive function $A : X \to G$ with

$$f(x) - A(x) \in K \text{ for every } x \in X \tag{1.2}$$

(i.e. $f = A + k$ with some $k : X \to K$)?

2000 *Mathematics Subject Classification.* 39B52.
Key words and phrases. Cauchy difference, measurability, additive function, σ-ideal.

It is known (from Example 2 in [11]; see also Remark 2 in [2]) that in the general situation this is not the case (cf. e.g. [1] and [2]). However there are assumptions on f such as continuity at a point or measurability (in some sense), with X being a linear topological space, which guarantee the desired form of f (see e.g. [1] – [4] and [6]).

In this paper we study the more general situation where the condition:

$$f(x+y)-f(x)-f(y)\in K$$

holds almost everywhere in X^2 with respect to some ideal in X^2. We consider the case of f being measurable (in the sense specified later); the case of f continuous at a point has been studied in [4].

Throughout the paper $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ denote the sets of positive integers, integers, rationals, and reals, respectively.

In the sequel we will need the following hypothesis:

(H_3) $(X,+)$ is a commutative semigroup with zero, uniquely divisible by 2 and endowed with a topology such that every neighbourhood of zero contains a subset V such that
- (i) $\frac{1}{2}x\in V$ for every $x\in V$,
- (ii) $X=\bigcup_{n\in\mathbb{N}}2^nV$, where $2^nV:=\{2^nx : x\in V\}$.

By (H_3') we will denote (H_3) with the expression "a subset V" replaced by "an open subset V". Note that from (ii) it results that $0\in V$.

For instance every topological linear space satisfies (H_3'). In [4] (pp. 118–119) (cf. also [6]) there are given some further examples and it is proved that there exist semitopological linear spaces which fulfil (H_3) and do not fulfil (H_3').

Given a topological group G and a normal subgroup K of G, in the factor group G/K we always take the factor topology, i.e. a set $U\subset G/K$ is open if the set $p^{-1}(U)$ is open in G, where $p: G\to G/K$ is the natural projection. G/K endowed with this topology is a topological group.

2. Preliminary definitions and lemmas

Let us start with the following two definitions.

Definition 2.1. *We say that a topological group $(Y,+)$ is σ-bounded provided for every neighbourhood $U\subset Y$ of zero there is a sequence $\{x_n\}_{n\in\mathbb{N}}\subset Y$ with*

$$Y=\bigcup_{n\in\mathbb{N}}U+x_n.$$

It is easily seen that every topological group having a countable dense subset is σ-bounded. For further details concerning the σ-bounded spaces refer to [7] (p. 88) and [8] (p. 125).

Definition 2.2. *Let X and D be nonempty sets, $D \subset X$, $M \subset 2^X$, and Y be a topological space. We say that a function $f : D \to Y$ is M-measurable if $f^{-1}(U) \in M$ for every open set $U \subset Y$.*

In what follows, given a function $f : X \to Y$ and $P \subset X$, by f_P we denote the restriction of f to the set P.

Now we prove two lemmas.

Lemma 2.3. *Suppose that $(Y, +)$ is a σ-bounded topological group, $(X, +)$ is a semigroup with zero endowed with a topology (Y and X need not to be commutative),*

(H_4) M is a family of subsets of X such that there is a σ-ideal $\Im_0 \subset 2^X$ with $X \notin \Im_0$ and

$$0 \in int\ \{x \in X : (x + B) \cap B \neq \emptyset\} \text{ for every } B \in M \setminus \Im_0, \tag{2.3}$$

and $P \in 2^X \setminus \Im_0$. Let $g : X \to Y$ be an additive function such that the function g_P is M-measurable. Then g is continuous at 0.

Proof. Fix a neighbourhood $U \subset Y$ of zero. There is an open neighbourhood $V \subset Y$ of zero such that $V - V \subset U$. Let $\{x_n\}_{n\in\mathbb{N}} \subset Y$ be a sequence with

$$Y = \bigcup_{n\in\mathbb{N}} V + x_n.$$

Then

$$P \subset X = g^{-1}(Y) = \bigcup_{n\in\mathbb{N}} g^{-1}(V + x_n).$$

Thus there exists $k \in \mathbb{N}$ with

$$B := g^{-1}(V + x_k) \cap P \in M \setminus \Im_0.$$

Put $W = int\ \{x \in X : (x + B) \cap B \neq \emptyset\}$. According to (2.3), $0 \in W$.

