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OSCILLATION AND MULTILINEAR STIELTJES INTEGRAL

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Abstract. In this note we consider oscillation of regulated functions. We improve and simplify the proof of the existence theorem for multilinear Stieltjes integral in the Riemann-Stieltjes and Moore-Pollard sense and introduce multilinear Henstock-Kurzweil-Stieltjes integral.

1. Introduction

Notations. Let X, Y and $X_j, j = 1, \ldots, p$ be linear normed spaces. Let $L(X_1, \ldots, X_p; Y)$ denote the linear normed space of bounded multilinear transformations $A : X_1 \times \cdots \times X_p \to Y$.

The existence of the Stieltjes multilinear integral of f_i relativ to g, in the case when the function g is of bounded semivariation, f_i are regulated functions, and $X_j, j = 1, \ldots, p$ are Banach spaces, was proved in [4]. In the present paper we simplify and improve the proof, assuming only Y to be a Banach space. Furthermore we suggest a definition of the multilinear Stieltjes integral in Henstock- Kurzweil sense.

Definition 1.1. *Let (M, d) be a metric space, $(X, |\cdot|)$ a linear normed space and A a subset of M. Let f be a mapping of A into X. The* oscillation *of f in A , is defined to be*

$$\omega(f, A) = \sup\{|f(t) - f(s)|,\ s, t \in A\}$$

Let a be a cluster point of A. The oscillation of f at the point a with respect to A is

$$\omega(f, a, A) = \inf_V \omega(A \cap V)$$

where V runs over the set of neighborhoods of a.

Definition 1.2. *A mapping $f : [a, b] \mapsto X$ is called a* regulated function *if it has one-sided limits at every point of $[a, b]$.*

REMARK 1.3. If $f : [a, b] \mapsto X$ is a regulated function and $s_0 \in (a, b)$ then the oscillation of the function f at the point s_0 is

$$\omega(f, s_0, [a, b]) =$$
$$\max\{|f(s_0 + 0) - f(s_0)|, |f(s_0) - f(s_0 - 0)|, |f(s_0 + 0) - f(s_0 - 0)|\},$$

and similarly

$$\omega(f, a, [a, b]) = |f(a + 0) - f(a)|, \quad \omega(f, b, [a, b]) = |f(b) - f(b - 0)|.$$

LEMMA 1.4. *Let $f : [a, b] \mapsto X$ be a regulated function and $\epsilon > 0$. Then there exists a subdivision E of the interval $[a, b]$,*

$$E = \{t_0, t_1, \ldots, t_n\}, \quad a = t_0 \leq t_1 \leq \cdots \leq t_n = b$$

such that the oscillation of f in each of the open intervals $I_i = (t_{i-1}, t_i)$ is $< \epsilon$.

PROOF. Given $\epsilon > 0$. For every $x \in [a, b]$ there is an open interval $V_x = (x - \delta_x, x + \delta_x)$ such that $|f(s) - f(t)| < \epsilon$ if either both s, t are in $(x - \delta_x, x)$ or both in $(x, x + \delta_x)$.
The intervals $U_x = (x - \delta_x/2, x + \delta_x/2)$ cover $[a, b]$. There exists a finite subfamily of such intervals $U_{x_i}, i = 1, \ldots, n - 1$, where x_i is an increasing sequence, which is a covering of $[a, b]$. We take $t_i = x_i, i = 1, \ldots, n-1, t_0 = a$ and $t_n = b$. Then either $x_i \in V(x_{i-1})$ or $x_{i-1} \in V(x_i)$ and hence

$$|f(s) - f(t)| < \epsilon \quad \text{i.e.} \quad \omega(f, (t_{i-1}, t_i)) < \epsilon.$$

□

COROLLARY 1.5. *Given $\epsilon > 0$. Let Q denote the set of the points at which the oscillation of a regulated function f is $\geq \epsilon$. Then Q is a finite set.*

2. SEMIVARIATION

DEFINITION 2.1. *Let $A \in L(X_1, \ldots, X_k, \ldots, X_p; Y)$, $\quad g : [a, b] \mapsto X_k$,*

$$P = \{t_0, t_1, \ldots, t_n\}, \quad a = t_0 \leq t_1 \leq \cdots \leq t_n = b.$$

The function g is of bounded semivariation *relative to A if there exists a positive constant M such that*

$$|\sum_{i=1}^{n} A[x_i^1, \ldots, x_i^{k-1}, g(t_i) - g(t_{i-1}), x_i^{k+1}, \ldots, x_i^p]\,|$$

is less than

$$M \cdot \max_i |x_i^1| \cdots \max_i |x_i^{k-1}| \cdot \max_i |x_i^{k+1}| \cdots \max_i |x_i^p|$$

for all subdivisions P of $[a, b]$ and all $x_i^j \in X_j$, $\quad j = 1, \ldots p$, $\quad j \neq k$, $\quad i = 1, \ldots n$.

$$P = \{t_0, t_1, \ldots, t_n\}, \quad a = t_0 \leq t_1 \leq \cdots \leq t_n = b.$$

The semivariation *of g relative to A, $SV(g,A,[a,b])$, is defined as*

$$\sup_P \Big\{ |\sum_{i=1}^{n} A[x_i^1,\dots,x_i^{k-1},g(t_i)-g(t_{i-1}),x_i^{k+1},\dots,x_i^p]|, \quad |x_i^j|\le 1, \quad x_i^j\in X_j \Big\}$$

The supremum is taken over all subdivisions P and all $x_i^j\in X_j, |x_i^j|\le 1$.

