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APPROXIMATE SYSTEMS WITH CONFLUENT BONDING MAPPINGS

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Abstract. If $\mathbf{X}=\{X_n, p_{nm}, \mathbb{N}\}$ is a usual inverse system with confluent (monotone) bonding mappings, then the projections are confluent (monotone). This is not true for approximate inverse system as shows Example 1.2. The main purpose of this paper is to show that the property of Kelley (smoothness) of the spaces X_n is a sufficient condition for the confluence (monotonicity) of the projections (Theorems 1.16 and 2.10).

1. Confluent bonding mappings

In this paper we shall consider the approximate inverse systems in the sense of S. Mardešić [10]. The basic definition and facts are given in Appendix.

A mapping $f: X \to Y$ is *confluent* provided every component of the inverse image $f^{-1}(C)$ of a continuum $C \subseteq Y$ is mapped onto C. Each monotone mapping is confluent.

Lemma 1.1. [4, Corollary 4.]. *Let $\boldsymbol{X}=\{X_a, p_{ab}, A\}$ be an inverse system of continua and confluent bonding mappings. Then the projections $p_a, a \in A$, are confluent.*

This is not true if $\mathbf{X}=\{X_n, p_{nm}, \mathbb{N}\}$ is an approximate inverse sequence as shows the following example.

Example 1.2. There exists an approximate inverse sequence $\mathbf{X}=\{X_n, p_{nm}, 0 \leq n \leq m < \infty\}$ such that the bonding mappings p_{nm} are confluent (monotone) but the projection p_0 is not confluent (monotone).

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Continuum X. Let $\mathbb{R}^2$ be the Euclidean plane endowed with the ordinary rectangular coordinate system Oxy. We define the continuum X as the union of the subsets $K, I_1, I_2, ..., I_n, ..., I_\infty$ such that

$$K = \{(1, y) : 1 \leq y \leq 3/2\},$$

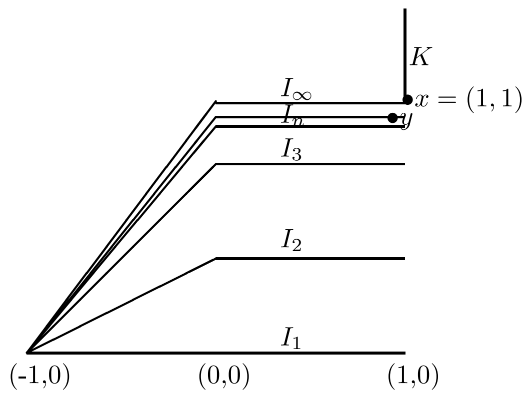
$$I_1 = \{(x, 0) : -1 \leq x \leq 1\},$$

$$I_n = \{(x, y) : -1 \leq x \leq 0, y = (1-\tfrac{1}{n})(x+1)\} \bigcup \{(x, y) : 0 \leq x \leq 1, y = 1-\tfrac{1}{n}\},$$

and

$$I_\infty = \{(x, y) : -1 \leq x \leq 0, y = x + 1\} \bigcup \{(x, y) : 0 \leq x \leq 1, y = 1\}.$$

The next picture shows approximately the continuum X.



Continuum X

Now we define a collection $\{H_n : n \in \mathbb{N}\}$ of homeomorphisms $H_n : X \to X$ by

$$H_n((x,y)) = \begin{cases} (x, (1 - \frac{1}{m-1})(x+1)), & (x,y) = (x, (1 - \frac{1}{m})(x+1)), \\ & \quad -1 \leq x \leq 0, 1 < m \leq n, \\ (x, 1 - \frac{1}{m-1}), & (x,y) = (x, 1 - \frac{1}{m}), 0 \leq x \leq 1, \\ & \quad 1 < m \leq n, \\ (x,y), & (x,y) \in I_m, m > n, \\ (2(1 - \frac{1}{n})x, 1 - \frac{1}{n}), & (x,y) \in I_1, 0 \leq x \leq \frac{1}{2}, \\ (\frac{1}{n}(2x-2) + 1, 1 - \frac{1}{n}), & (x,y) \in I_1, \frac{1}{2} \leq x \leq 1, \\ (x,y), & (x,y) \in K. \end{cases}$$

The approximate sequence $\mathbf{X}$ consists of the following:

1. The spaces $X_0, X_1, ..., X_n, ...$ are homeomorphic to X.
2. The bonding mapping $p_{nm} : X_m \to X_n$ is defined by $p_{nm}((x, y)) = (x, y)$ for each $n, m \geq 1$.
3. Let $n \geq 1$. The mapping p_{0n} is defined by $p_{0n}((x, y)) = H_n(x, y)$.