Take $x \in W$. There are $z, w \in B$ with $x+z = w$. Hence $g(x)+g(z) = g(w)$ and consequently $g(x) = g(w) - g(z)$. So we have shown that

$$g(W) \subset (V + x_k) - (V + x_k) \subset U,$$

which means that g is continuous at zero. ∎

Lemma 2.4. *Let X, M be as in Lemma* 2.3 Suppose that $(G, +)$ is a commutative topological group, K is a subgroup of G, $f : X \to G$ is a function satisfying (1.1), f is M-measurable, X is divisible by 2 (not necessarily uniquely),

$$2^n B \in \Im_0 \text{ for every } n \in \mathbb{N}, B \in M \cap \Im_0, \tag{2.4}$$

(H_5) for every $n \in \mathbb{N}$ the set $2^{-n}K := \{x \in G : 2^n x \in K\}$ is at most countable and, for every neighbourhood $W \subset G$ of zero, $G = \bigcup_{n\in\mathbb{N}} 2^n W$.

Then the function $h := p \circ f : X \to G/K$ is continuous at zero.

Proof. Fix a neighbourhood $U \subset G/K$ of zero. There is an open neighbourhood $V \subset G/K$ of zero such that $V - V \subset U$. Put $W = p^{-1}(V)$. Then W is an open neighbourhood of 0 in G and

$$X = f^{-1}(G) = f^{-1}(\bigcup_{n\in\mathbb{N}} 2^n W) = \bigcup_{n\in\mathbb{N}} f^{-1}(2^n W).$$

Whence, for some $m \in \mathbb{N}$, $B := f^{-1}(2^m W) \notin \Im_0$.

Take $x \in f^{-1}(2^m W)$. Then $f(x) \in 2^m W$. Further, there is $z \in X$ with $x = 2^m z$ and, by induction, from (1.1) we get $f(x) - 2^m f(z) \in K$. Thus

$$2^m f(z) \in K + f(x) \subset K + 2^m W,$$

which means that $f(z) \in W + 2^{-m}K$. Hence $x = 2^m z \in 2^m f^{-1}(W + 2^{-m}K)$. So we have shown that

$$f^{-1}(2^m W) \subset 2^m f^{-1}(W + 2^{-m}K). \tag{2.5}$$

Suppose that $f^{-1}(W + y) \in \Im_0$ for every $y \in 2^{-m}K$. Then, by (2.4), $2^m f^{-1}(W + y) \in \Im_0$ for every $y \in 2^{-m}K$ and consequently $2^m f^{-1}(W + 2^{-m}K) \in \Im_0$, because $2^{-m}K$ is countable. Whence, in view of (2.5), $B \in \Im_0$, which brings a contradiction.

In this way we have proved that for some $y_0 \in 2^{-m}K$

$$D := f^{-1}(W + y_0) \in M \setminus \Im_0.$$

Thus, according to (2.3), $0 \in T := int\ \{ x \in X : (x+D) \cap D \neq \emptyset \}$. Moreover, since h is additive,

$$h(T) \subset h(D) - h(D) = p \circ f(D) - p \circ f(D) \subset (V + p(y_0)) - (V + p(y_0)) \subset U$$

(cf. the last part of the proof of Lemma 2.3). This yields the statement. ∎

Remark 1. We have the following examples of families M satisfying (H_4):

1. X is a locally compact topological group, M is the family of Haar measurable subsets of X and $\Im_0 = \{B \subset X : B$ is locally of Haar measure zero$\}$ (see e.g. [14]);
2. X is a group endowed with a topology such that every non-empty open set is of the second category of Baire and every translation is continuous, $M = \{B \subset X : B$ has the Baire property$\}$ and $\Im_0 = \{B \subset X : B$ is of the first category$\}$ (see [13] and [5], Proposition 1);
3. X is a Polish abelian group, $M = \{B \subset X : B$ is Christensen measurable$\}$ and $\Im_0 = \{B \subset X : B$ is a Christensen zero set$\}$ (see [9]);
4. X is an abelian semigroup with 0 endowed with a topology generated by a complete metric and such that all translations are continuous, $M = \{B \subset X : B$ is universally measurable$\}$ and $\Im_0$ is the σ-ideal generated by the family $\Im_1 = \{B \in M : 0 \notin int\{x \in X : (x+B) \cap B \neq \emptyset\}\}$ (see e.g. [7], Theorem 7.1 and [8], Theorem 1);

5. X is a semigroup with 0 endowed with a topology such that all translations are continuous at 0, $\Im_0 = \{\emptyset\}$, and $M = \{U \setminus B : B \in L$ and $U \subset X$ is a non-empty open set$\}$, where $L \subset 2^X$ is an ideal such that, for every neighbourhood $W \subset X$ of zero, $y \in X$, $A \in L$, we have $y + A, 2A \in L$ and $y + W \notin L$ (see [5], Proposition 1).