Remark 2.2. It is obvious that if g is of bounded variation than g is also of bounded semivariation.

The proofs of the next two lemmas follow from Definition 2.1.

Lemma 2.3. *If $[a_i,b_i]$, $i=1,\dots n$, are non-overlapping intervals such that*

$$\bigcup_i^n [a_i,b_i]\subseteq [a,b]$$

then

$$|\sum_{i=1}^{n} A[x_i^1,\dots,x_i^{k-1},g(b_i)-g(a_i),x_i^{k+1},\dots,x_i^p]\,|$$

is less than

$$\max_i |x_i^1|\cdots \max_i |x_i^{k-1}|\cdot \max_i |x_i^{k+1}|\cdots \max_i |x_i^p|\cdot SV(g,A,[a,b]).$$

Lemma 2.4. *If $c\le a\le b\le d$, then $SV(g,A,[a,b])\le SV(g,A,[c,d])$*

Lemma 2.5. *Let Y and $X_j, j=1,\dots,p$, be linear normed spaces. Let $A\in L(X_1,\dots,X_k,\dots,X_p;Y)$, let $g:[a,b]\mapsto X_k$ be a function of bounded semivariation and let*

$$P=\{t_0,t_1,\dots,t_n\}, \quad a=t_0\le t_1\le\cdots\le t_n=b.$$

Suppose that the vectors $v_i^j,u_i^j\in X_j$, $j\ne k$, $i=1,\dots n$, satisfy

$$|v_i^j-u_i^j|<\epsilon,$$

and denote $M_j=\sup_i\{1,|u_i^j|,|v_i^j|\}$.
Then the sum

$$|S| = |\sum_{i=1}^{n}\big\{A[v_i^1,\dots,v_i^{k-1},g(t_i)-g(t_{i-1}),v_i^{k+1},\dots,v_i^p] - A[u_i^1,\dots,u_i^{k-1},g(t_i)-g(t_{i-1}),u_i^{k+1},\dots,u_i^p]\ \}\,|$$

is less than $\epsilon\cdot M\cdot SV(g,A,[a,b])$, where $M=p\cdot M_1\cdots M_{k-1}\cdot M_{k+1}\cdots M_p$.

PROOF. Since A is a multilinear operator we can rewrite S .

$$|S| = |\sum_{i=1}^{n} \{A[v_i^1, v_i^2 \ldots, v_i^{k-1}, g(t_i) - g(t_{i-1}), v_i^{k+1}, \ldots, v_i^p] \\ - A[u_i^1, v_i^2 \ldots, v_i^{k-1}, g(t_i) - g(t_{i-1}), v_i^{k+1}, \ldots, v_i^p]\} \\ + \sum_{i=1}^{n} \{A[u_i^1, v_i^2 \ldots, v_i^{k-1}, g(t_i) - g(t_{i-1}), v_i^{k+1}, \ldots, v_i^p] \\ - A[u_i^1, u_i^2 \ldots, v_i^{k-1}, g(t_i) - g(t_{i-1}), v_i^{k+1}, \ldots, v_i^p]\} \\ \cdots \\ + \sum_{i=1}^{n} \{A[u_i^1, u_i^2 \ldots, u_i^{k-1}, g(t_i) - g(t_{i-1}), u_i^{k+1}, \ldots, v_i^p] \\ - A[u_i^1, \ldots, u_i^{k-1}, g(t_i) - g(t_{i-1}), u_i^{k+1}, \ldots, u_i^p] \} |.$$

So we have

$$|S| \le \epsilon \cdot M_2 \cdots M_{k-1} \cdot M_{k+1} \cdots M_p \cdot SV(g, A, [a, b]) \\ + M_1 \cdot \epsilon \cdot \cdots M_{k-1} \cdot M_{k+1} \cdots M_p \cdot SV(g, A, [a, b]) \\ \cdots \\ + M_1 \cdots M_{k-1} \cdot M_{k+1} \cdots M_{p-1} \cdot \epsilon \cdot SV(g, A, [a, b]) \\ < \epsilon \cdot M \cdot SV(g, A, [a, b]).$$

∎

LEMMA 2.6. *Let Y and $X_j, j = 1, \ldots, p$, be linear normed spaces. Let $A \in L(X_1, \ldots, X_k, \ldots, X_p; Y)$, let $g : [a, b] \mapsto X_k$ be a function of bounded semivariation and let*

$$P = \{t_0, t_1, \ldots, t_n\}, \quad a = t_0 < t_1 < \cdots < t_n = b.$$

Suppose that the vectors $v_i^j, u_i^j, y_i^j, x_i^j \in X_j, \quad j \ne k, \quad i = 1, \ldots n$, satisfy

$$|v_i^j - u_i^j| < \epsilon, \quad |y_i^j - x_i^j| < \epsilon,$$

and that $M_j = \sup_i\{1, |u_i^j|, |v_i^j|, |x_i^j|, |y_i^j|\}$.
Then the sum

$$|S| = |\sum_{i=1}^{n} \{A[v_i^1, \ldots, v_i^{k-1}, g(t_i - 0) - g(t_{i-1} + 0), v_i^{k+1}, \ldots, v_i^p] \\ - A[u_i^1, \ldots, u_i^{k-1}, g(t_i - 0) - g(t_{i-1} + 0), u_i^{k+1}, \ldots, u_i^p] \} \\ + \sum_{i=1}^{n-1} \{A[y_i^1, \ldots, y_i^{k-1}, g(t_i + 0) - g(t_i - 0), y_i^{k+1}, \ldots, y_i^p] \\ - A[x_i^1, \ldots, x_i^{k-1}, g(t_i + 0) - g(t_i - 0), x_i^{k+1}, \ldots, x_i^p] \} |$$

is less than $\epsilon \cdot M \cdot SV(g, A, [a, b])$, where $M = p \cdot M_1 \cdots M_{k-1} \cdot M_{k+1} \cdots M_p$.