$\mathbf{X} = \{X_n, p_{nm}, 0 \leq n \leq m < \infty\}$ *is an approximate inverse sequence.*

The limit of the sequence $\mathbf{X}$ *is the continuum* X. This follows from the fact that the sequence $\mathbf{X}$ has the subsequence $\{X_n, p_{nm}, 1 \leq n \leq m < \infty\}$ which is a usual inverse sequence with limit homeomorphic to X. Now, applying [11, Theorem (1.19)] we conclude that $\lim \mathbf{X}$ is homeomorphic to X. We denote its segments by $I_1^\infty, ..., I_n^\infty, ..., I_\infty^\infty$.

The bonding mappings are onto and monotone. Moreover, $p_{nm} : X_m \to X_n$ are homeomorphisms for each $m, n \geq 0$.

The projection $p_0 : \lim \mathbf{X} \to X_0$ *is onto but not monotone.* Let x be any point of $\lim \mathbf{X} = X$. It is clear that, for $n \geq 1$, we have the subnet $(p_n(x) : n \geq 1)$ such that $p_{nm} p_m(x) = p_n(x), m \geq n$. If $x \notin I_1^\infty \setminus \{(-1, 0)\}$, then $p_0(p_k(x)) = p_0(p_m(x))$ for all k, m. Thus, if $x = (x_1, y_1)$, then $p_0(x) = (x_1, y_1')$, where $y_1' = y_1$ for $(x_1, y_1) \in I_\infty$, and $(x_1, y_1') \in I_{m-1}$ for $(x_1, y_1) \in I_m, 1 < m < \infty$. For the points of $I_1^\infty \setminus \{(-1, 0)\}$ we have the following cases.

1) Let x be a point of the form $(x_1, 0), -1 < x_1 \leq 0$. Then $p_n(x_1, 0) = (x_1, 0)$ and $p_{0n}(p_n(x_1, 0)) = p_{0n}(x_1, 0) = (x_1, (1 - \frac{1}{n})(x_1 + 1))$. The sequence $\{(x_1, (1-\frac{1}{n})(x_1+1)) : n \in \mathbb{N}\}$ converges to the point $(x_1, x_1 + 1)$. This means that $p_0^{-1}(x_1, x_1 + 1)$ contains x. On the other hand, $p_0^{-1}(x_1, x_1 + 1)$ contains the point $(x_1, x_1 + 1) \in \lim \mathbf{X}$.

2) If x is a point of the form $(x_1, 0), 0 < x_1 \leq 1/2$, then $p_n(x_1, 0) = (x_1, 0)$ and $p_{0n}(p_n(x_1, 0)) = p_{0n}(x_1, 0) = (2(1 - \frac{1}{n})x_1, 1 - \frac{1}{n})$. The sequence $\{(2(1 - \frac{1}{n})x_1, 1 - \frac{1}{n}) : n \in \mathbb{N}\}$ converges to the point $(2x_1, 1, 0)$. This means that $p_0^{-1}(2x_1, 1)$ contains x. On the other hand, $p_0^{-1}(2x_1, 1)$ contains the point $(2x_1, 1) \in \lim \mathbf{X}$.

3) Finally, let $x = (x_1, 0), 1/2 \leq x_1 \leq 1$. Then $p_{0n}(p_n(x_1, 0)) = p_{0n}(x_1, 0) = (\frac{1}{n}(2x-2)+1, 1-\frac{1}{n})$. The sequence $\{(\frac{1}{n}(2x-2)+1, 1-\frac{1}{n}) : n \in \mathbb{N}\}$ converges to the point $(1, 1)$. We conclude that $p_0^{-1}(1, 1)$ contains the whole segment $\{(x, 0) : 1/2 \leq x \leq 1\}$ and the point $(1, 1)$. Thus, $p_0^{-1}(1, 1)$ is not connected and p_0 *is not monotone*.