It is easily seen that if X (in these examples) is a real linear space and the topology on it is semilinear (see [13]), then (2.4) holds, too.

Remark 2. The functions g and h in the statements of Lemmas 2.3 and 2.4 need not to be continuous at points $x \neq 0$. Namely, for Lemma 2.3 we take $(Y, +) = (\mathbb{R}, +)$ with the usual topology and $(X, +) = ([0, +\infty), +)$ with the topology generated by the basis

$$T = \{[a, b) + k\mathbb{N} : k \in \mathbb{N}, a, b \in (0, +\infty), a < b\} \;\cup\; \{[a, b) : 0 < a < b < 1\}.$$

Then it is easy to see that the topology on X is Hausdorff and every neighbourhood of a point $x \geq 1$ contains a subset of the form $[x, b) + k\mathbb{N}$ with some $b \in (x, +\infty)$ and $k \in \mathbb{N}$. Thus the function $g : X \to Y$, given by: $g(x) = x$ for $x \in X$, is additive, continuous at 0, and discontinuous at every point $x \geq 1$. Moreover it is M-measurable with $M = \{B \subset [0, +\infty) : B$ has the Baire property with respect to the usual topology in $\mathbb{R}\}$. Next, (H_4) holds with $\Im_0$ being the family of first category (with respect to the usual topology in $\mathbb{R}$) subsets of $[0, +\infty)$ (cf. example 2 of Remark 1).

Taking $G = Y$, $K = \sqrt{2}\mathbb{Z}$ and $f = g$, we get an example for Lemma 2.4. In fact, since, for each $k \in \mathbb{N}$, the set $\{km - \sqrt{2}n : n, m \in \mathbb{N}\}$ is dense in $\mathbb{R}$ (with the usual topology), the set $p(k\mathbb{N})$ is dense in G/K and consequently $p([a, b) + k\mathbb{N}) = G/K$ for every $a, b \in (0, +\infty)$, $a < b$. Hence $h = p \circ f$ is discontinuous at every point $x \in X, x \geq 1$.

Remark 3. For instance, every real linear topological space and the multiplicative group of non-zero complex numbers (with the usual topology) satisfy (H_5) with any countable subgroup K.

3. The main theorems

We need one more definition.

Definition 3.1. *Let $(Y, +)$ be a commutative semigroup. A family $T \subset 2^Y$ is translation invariant (abbreviated in the sequel to t.i.) in Y provided*

$$x + B \notin T \text{ for every } B \in 2^X \setminus T, x \in Y$$

and

$$x + B \in T \text{ for every } B \in T, x \in Y.$$

Given a non-empty set Y and $\Im \subset 2^Y$ we put

$$\Omega(\Im) = \{D \subset Y^2 : \text{there is } B_D \in \Im \text{ with } D[x] \in \Im \text{ for } x \in Y \setminus B_D\},$$

where $D[x] := \{y \in Y : (y, x) \in D\}$ (cf. [10] and [5]). (The condition defining $\Omega(\Im)$ is an abstract equivalent of the Fubini Theorem). Further, we say that a property $P(x), x \in D \subset Y$ holds $\Im$-almost everywhere (abbreviated to $\Im$-a.e.) in Y provided there is a set $B \in \Im$ such that the property holds for every $x \in D \setminus B$.

Now, we have all tools to prove the following.

THEOREM 3.2. *Suppose that* (H_1)–(H_4) *and one of the following two conditions are valid.*

(i) *G is σ-bounded.*

(ii) *G is commutative and (H_5) and (2.4) hold.*

Let $\Im \subset 2^X$ be a t.i. ideal in X with

$$(D \cup E) \setminus B \in M \text{ for every } D \in M, B, E \in \Im, \tag{3.6}$$

and $f : X \to G$ be a function satisfying

$$f(x + y) - f(x) - f(y) \in K \quad \Omega(\Im) - a.e. \text{ in } X^2$$

which is M-measurable. Then there exists an additive function $A : X \to G$ with

$$f(x) - A(x) \in K \quad \Im - a.e. \text{ in } X. \tag{3.7}$$

Furthermore, if (H_3') holds, then A can be chosen continuous at 0 and, if, additionally, the following two conditions are valid:

$$\text{the translation } X \ni x \to x + y \text{ is continuous at 0 for every } y \in X, \tag{3.8}$$

$$W + y \notin \Im \text{ for every } y \in X \text{ and } W \subset X \text{ with } 0 \in int\, W, \tag{3.9}$$

then such A (continuous at 0) is unique.