PROOF. Let $\epsilon_1 > 0$. Since A is a bounded operator we can chose points t_i' and t_i'' such that $t_i \in (t_i', t_i'')$, $t_i'' < t_{i+1}$ and that the sum

$$\begin{aligned}|S_1| = |\sum_{i=1}^{n} \{ & A[v_i^1, \dots, v_i^{k-1}, g(t_i') - g(t_{i-1}''), v_i^{k+1}, \dots, v_i^p] \\ & - A[u_i^1, \dots, u_i^{k-1}, g(t_i') - g(t_{i-1}''), u_i^{k+1}, \dots, u_i^p] \} \\ + \sum_{i=1}^{n-1} \{ & A[y_i^1, \dots, y_i^{k-1}, g(t_i'') - g(t_i'), y_i^{k+1}, \dots, y_i^p] \\ & - A[x_i^1, \dots, x_i^{k-1}, g(t_i'') - g(t_i'), x_i^{k+1}, \dots, x_i^p] \} |\end{aligned}$$

differs from S by less than ϵ_1. It follows from Lemma 2.5 that $|S_1| < \epsilon \cdot M \cdot SV(g, A, [a, b])$, so we have

$$|S| \leq |S - S_1| + |S_1| < \epsilon_1 + \epsilon \cdot M \cdot SV(g, A, [a, b]).$$

Since ϵ_1 is arbitrary small, we have that

$$|S| \leq |S - S_1| + |S_1| < \epsilon \cdot M \cdot SV(g, A, [a, b]).$$

□

3. Multilinear Stieltjes integral

DEFINITION 3.1. *Let Y and $X_j, j = 1, \dots, p$, be linear normed spaces. Let $A \in L(X_1, \dots, X_k, \dots, X_p; Y)$, let $g : [a, b] \mapsto X_k$ and let $f_j : [a, b] \mapsto X_j$, $j = 1, \dots, p$, $j \neq k$ .For the partition*

$$P = \{t_0, t_1, \dots, t_n\}, \quad a = t_0 \leq t_1 \leq \dots \leq t_n = b,$$

we denote $\max\{|t_i - t_{i-1}|\}$ *by* $|P|$.
Let $s_i^j, j = 1, \dots, p, j \neq k$, be $p - 1$ points arbitrarily taken from the interval $[t_{i-1}, t_i]$, by $S(P)$ we denote the Stieltjes sum

$$S(P) = \\ = \sum_{i=1}^{n} \left\{ A[f_1(s_i^1), \dots, f_{k-1}(s_i^{k-1}), g(t_i) - g(t_{i-1}), f_{k+1}(s_i^{k+1}), \dots, f_p(s_i^p)] \right\}.$$

We say that the Stieltjes integral on $[a, b]$ of $f_1, \dots, f_{k-1}, f_{k+1}, \dots, f_p$ with respect to g and A exists in the Riemann sense and has the value I if, for every $\epsilon > 0$ there exist $\delta > 0$ such that

$$|P| < \delta \Rightarrow |I - S(P)| < \epsilon$$

for any choice of the points $t_i \in [a,b]$ *and* $s_i^j \in [t_{i-1}, t_i]$.
We denote

$$I = (RS) \int\limits_{[a,b]}^{A} (f_1, \dots, f_{k-1}, dg, f_{k+1}, \dots, f_p).$$

DEFINITION 3.2. *We say that the Stieltjes integral exists in the Moore-Pollard sense and has the value* I *if, for every* $\epsilon > 0$ *there exist a subdivision* P_0 *such that for every refinement* $P \supseteq P_0$ *we have*

$$|I - S(P)| < \epsilon.$$

We denote

$$I = (MP) \int\limits_{[a,b]}^{A} (f_1, \dots, f_{k-1}, dg, f_{k+1}, \dots, f_p).$$

In the case when g is a regulated function we can define Stieltjes integral in the Young-Moore-Pollard sense.

DEFINITION 3.3. *Let* $g : [a,b] \mapsto X_k$ *be a regulated function. Let*

$$P = \{t_0, t_1, \dots, t_n\}, \quad a = t_0 < t_1 < \cdots < t_n = b.$$

Let $s_i^j, j \neq k$, *be* $p-1$ *points in the* open *interval* (t_{i-1}, t_i). *By* $YS(P)$ *we denote the sum*

$$\sum_{i=1}^{n} A[f_1(s_i^1), \dots, f_{k-1}(s_i^{k-1}), g(t_i - 0) - g(t_{i-1} + 0), f_{k+1}(s_i^{k+1}), \dots, f_p(s_i^p)] +$$
$$\sum_{i=1}^{n-1} A[f_1(t_i), \dots, f_{k-1}(t_i), g(t_i + 0) - g(t_i - 0), f_{k+1}(t_i), \dots, f_p(t_i)] +$$
$$A[f_1(b), \dots, f_{k-1}(b), g(b) - g(b-0), f_{k+1}(b), \dots, f_p(b)] +$$
$$A[f_1(a), \dots, f_{k-1}(a), g(a+0) - g(a), f_{k+1}(a), \dots, f_p(a)].$$

We say that the Stieltjes integral on $[a,b]$ *of* $f_1, \dots, f_{k-1}, f_{k+1}, \dots, f_p$ *with respect to* g *and* A *exists in the Young-Moore-Pollard sense and has the value* I *if, for every* $\epsilon > 0$ *there exist a subdivision* P_0 *such that for every refinement* $P \supseteq P_0$ *we have*

$$|I - YS(P)| < \epsilon$$

We denote

$$I = (Y) \int\limits_{[a,b]}^{A} (f_1, \dots, f_{k-1}, dg, f_{k+1}, \dots, f_p).$$

Similarly we define integrals

$$\int\limits_{[a,b]}^{A} (d_1 f_1, d_2 f_2, \ldots, d_p f_p),$$

where $d_i f_i$ denotes f_i or df_i, see A. Halilovic [2].

