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A QUALITATIVE UNCERTAINTY PRINCIPLE FOR CERTAIN HYPERGROUPS

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Abstract. It is known that if the supports of a function $f \in L^1(R^n)$ and its Fourier transform have finite measure then $f = 0$ almost everywhere. We study generalizations of this property for certain classes of locally compact hypergroups.

1. Introduction

It is known that for a locally compact abelian group G, if $f \in L^1(G)$ and the product of the measure of the support of f and its Fourier transform $\hat{f}$ is less than one then $f = 0$ a.e. This result has been generalized to a commutative hypergroup by Kumar [9] (see also Voit [18]). For the definition and notations of hypergroups which is same as 'convos' see [7] and [1] and also the survey [10],[13]. In this paper we first show that for finite hypergroups this property is equivalent to commutativity of K. This result seems to be new even for groups.

For a locally compact hypergroup K equipped with left Haar measure m, such that the C^*-algebra $C^*(K)$ of K is type I, let $\hat{K}$ be the dual space with Plancherel measure μ (see [2], [7]), For $f \in L^1(K)$, let

$$A_f = \{x \in K : f(x) \neq 0\} \text{ and } B_f = \{\pi \in \hat{K} : \pi(f) \neq 0\}.$$

We first show that a finite hypergroup K is commutative if and only if for $f \in L^1(K), m(A_f)\mu(B_f) < 1 \Rightarrow f = 0$.

A hypergroup K is said to have QUP (Qualitative Uncertainty Principle) if $m(A_f) < \infty, \mu(B_f) < \infty \Rightarrow f = 0$ a.e. It has been shown that a central

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hypergroup K (not necessarily commutative) ([4], [14]) with non-compact connected component of $Z = G(K) \cap Z(K)$ has QUP.

2. Main Results

Let K be a finite hypergroup. By ([7], 7.1A) there exists a left Haar measure m for which $m(\{e\}) = 1$ and $m(\{x\}) = \frac{1}{p_{\check{x}^*}\ p_x(e)}$ for each $x \in K$. It is easy to see from ([14], Theorem 1.3 and Theorem 1.7) that a Plancherel measure μ on $\hat{K}$ is given by $\mu(A) = \sum_{\rho\ \in\ A} c_\rho^{-1}$ for $A \subseteq \hat{K}$, where $d_\rho c_\rho \leq 1$, d_ρ being the dimension of representation space $H(\rho)$ of ρ and $c_\rho = \sum_{p\ \in\ A} |\langle \rho(x)u, u\rangle|^2\ m(\{x\})$ for $u \in \mathcal{H}(\rho)$, with $||u|| = 1$. The following theorem rules out any possibility of generalization of ([9], Cor, 2) or ([18], Theorem 2.3) to non-abelian groups or hypergroups.

Theorem 2.1. *If K is a finite hypergroup, then the following are equivalent*

(i) *K is commutative*
(ii) *For $f \in L^1(K)$, $m(A_f)\mu(B_f) < 1 \Rightarrow f = 0$.*

Proof. (i) $\Rightarrow$ (ii) has been proved in ([9], Cor, 2) For (ii) $\Rightarrow$ (i), if K is noncommutative, then there exists $\pi \in \hat{K}$ such that $d_\pi \geq 2$. Let $f = \delta_e \in L^1(K)$, clearly $A_f = \{e\}$ so $m(A_f) = 1$ and

$$B_f = \hat{K} \text{ thus } \mu(B_f) = \sum_{\pi\ \in\ \hat{K}} c_\pi^{-1}.$$

Now by ([15], Theorem 4.4), $1 = \xi_{\{e\}}(e) = \sum_{\pi\ \in\ \hat{K}} k_\pi tr(\pi(\xi_{\{e\}}))$. Also

$$\langle \pi(\delta_{\{e\}}\xi, \eta\rangle = \sum_{x\ \in\ K} \langle \pi(x)\xi, \eta\rangle \frac{1}{p_{\check{x}^*}\ p_x(e)}\ \delta_{\{e\}}(x) = \langle \xi, \eta\rangle,$$

so $tr\ \pi(\xi_{\{e\}}) = d_\pi$. Thus

$$1 = \sum k_\pi\ d_\pi.$$

Now

$$m(A_f)\ m(B_f) = \sum_{\pi\ \in\ B_f} c_\pi^{-1} = \sum_{\pi\ \in\ B_f} k_\pi < \sum_{\pi\ \in\ \hat{K}} k_\pi d_\pi = 1,$$

but $f \neq 0$. ∎

Let K be a hypergroup with a left Haar measure m such that the C^*-algebra $C^*(K)$ of K is of type I. There exists a unique measure μ on $\hat{K}$, the so called Plancherel measure, such that for any $f \in L^1(K) \cap L^2(K)$.