PROOF. Put $g_0 = p \circ f$. Then

$$g_0(x + y) = g_0(x) + g_0(y) \quad \Omega(\Im) - \text{a.e. in } X^2.$$

Thus, according to Theorem 1 in [5], there exists an additive function $g : X \to G/K$ and $B \in \Im$ such that $g(x) = g_0(x)$ for $x \in X \setminus B$. Let $h : X \to G$ be such that $h(x) = f(x)$ for $x \in X \setminus B$ and $h(x) \in g(x)$ for $x \in B$. It is easily seen that, by (3.6), g and h are M-measurable. Hence, in view of Lemmas 2.3 and 2.4, g is continuous at 0 and, by Lemma 1 in [3], there exists $k : X \to G$, continuous at 0 and such that $k(x) \in g(x)$ for $x \in X$. Next, on account of Theorem 2.1 in [4], there is an additive $A : X \to G$ such that $k(x) - A(x) \in K$ for $x \in X$, which yields (3.7); moreover, if (H_3') holds, then A can be chosen continuous at 0. It remains to show the uniqueness.

Suppose that (3.8) and (3.9) hold and $A_0 : X \to G$ is also an additive function which is continuous at 0 and satisfies

$$f(x) - A_0(x) \in K \quad \Im - \text{a.e. in } X.$$

Then $A(x) - A_0(x) \in K$ $\Im$-a.e. in X. Fix neighbourhoods $V_0, V, U \subset G$ of zero such that $U \cap K = \{0\}$, $V_0 - V_0 \subset V$, and $V - V \subset U$. Since A and A_0 are continuous at 0, there is a neighbourhood $W \subset X$ of 0 with $A(W), A_0(W) \subset V_0$. Further, there is $B_0 \in \Im$ with $A(x) - A_0(x) \in K$ for $x \in X \setminus B_0$. Put $D = W \setminus B_0$ and

$$D_0 := \{x \in X : (x + D) \cap D \neq \emptyset\}.$$

Then we have

$$A(x) - A_0(x) \in [(V_0 - V_0) - (V_0 - V_0)] \cap K \subset U \cap K = \{0\} \text{ for } x \in D_0$$

and, by Proposition 1 in [5], $0 \in \mathit{int}\, D_0$.

In this way we have proved that there is a neighbourhood $W_0 \subset X$ of 0 such that $A_0(x) = A(x)$ for every $x \in W_0$. Take $x \in X$. According to (H_3) there are $n \in \mathbb{N}$ and $y \in W_0$ with $x = 2^n y$. Thus

$$A_0(x) = 2^n A_0(y) = 2^n A(y) = A(x).$$

Consequently $A = A_0$. ∎

Remark 4. The examples 1–3 and 5 in Remark 1 satisfy (3.6) and (3.9).

Remark 5. Let M and $\Im_0$ be as in example 4 of Remark 1. Then $M_0 = \{(D \setminus B) \cup E : D \in M \text{ and } B, E \in \Im_0\}$ satisfies (H_4) and (3.6) (with M replaced by M_0).

For the proof of our last theorem we need a proposition.

Proposition 3.3. *Suppose that $(X, +)$ is a group, $\Im$ is a t.i. ideal in X, $\mathbb{F}$ is a field (not necessarily commutative), $K \subset \mathbb{F} \setminus \{0\}$, $K \neq \emptyset$, and $f : X \to \mathbb{F}$ satisfies*

$$f(x + y) \in K f(x) f(y) \quad \Omega(\Im)\text{-a.e. in } X^2. \tag{3.10}$$

Then either $f(x) = 0$ $\Im$-a.e. in X or there exists a function $f_0 : X \to \mathbb{F} \setminus \{0\}$ such that $f(x) = f_0(x)$ $\Im$-a.e. in X and

$$f_0(x + y) f_0(y)^{-1} f_0(x)^{-1} \in K \quad \Omega(\Im)\text{-a.e. in } X^2. \tag{3.11}$$

Proof. Let $P = \{(x, y) \in X^2 : f(x+y) \notin K f(x) f(y)\}$. Then $P \in \Omega(\Im)$. First suppose that $D_0 := f^{-1}(\{0\}) \in \Im$. Take $z_0 \in \mathbb{F} \setminus \{0\}$ and define $f_0 : X \to \mathbb{F} \setminus \{0\}$ by $f_0(x) = f(x)$ for $x \in X \setminus D_0$ and $f_0(D_0) = \{z_0\}$. Clearly $f(x) = f_0(x)$ $\Im$-a.e. in X. Next, for every $(x, y) \in X^2 \setminus [(D_0 \times X) \cup (X \times D_0) \cup P]$ we have $f(x + y) \in K f(x) f(y)$, which means that $f(x + y) \neq 0$ and consequently

$$f_0(x + y) = f(x + y) \in K f(x) f(y) = K f_0(x) f_0(y).$$

Since $(D_0 \times X) \cup (X \times D_0) \cup P \in \Omega(\Im)$, (3.11) holds.