In the next theorem we assume only Y to be a Banach space.

THEOREM 3.4. *Let Y be a Banach space and let $X_j, j = 1, \ldots, p$, be linear normed spaces over the same field. Let $A \in L(X_1, \ldots, X_k, \ldots, X_p; Y)$, let $g : [a,b] \mapsto X_k$ be a regulated function of bounded semivariation and let $f_j : [a,b] \mapsto X_j,\ j = 1, \ldots, p,\ j \neq k$ be regulated functions. Then*
(i) The Stieltjes integral

$$I = (Y) \int\limits_{[a,b]}^{A} (f_1, \ldots, f_{k-1}, dg, f_{k+1}, \ldots, f_p)$$

exists in the Young-Moore-Pollard sense.
(ii) The Stieltjes integral

$$I = (MP) \int\limits_{[a,b]}^{A} (f_1, \ldots, f_{k-1}, dg, f_{k+1}, \ldots, f_p)$$

exists in the Moore-Pollard sense if and only if the functions $g : [a,b] \mapsto X_k$ and $f_j : [a,b] \mapsto X_j,\ j = 1, \ldots, p,\ j \neq k$, satisfy conditions (b) and (c) below.
(iii) The Stieltjes integral

$$I = (RS) \int\limits_{[a,b]}^{A} (f_1, \ldots, f_{k-1}, dg, f_{k+1}, \ldots, f_p).$$

exists in the ordinary Riemann-Stieltjes sense if and only if the functions $g : [a,b] \mapsto X_k$ and $f_j : [a,b] \mapsto X_j,\ j = 1, \ldots, p,\ j \neq k$, satisfy conditions $(a), (b)$ and (c) below.
The conditions $(a) - (c)$ are

$$(a)\ A[f_1(s_1), \ldots, f_{k-1}(s_{k-1}), g(t+0) - g(t-0), f_{k+1}(s_{k+1}), \ldots, f_p(s_p)]$$
$$= A[f_1(t), \ldots, f_{k-1}(t), g(t+0) - g(t-0), f_{k+1}(t), \ldots, f_p(t)]$$

for all 3^{p-1} combinations obtained by taking $f_j(s_j) \in \{f_j(t-0), f_j(t), f_j(t+0)\},\ j = 1, \ldots, p,\ j \neq k$, for every $t \in (a,b)$.

$$(b)\ A[f_1(s_1), \ldots, f_{k-1}(s_{k-1}), g(t+0) - g(t), f_{k+1}(s_{k+1}), \ldots, f_p(s_p)]$$
$$= A[f_1(t), \ldots, f_{k-1}(t), g(t+0) - g(t), f_{k+1}(t), \ldots, f_p(t)]$$

for all 2^{p-1} *combinations obtained by taking* $f_j(s_j) \in \{f_j(t), f_j(t+0)\}$, $j = 1, \ldots, p,\ j \neq k$, *for every* $t \in [a, b)$.

$$(c)\ A[f_1(s_1), \ldots, f_{k-1}(s_{k-1}), g(t) - g(t-0), f_{k+1}(s_{k+1}), \ldots, f_p(s_p)]$$

$$= A[f_1(t), \ldots, f_{k-1}(t), g(t) - g(t-0), f_{k+1}(t), \ldots, f_p(t)]$$

for all 2^{p-1} *combinations obtained by taking* $f_j(s_j) \in \{f_j(t-0), f_j(t)\}$, $j = 1, \ldots, p,\ j \neq k$, *for every* $t \in (a, b]$.

Proof. *Existence in the Young-Moore-Pollard sense.* Given $\epsilon > 0$. It follows from Lemma 1.4 that there exists a subdivision E of the interval $[a, b]$,

$$E = \{t_0, t_1, \ldots, t_n\}, \quad a = t_0 \leq t_1 \leq \cdots \leq t_n = b,$$

such that the oscillation of f_j in each of the open intervals $I_i = (t_{i-1}, t_i)$, $i = 1, \ldots, n$, is less than ϵ. Let P be any refinement of E. We compare Y(E) and Y(P). Let $s_i^j, j = 1, \ldots, p, j \neq k$, be $p-1$ points arbitrarily chosen from the interval $[t_{i-1}, t_i]$. We suppose that $t_{i-1} = z_{i,0} < z_{i,1} \cdots < z_{i,r(i)} = t_i$ are new points in the interval $[t_{i-1}, t_i]$ and that $u_{i,e}^j \in [z_{i,e-1}, z_{i,e}]$, $j = 1, \ldots, p,\ j \neq k$, $e = 1, \ldots, r_i$. We consider the difference $S(P) - S(E)$ in the intervals $[t_{i-1}, t_i]$. Since the points t_i are in P and in E, the terms

$$A[f_1(t_i), \ldots, f_{k-1}(t_i), g(t_i+0) - g(t_i-0), f_{k+1}(t_i), \ldots, f_p(t_i)],$$

$$A[f_1(b), \ldots, f_{k-1}(b), g(b) - g(b-0), f_{k+1}(b), \ldots, f_p(b)]$$

and

$$A[f_1(a), \ldots, f_{k-1}(a), g(a+0) - g(a), f_{k+1}(a), \ldots, f_p(a)]$$

vanish, so we have.