$$\int_K |f(x)|^2 \, dm(x) = \int_{\hat{K}} tr(\pi(f) \, \pi(f^*)) d\mu(\pi).$$

If K is commutative or a central hypergroup (in particular if K is compact) then there is a Plancherel measure on $\hat{K}$ ([7], 7.3I) ([14], Theorem 1.7),([4], Theorem 3.2). K is said to satisfy QUP if for each $f \in L^1(K), m(A_f) < \infty$ and $\mu(B_f) < \infty \Rightarrow f = 0$ a.e.

Remark 2.2. It is easy to see that if K is compact hypergoup or a discrete hypergroup then QUP is violated. More generally it can be shown that QUP is violated in case of hypergroups constructed by substitution of compact and discrete hypergroups (See [19] for construction of substitution hypergroup). In particular QUP is violated in case of join of a compact and a discrete hypergroup (([7], 10.5), [16]). In general, if K is a commutative hypergroup with compact identity component $\{K_0\}$ then QUP is violated. This can be proved as follows:

By ([17], Cor 1.4) K/K_0 is totally disconnected and so is 0-dimensional. Let U be a neighborhood of K_0 such that U^- is compact then U contains an open and closed neighbourhood V which is compact and $U * V \subset U$. So $F = \cup \, V^n$ is a compact open subhypergroup of K/K_0. Then $F_0 = p^{-1}(F)$ is compact open subhypergroup of K. Let $f = \xi_{F_0}$, then $m(A_f) = mF_0$ and $B_f = \{\gamma : \gamma|F_0 = 1\}$. Using Plancherel theorem ([7], 7.3I), it follows that $\mu(B_f) = m(F_0)$. Hence QUP is violated.

Example 2.3. For a prime p, let Ω_p and $\triangle_p$ be the set of p-adic numbers (resp. p-adic integers respectively.) Moreover, let $\wedge_k = \{z \in \Omega_p;\ z_n = 0$ for $n < k\}$ (see [5], §10). Let

$$G = \left\{ \begin{bmatrix} 1 & x_1 & z \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{bmatrix} : x_1, x_2 \in \triangle_p, z \in \Omega_p \right\}$$

and

$$H = \left\{ \begin{bmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} : t \in \triangle_p \right\}$$

and

$$K = G//H.$$

Let $x = (x_1, x_2, z),\ y = (y_1, y_2, w) \in G$ where $x_i, y_i \in \triangle_p$ and $z, w \in \Omega_p$. Then $HxH = HyH$ if and only if $x_2 = y_2, z - w \in x_2 \cdot \triangle_p$. We write

$HxH = (x_2, [z]_{x_2})$. K can be identified with $\bigcup_{K=0}^{\infty} (\wedge_k \backslash \wedge_{k-1}) \times (\Omega_p \backslash \wedge_k)$

$$\delta_{HxH} * \delta_{HyH} = \int_H \delta_{HxtyH}\, dt,$$

where

$$HxtyH = (x_2 + y_2,\ [z + w + (t + x_1)y_2]_{x_2+y_2}).$$

Thus,

$$\delta_{HxH} * \delta_{HyH} = \begin{cases} \delta_{HxyH} & \text{if } y_2 \in (x_2 + y_2)\triangle_p \\ \int_X \delta(x_2 + y_2, [z + w]_{x_2+y_2} + \dot{t})d\dot{t}, & \text{otherwise} \end{cases}$$

where $X = y_2\triangle_p/(x_2 + y_2)\triangle_p$ and $d\dot{t}$ is the normalized Haar measure on X. Clearly K is commutative and $G(K)$ (maximal subgroup) $= \{(0, [z]_0); z \in \Omega_p\}$. Since $\wedge_k$ are the only proper subgroups of Ω_p. The connected component of K is compact. Thus QUP is violated in this case.