It remains to consider the case $D_0 \notin \Im$. Then there exists $z \in D_0$ such that $P[z] \in \Im$. Note that

$$f(x+z) \in Kf(x)f(z) = \{0\} \text{ for } x \in X \setminus P[z]$$

and $P[z] + z \in \Im$. Thus $f(x) = 0$ $\Im$-a.e. in X. This completes the proof. □

Finally we have the following.

THEOREM 3.4. *Suppose that $(X, +)$ is a group satisfying (H_3), (H_4) holds, $\mathbb{F}$ is a field (not necessarily commutative) endowed with a topology such that $(\mathbb{F} \setminus \{0\}, \cdot)$ is a topological group, K is a multiplicative normal and discrete subgroup of $\mathbb{F}$, one of conditions (i),(ii) of Theorem* 3.2 (with $G = \mathbb{F} \setminus \{0\}$) holds, $\Im$ is as in Theorem 3.2, and $f : X \to \mathbb{F}$ is a function satisfying (3.10) which is M-measurable. Then either $f(x) = 0$ $\Im$-a.e. in X or there exists a solution $g : X \to \mathbb{F} \setminus \{0\}$ of the functional equation

$$g(x+y) = g(x)g(y)$$

such that $f(x)g(x)^{-1} \in K$ $\Im$-a.e. in X.

Furthermore, if (H_3') holds, then g can be chosen continuous at 0 and if, additionally, (3.8) and (3.9) are valid, such g (continuous at 0) is unique.

PROOF. Suppose that $\{x \in X : f(x) \neq 0\} \notin \Im$. According to Proposition 3.3 there is $f_0 : X \to \mathbb{F} \setminus \{0\}$ such that $f(x) = f_0(x)$ $\Im$-a.e. in X and (3.11) holds. Now, in view of (3.6), it suffices to use Theorem 3.2 for f_0 (with $(G, +) = (\mathbb{F} \setminus \{0\}, \cdot)$). This completes the proof. □

REFERENCES

[1] K. Baron and PL. Kannappan, *On the Pexider difference.* Fund. Math. 134 (1990), 247–254.

[2] K. Baron and PL. Kannappan, *On the Cauchy difference.* Aequationes Math. 46 (1993), 112–118.

[3] J. Brzdęk, *On the Cauchy difference.* Glasnik Mat. 27 (47) (1992), 263–269.

[4] J. Brzdęk, *The Cauchy and Jensen differences on semigroups.* Publ. Math. Debrecen 48 (1996), 117-136.

[5] J. Brzdęk, *On almost additive functions.* Bull. Austral. Math. Soc. 54 (1996), 281–290.

[6] J. Brzdęk and A. Grabiec, *Remarks to the Cauchy difference.* In: Stability of mappings of Hyers-Ulam type, ed. by Th. M. Rassias and J. Tabor, Hadronic Press, 1994, 23–30.

[7] J. P. R. Christensen, *Topology and Borel structure.* [North-Holland Mathematical Studies 10], North-Holland, Amsterdam - London, American Elsevier, New York, 1974.

[8] J. P. R. Christensen, *Borel structure in groups and semigroups.* Math. Scand. 28 (1971), 124–128.

[9] P. Fisher and Z. Słodkowski, *Christensen zero sets and measurable convex functions.* Proc. Amer. Math. Soc. 79 (1980), 449–453.

[10] R. Ger, *Almost additive functions on semigroups and a functional equation.* Publ. Math. Debrecen 26 (1979), 219–228.

[11] G. Godini, *Set-valued Cauchy functional equation.* Rev. Roumaine Math. Pure Appl. 20 (1975), 1113–1121.

[12] D. H. Hyers and Th. M. Rassias, *Approximate homomorphisms.* Aequationes Math. 44 (1992), 125–153.

[13] Z. Kominek and M. Kuczma, *Theorems of Bernstein-Doetsch, Piccard and Mehdi and semilinear topology.* Arch. Math. (Basel) 52 (1989), 595–602.

[14] K. Stromberg, *An elementary proof of Steinhaus's theorem.* Proc. Amer. Math. Soc. 36 (1972), 308.

Department of Mathematics,
Pedagogical Academy, Podchorążych,
30–084 Cracow, Poland

Received: 10.07.96.