$$\begin{aligned}\Delta_i =& A[f_1(s_i^1), \ldots, f_{k-1}(s_i^{k-1}), g(t_i-0) - g(t_{i-1}+0), f_{k+1}(s_i^{k+1}), \ldots, f_p(s_i^p)] \\ &- \sum_{e=1}^{r(i)} A[f_1(u_{i,e}^1), \ldots, f_{k-1}(u_{i,e}^{k-1}), g(z_{i,e}-0) - \\ &\qquad - g(z_{i,e-1}+0), f_{k+1}(u_{i,e}^{k+1}), \ldots, f_p(u_{i,e}^p)] \\ &- \sum_{e=1}^{r(i)-1} A[f_1(z_{i,e}), \ldots, f_{k-1}(z_{i,e}), g(z_{i,e}+0) - \\ &\qquad - g(z_{i,e}-0), f_{k+1}(z_{i,e}), \ldots, f_p(z_{i,e})].\end{aligned}$$

Inserting

$$\begin{aligned}g(t_i-0) - g(t_{i-1}+0) &= g(z_{i,r(i)}-0) - g(z_{i,r(i)-1}+0) + \\ &+ \sum_{e=1}^{r(i)-1} [g(z_{i,e}-0) - g(z_{i,e-1}+0) + g(z_{i,e}+0) - g(z_{i,e}-0)]\end{aligned}$$

we obtain

$$\Delta_i = \sum_{e=1}^{r(i)} \Big\{ A[f_1(s_i^1), \dots, f_{k-1}(s_i^{k-1}), g(z_{i,e}-0)- g(z_{i,e-1}+0), f_{k+1}(s_i^{k+1}), \dots, f_p(s_i^p)]$$
$$-A[f_1(u_{i,e}^1), \dots, f_{k-1}(u_{i,e}^{k-1}), g(z_{i,e}-0)- g(z_{i,e-1}+0), f_{k+1}(u_{i,e}^{k+1}), \dots, f_p(u_{i,e}^p)] \Big\}$$
$$+ \sum_{e=1}^{r(i)-1} \Big\{ A[f_1(s_i^1), \dots, f_{k-1}(s_i^{k-1}), g(z_{i,e}+0)- g(z_{i,e}-0), f_{k+1}(s_i^{k+1}), \dots, f_p(s_i^p)]$$
$$-A[f_1(z_{i,e}), \dots, f_{k-1}(z_{i,e}), g(z_{i,e}+0)- g(z_{i,e}-0), f_{k+1}(z_{i,e}), \dots, f_p(z_{i,e})] \Big\}.$$

Since

$$S(P) - S(E) = \sum \Delta_i,$$

and the oscillation in the intervals (t_{i-1}, t_i) is less than ϵ, by Lemmas 2.3- 2.6 we have

$$|S(P) - S(E)| \leq \epsilon \cdot M \cdot SV(g, A, [a, b]).$$

Since $S(P) \in Y$, and Y is a Banach space, the integral exists in the Young-Moore-Pollard sense.

Existence in the Moore-Pollard sense. Given $\epsilon > 0$. It follows from Lemma 1.4 that there exists a subdivision E of the interval $[a, b]$,

$$E = \{y_0, y_1, \dots, y_m\}, \quad a = y_0 \leq y_1 \leq \dots \leq y_m = b$$

such that the oscillation of f_j in every of the open intervals $I_l = (y_{l-1}, y_l)$, $l = 1, \dots, m$, is less than ϵ. It follows from the conditions (b) and (c) that there exists $\delta > 0$ such that for $t = y_l$ we have

$$\begin{aligned} &|A[f_1(s_1''), \dots, f_{k-1}(s_{k-1}''), g(u) - g(t), f_{k+1}(s_{k+1}''), \dots, f_p(s_p)]- \\ &A[f_1(s_1'), \dots, f_{k-1}(s_{k-1}'), g(v) - g(t), f_{k+1}(s_{k+1}'), \dots, f_p(s_p')]| < \frac{\epsilon}{2m} \end{aligned} \tag{3.1}$$

if $s_j'', s_j' \in [t, t+\delta]$, $j = 1, \dots, p$, $j \neq k$, $u, v \in (t, t+\delta]$, and

$$\begin{aligned} &|A[f_1(s_1''), \dots, f_{k-1}(s_{k-1}''), g(t) - g(u), f_{k+1}(s_{k+1}''), \dots, f_p(s_p)]- \\ &A[f_1(s_1'), \dots, f_{k-1}(s_{k-1}'), g(t) - g(v), f_{k+1}(s_{k+1}'), \dots, f_p(s_p')]| < \frac{\epsilon}{2m} \end{aligned} \tag{3.2}$$

if $s_j'', s_j' \in [t-\delta, t]$, $j = 1, \dots, p$, $j \neq k$, $u, v \in [t-\delta, t)$.