We don't know whether QUP is true in general for hypergroups having non-compact connected component or not. The QUP has been proved in [9] if the map $x \to m(x * C)(0 < m(C) < \infty)$ is continuous on K. However, we prove this result for central hypergroups without this technical condition. The technique used is similar to one used in ([3], Theorem 1.2). Notations used are same as in [4]. The following Lemma can be proved as in ([11], Lemma 1) and ([3], Lemma 1.1) using definition of induced representations given in [4].

Lemma 2.4. *Let H be a closed unimodular subgroup of a σ-compact hypergroup K. Given $f \in L^1(K)$, there exists a zero set M in K such that for every $y \in K \sim M$ and every representation ρ of $H, y^f|_H \in L^1(H)$ and $\rho(y^f|_H) \neq 0$ implies $ind_H^K \rho(f) \neq 0$.*

Recall that a hypergroup K (not necessarily commutative) is said to central hypergroup if $K/(Z(K) \cap G(K))$ is compact where $G(K)$ is the maximal subgroup of K and $Z(K)$ is the centre of K.

Theorem 2.5. *Let K be a σ-compact central hypergroup with connected component of $Z = G(K) \cap Z(K)$ non-compact then K has QUP.*

Proof. Let $f \in L^1(K)$ with $m(A_f) < \infty, \mu(B_f) < \infty$. It follows from above lemma and ([8], 4.1), ([4], 1.4) that there exists a zero set M in K such that for all $y \in K \sim M$, we have

(i) $y^f|_Z \in L^1(Z)$ and $m_Z(A_{y^f|_Z}) < \infty$

(ii) For every $\lambda \in \hat{Z}$, $\lambda(y^f|_Z) \neq 0 \Rightarrow \text{ind}_Z^K \lambda(f) \neq 0$

By ([2], 3.3.7, 3.3.9), $\pi \to ||\pi(f)||$ is a function in $C_0(\hat{K})$, in a particular $\{\pi : \pi(f)|| \geq \alpha\},\ \alpha > 0$ is a compact set, hence

$$B_f \subset \bigcup_{n\in\mathbf{N}} \{\pi \in \hat{K} : ||\pi(f)|| \geq n\} \cup \bigcup_{n\in\mathbf{N}} \{\pi \in \hat{K} : ||\pi(f)|| \geq \frac{1}{n}\}.$$

Thus B_f is a σ-bounded set. Now by ([14], 1.3)

$$\mu(B_f) = \int_{\hat{Z}} |B_f \cap r^{-1}(\lambda)| d\lambda,$$

where $|B_f \cap r^{-1}(\lambda)|$ is the cardinality of the set $B_f \cap r^{-1}(\lambda)$. Also $\mu_Z(B_{y^f|_Z}) = \int_{\hat{Z}} \xi_{B_{y^f|_Z}}(\lambda) d\lambda$. If $\lambda \in B_{y^f|_Z}$ then $\lambda(y^f|_Z) \neq 0 \Rightarrow \mathrm{ind}_Z^K \lambda(f) \neq 0$. But by ([4], 2.8) $\mathrm{ind}_Z^K \lambda \simeq \bigoplus_{\rho \in r^{-1}(\lambda)} d_\rho \rho$, so there exist $\rho \in r^{-1}(\lambda)$ such that $\rho(f) \neq 0$ i.e. $\rho \in B_f \cap r^{-1}(\lambda)$. Thus

$$\xi_{B_{y^f|_Z}}(\lambda) \leq \left|B_f \cap r^{-1}(\lambda)\right| \Rightarrow \mu_Z(B_{y^f|_Z}) \leq \mu(B_f) < \infty$$

But by hypothesis and [6], $m_Z(A_{y^f|_Z}) < \infty$ and $\mu_Z(B_{y^f|_Z}) < \infty \Rightarrow y^f|_Z = 0$ a.e. since this is true for almost all $y \in K$, we get $f = 0$ a.e.

The following generalizes ([12], 1.2) to hypergroups.

Corollary 2.6. *Let K be a compact hypergroup. Then the direct product $R^d \times K$ has QUP.*

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