Let now $P_0 = \{t_0, t_1, \ldots, t_n\}$, $\quad a = t_0 \le t_1 \le \cdots \le t_n = b$, be a subdivision of $[a, b]$ such that $P_0 \supseteq E$ and $|P_0| < \delta$. Let $s_i^j, j = 1, \ldots, p, j \ne k$, be $p-1$ points arbitrarily chosen from the interval $[t_{i-1}, t_i]$

Let P be an arbitrary refinement of P_0. We suppose that $t_{i-1} = z_{i,0} \le z_{i,1} \cdots \le z_{i,r(i)} = t_i$ are new points in the interval $[t_{i-1}, t_i]$ and that $u_{i,e}^j \in [z_{i,e-1}, z_{i,e}]$, $j = 1, \ldots, p$, $j \ne k$, $e = 1, \ldots, r_i$. We consider the difference $S(P) - S(P_0)$ in the intervals $[t_{i-1}, t_i]$.

Let

$$\begin{aligned}\Delta_i =& A[f_1(s_i^1), \ldots, f_{k-1}(s_i^{k-1}), g(t_i) - g(t_{i-1}), f_{k+1}(s_i^{k+1}), \ldots, f_p(s_i^p)] \\ & - \sum_{e=1}^{r(i)} \Big\{ A[f_1(u_{i,e}^1), \ldots, f_{k-1}(u_{i,e}^{k-1}), g(z_{i,e}) - \\ & \qquad g(z_{i,e-1}), f_{k+1}(u_{i,e}^{k+1}), \ldots, f_p(u_{i,e}^p)] \Big\}.\end{aligned}$$

Inserting

$$g(t_i) - g(t_{i-1}) = \sum_{e=1}^{r(i)} [g(z_{i,e}) - g(z_{i,e-1})]$$

we obtain

$$\begin{aligned}\Delta_i =& \sum_{e=1}^{r(i)} \Big\{ A[f_1(s_i^1), \ldots, f_{k-1}(s_i^{k-1}), g(z_{i,e}) - \\ & \qquad g(z_{i,e-1}), f_{k+1}(s_i^{k+1}), \ldots, f_p(s_i^p)] \\ & - A[f_1(u_{i,e}^1), \ldots, f_{k-1}(u_{i,e}^{k-1}), g(z_{i,e}) - g(z_{i,e-1}), f_{k+1}(u_{i,e}^{k+1}), \ldots, f_p(u_{i,e}^p)] \Big\}.\end{aligned}$$

Hence

$$\begin{aligned}S(P_0) - S(P) =& \sum_{i=1}^{n} \Delta_i \\ =& \sum_{i=1}^{n} \sum_{e=1}^{r(i)} \Big\{ A[f_1(s_i^1), \ldots, f_{k-1}(s_i^{k-1}), g(z_{i,e}) - \\ & \qquad g(z_{i,e-1}), f_{k+1}(s_i^{k+1}), \ldots, f_p(s_i^p)] \\ & - A[f_1(u_{i,e}^1), \ldots, f_{k-1}(u_{i,e}^{k-1}), g(z_{i,e}) - g(z_{i,e-1}), f_{k+1}(u_{i,e}^{k+1}), \ldots, f_p(u_{i,e}^p)] \Big\} \\ =& \sum{}' + \sum{}''. \end{aligned} \tag{3.3}$$

Some of the $z_{i,e}$ are in E, i.e. coincide with y_l, and we denote by $\sum'$ the sum of terms over those intervals, where at least one of the endpoints $z_{i,e}$ is in E. According to (3.1) and (3.2) we have

$$|\sum{}' | < 2m \cdot \frac{\epsilon}{2m} = \epsilon. \tag{3.4}$$

The oscillation of the functions f_j over every interval, which build the sum denoted by $\sum''$, is less than ϵ. Hence by Lemmas 2.3 and 2.5 we have

$$|\sum{}'' | < \epsilon \cdot M \cdot SV(g, A, [a, b]). \tag{3.5}$$

It follows from (3.3), (3.4) and (3.5) that

$$|S(P_0) - S(P)| = |\sum{}' + \sum{}'' | \leq |\sum{}' | + |\sum{}'' | \leq \epsilon + \epsilon \cdot M \cdot SV(g, A, [a, b])$$

Since $S(P) \in Y$, and Y is a Banach space, the integral exists in Moore-Pollard sense. *Necessity.* Suppose that one of the conditions $(b), (c)$ for example (c) does not hold in a point t . We consider a subdivision P_n which includes the interval $[t - 1/n, t]$. We can chose associated points s_n^j so that

$$\begin{aligned}\Delta_n = |A[f_1(s_n^1), \dots, f_{k-1}(s_n^{k-1}), g(t) - g(t-0), f_{k+1}(s_n^{k+1}), \dots, f_p(s_n^p)] \\ -A[f_1(t), \dots, f_{k-1}(t), g(t) - g(t-0), f_{k+1}(t), \dots, f_p(t)]|\end{aligned}$$

does not converge to 0 when $n \to \infty$. We compare two sums $S_1(P_n)$ and $S_2(P_n)$ which agree excepting that in the interval $[t - 1/n, t]$ we take different associated points, $s^j = s_n^j$ for S_1 and $s^j = t$ for S_2. So we have that $|S_1 - S_2| = \Delta_n$ does not converge to 0 when $n \to \infty$. Consequently $(MP)\int$ does not exist.

Existence in the Riemann-Stieltjes sense. If the conditions $(a), (b)$ are fulfilled then the integral exists in the Moore-Pollard sense, and let I denote its value. By Definition 3.2, for $\epsilon > 0$, there exists a subdivision

$$E = \{y_0, y_1, \dots, y_m\}, \quad a = y_0 < y_1 < \dots, < y_m = b$$

such that

$$P' \supseteq E \Rightarrow |S(P') - I| < \epsilon. \tag{3.6}$$

It follows, from the conditions $(a), (b)$ and (c), that there exists $\delta > 0$ such that for $t = y_l$ we have

$$\begin{aligned}&|A[f_1(s_1''), \dots, f_{k-1}(s_{k-1}''), g(u) - g(t), f_{k+1}(s_{k+1}''), \dots, f_p(s_p)] - \\ &A[f_1(s_1'), \dots, f_{k-1}(s_{k-1}'), g(v) - g(t), f_{k+1}(s_{k+1}'), \dots, f_p(s_p')]| < \frac{\epsilon}{2m}\end{aligned} \tag{3.7}$$

if $s_j'', s_j' \in [t, t + \delta]$, $j = 1, \dots, p$, $j \neq k$, $u, v \in (t, t + \delta]$,

$$\begin{aligned}&|A[f_1(s_1''), \dots, f_{k-1}(s_{k-1}''), g(t) - g(u), f_{k+1}(s_{k+1}''), \dots, f_p(s_p)] - \\ &A[f_1(s_1'), \dots, f_{k-1}(s_{k-1}'), g(t) - g(v), f_{k+1}(s_{k+1}'), \dots, f_p(s_p')]| < \frac{\epsilon}{2m}\end{aligned} \tag{3.8}$$

if $s_j'', s_j' \in [t - \delta, t]$, $j = 1, \dots, p$, $j \neq k$, $u, v \in [t - \delta, t)$, and

$$
\begin{aligned}
&|A[f_1(s''_1),\ldots,f_{k-1}(s''_{k-1}),g(u')-g(u),f_{k+1}(s''_{k+1}),\ldots,f_p(s_p)]- \\
&A[f_1(s'_1),\ldots,f_{k-1}(s'_{k-1}),g(v')-g(v),f_{k+1}(s'_{k+1}),\ldots,f_p(s'_p)]| < \frac{\epsilon}{2m}
\end{aligned} \tag{3.9}
$$

if $s''_j, s'_j \in [t-\delta, t+\delta]$, $j=1,\ldots,p$, $j \neq k$, $u,v \in [t-\delta,t)$, $u',v' \in (t,t+\delta]$. For δ so determined, we consider any subdivision P with $|P|<\delta$. Suppose

$$P = \{t_0,t_1,\ldots,t_n\}, \quad a=t_0 \le t_1 \le \cdots \le t_n = b,$$

and let $s_i^j, j=1,\ldots,p, j\neq k$, be $p-1$ points arbitrarily taken from the interval $[t_{i-1},t_i]$. Suppose $P_1 = P\cup E$. We define associated points $s_{1,i}^j = s_i^j$ in any interval $[t_{i-1},t_i]$ which contains no points of E. In the intervals which contains y_l as an end point we chose associated points to be equal y_l. We can assume that δ is less than $\min|y_l - y_{l-1}|$ so that there is maximum one point y_l in any interval $[t_{i-1},t_i]$. Because of (3.9) we have

$$|S(P)-S(P_1)| < 2m\cdot\frac{\epsilon}{2m}. \tag{3.10}$$

Since $P_1 \supseteq E$ we have

$$|I - S(P_1)| < \epsilon. \tag{3.11}$$

It follows from (3.10) and (3.11) that

$$|I - S(P)| < 2\epsilon.$$

It means that the integral exists in the Riemann-Stieltjes sense. We can prove the necessity of the conditions $(a),(b)$ and (c) in the same way as in (ii). For the condition (a) we consider the intervals $(t-1/n, t+1/n)$. □

Remark. For the necessity in Theorem 3.4 we do not need the assumption that Y is a Banach space, but only that Y is a linear normed space, so we have the following theorem.

THEOREM 3.5. *Let $X_j, j=1,\ldots,p$, and Y be linear normed spaces over the same field. Let $A\in L(X_1,\ldots,X_k,\ldots,X_p;Y)$ and let $f_j:[a,b]\mapsto X_j$, $j=1,\ldots,p$, $j\neq k$, and $g:[a,b]\mapsto X_k$ be regulated functions. Then the Stieltjes integral*

$$I = (MP)\int_{[a,b]}^{A} (f_1,\ldots,f_{k-1},dg,f_{k+1},\ldots,f_p)$$

exists in the Moore-Pollard sense only if the functions $g:[a,b]\mapsto X_k$ and $f_j:[a,b]\mapsto X_j$, $j=1,\ldots,p$, $j\neq k$, satisfy conditions (b) and (c) in

Theorem 3.4;
The Stieltjes integral

$$I = (RS)\int\limits_{[a,b]}^{A} (f_1, \dots, f_{k-1}, dg, f_{k+1}, \dots, f_p).$$

exists in the ordinary Riemann-Stieltjes sense only if the functions $g : [a,b] \mapsto X_k$ *and* $f_j : [a,b] \mapsto X_j,\ j = 1, \dots, p,\ j \neq k$, *satisfy conditions* $(a), (b)$ *and* (c) *in Theorem 3.4.*

Example. Let $M_{m,n}$ denote the linear normed space of all $m \times n$ matrices. We define $i : [-1,1] \mapsto R$ to be

$$i(t) = \begin{cases} 1, & \text{if } t \text{ is a rational number;} \\ 2, & \text{if } t \text{ is an irrational number.} \end{cases}$$

Let a, b and c be real numbers and $b \neq 0$. We define functions $f_1 : [-1,1] \mapsto M_{2,3}$, $g : [-1,1] \mapsto M_{3,2}$ and $f_3 : [-1,1] \mapsto M_{2,4}$ as follows

$$f_1(t) = \begin{bmatrix} b \cdot i(t) & 0 & c \\ b \cdot i(t) & 0 & c \end{bmatrix},$$

$$g(t) = \begin{bmatrix} a & a \\ a & i(t) \\ a & t \end{bmatrix},$$

$$f_3(t) = \begin{bmatrix} i(t) & i(t) & i(t) & i(t) \\ d & d & d & d \end{bmatrix}.$$

The functions f_1, g and f_3 have common discontinuities at all points of the interval $[-1,1]$. We define a multilinear operator $A : M_{2,3} \times M_{3,2} \times M_{2,4} \mapsto M_{2,4}$ as ordinary matrix multiplication, $A(X_{2,3}, X_{3,2}, X_{2,4}) = X_{2,3} \cdot X_{3,2} \cdot X_{2,4}$, where $X_{i,j}$ is a matrix in $M_{i,j}$. Let

$$P = \{t_0, t_1, \dots, t_n\}, \quad -1 = t_0 \leq t_1 \leq \cdots \leq t_n = 1.$$

In every interval $[t_{i-1}, t_i]$ we choose two arbitrary points s_i^1, s_i^3 and form the Stieltjes sum

$$S(P) = \sum_{i=1}^{n} f_1(s_i^1) \cdot [g(t_i) - g(t_{i-1})] \cdot f_3(s_i^3)$$

$$= \sum_{i=1}^{n} \begin{bmatrix} cd(t_i - t_{i-1}) & cd(t_i - t_{i-1}) & cd(t_i - t_{i-1}) & cd(t_i - t_{i-1}) \\ cd(t_i - t_{i-1}) & cd(t_i - t_{i-1}) & cd(t_i - t_{i-1}) & cd(t_i - t_{i-1}) \end{bmatrix}.$$

By Definition 3.1 we have that the RS integral exists and have the value

$$I = (RS) \int_{[-1,1]}^{A} (f_1, dg, f_3) = \lim_{|P| \to 0} S(P) = \begin{bmatrix} 2cd & 2cd & 2cd & 2cd \\ 2cd & 2cd & 2cd & 2cd \end{bmatrix}$$

although the functions f_1, g and f_3 *have a common discontinuity* in every point in the interval $[-1, 1]$.

Multilinear Stieltjes integral in the Henstock-Kurzweil sense.

Let Y and $X_j, j = 1, \ldots, p$, be linear normed spaces. Let $A \in L(X_1, \ldots, X_p; Y)$ and let $f_j : [a, b] \mapsto X_j,\ j = 1, \ldots, p$. Let

$$P = \{t_0, t_1, \ldots, t_n\}, \quad a = t_0 < t_1 < \cdots < t_n = b,$$

be a partition of $[a, b]$. If we consider multilinear Stieltjes integral in the general case, we need an ordered set J which indicates functions and those coordinates where we consider "df ". For example, if $J = (0, 1, 0, 1, 0)$ then we consider the multilinear Stieltjes integral

$$\int_{[a,b]}^{A} (f_1, df_2, f_3, df_4, f_5).$$

If we consider the integral in the Riemann sense then the integral is the limit of the sums

$$\sum_{i=1}^{n} A[f_1(s_i^1), f_2(t_i) - f_2(t_{i-1}), f_3(s_i^3), f_4(t_i) - f_4(t_{i-1}), f_5(s_i^5)],$$

where $s_i^1, s_i^3, s_i^5 \in [t_{i-1}, t_i]$, and in the Henstock sense (see definition below) the integral is the limit of the sums

$$\sum_{i=1}^{n} A[f_1(s_i), f_2(t_i) - f_2(t_{i-1}), f_3(s_i), f_4(t_i) - f_4(t_{i-1}), f_5(s_i)],$$

where $s_i \in [t_{i-1}, t_i]$.

Let the "indicator", set $J = \{e_1, e_2, \ldots, e_p\}$, where $e_j = 1$ or $e_j = 0$, be given. For given J we denote

$$F_i^j = \begin{cases} f_j(s_i), & \text{if } e_j = 0; \\ f_j(t_i) - f_j(t_{i-1}), & \text{if } e_j = 1. \end{cases}$$

We define Henstock-Stieltjes sum to be

$$HS(P) = \sum_{i=1}^{n} A[F_i^1, \ldots, F_i^p].$$

Definition 3.6. *We say that the multilinear Stieltjes integral on* $[a, b]$ *exists in the Henstock-Kurzweil sense and has the value* I*, if for every* $\epsilon > 0$ *there exists* $\delta(s) > 0$*, such that whenever a partition* P *and points* s_i *satisfy*

$$s_i \in [x_i, x_{i-1}] \subset (s_i - \delta(s_i), s_i + \delta(s_i))$$

for $i = 1, \dots, n$*, we have*

$$|I - \sum_{i=1}^{n} A[F_i^1, \dots, F_i^p]| < \epsilon.$$

We write

$$I = (HS) \int_{[a,b]}^{A} (d_1 f_1, \dots, d_p f_p),$$

where the symbol $d_j f_j$ *is defined as follows*

$$d_j f_j = \begin{cases} f_j, & \text{if } e_j = 0; \\ df_j, & \text{if } e_j = 1. \end{cases}$$

Remark The Stieltjes integral which we consider in the example obviously exist in the Henstock-Kurzweil sense. We compare Stieltjes integral in the Riemann, Moore-Pollard, Young and Henstock-Kurzweil sense in [5].

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