The projection $p_0 : \lim \mathbf{X} \to X_0$ *is not confluent.* Namely, we have $p_0^{-1}(K) = \{(1, y) : 1 \leq y \leq 3/2\} \bigcup \{(x, 0) : 1/2 \leq x \leq 1\}$. We infer that $p_0^{-1}(K)$ has two components $Q = \{(1, y) : 1 \leq y \leq 3/2\} = K$ and $R = \{(x, 0) : 1/2 \leq x \leq 1\}$. It is clear that $p_0(Q) = K$. On the other hand $p_0(R) = \{(1, 1)\} \neq K$ since $R \subseteq p_0^{-1}(1, 1)$. Hence, the projection p_0 is not confluent.

□

In the sequel we shall show that the property of Kelley is a sufficient condition for the confluence of the projections.

Definition 1.3. *A metric continuum* X *is said to have the property of Kelley* [13, p. 538] *provided that given any* $\epsilon > 0$ *there exists* $\delta = \delta(\epsilon) > 0$ *such* *that if* $a, b \in X, d(a, b) < \delta$ *and* $a \in A \in C(X)$*, then there exists* $B \in C(X)$ *such that* $b \in B$ *and* $H(A, B) < \epsilon$*.*

Let us recall that H is the Hausdorff metric on 2^X [13, p. 1] which induces the Hausdorff metric topology for 2^X. The Vietoris topology for 2^X is explained on p. 59. In the sequel we shall use the following theorem.

THEOREM 1.4. [13, Theorem (0.13)]. *The Vietoris topology for* 2^X *and the Hausdorff metric topology for* 2^X *are the same.*

REMARK 1.5. Each locally connected metric continuum has the property of Kelley [13, Example (6.11), p. 538].

LEMMA 1.6. *The continuum* X *in Example* 1.2 *does not have the property of Kelley.*

PROOF. Consider the continuum $A = K$ and the point $x = (1, 1)$. Let $V = \langle V_1, V_2 \rangle$ be a neighborhood of A (in $C(X)$), where: $V_1 = \{(x, y) : 0.9 < x < 1.1, 1/100 < y < 1.1\} \bigcap X$, $V_2 = K \setminus \{x\}$. Take any neighborhood U of x and any point $y \in U \bigcap V_1$. Each subcontinuum B of X containing y which intersects V_1 and V_2 must contains the point $(-1, 0)$. It is clear that $B \not\subseteq V_1 \bigcup V_2$. Hence, $B \notin \langle V_1, V_2 \rangle = V$. Thus, X does not have the property of Kelley. Let us observe that X is not locally connected at any point of the set $\{(x, 1) : 0 \leq x \leq 1\}$. □

REMARK 1.7. Let us observe that the continuum X is a dendroid, i.e., an arcwise connected and hereditarily unicoherent continuum. By virtue of [5] it follows that if a dendroid has the property of Kelley, then it is smooth and, consequently, locally connected. It is clear that the continuum X is not locally connected. We infer that X does not have the property of Kelley.

DEFINITION 1.8. *Let* X *be a continuum. For each* $a \in X$*, let* $\alpha_X(a) = \{A \in C(X), a \in A\}$*.*

If X is a metric continuum, then $\alpha_X(a)$ is a continuum of $C(X)$ [15, p. 292]. Moreover, $\alpha_X(a)$ is arcwise connected. This means that $\alpha_X(a) \in C(X)$ and we have the mapping $\alpha_X : X \to C^2(X)$.

THEOREM 1.9. [15, Theorem 2.2]. *The mapping* $\alpha_X : X \to C^2(X)$ *is continuous if and only if* X *has the property of Kelley.*

THEOREM 1.10. [15, Theorem 4.3]. *Let* $f : X \to Y$ *be a confluent mapping. If* X *has the property of Kelley, then* Y *has the property of Kelley.*

COROLLARY 1.11. *Let* $f : X \to Y$ *be a monotone mapping. If* X *has the property of Kelley, then* Y *has the property of Kelley.*

In the sequel we shall frequently consider the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \Big\downarrow \alpha_X & & \Big\downarrow \alpha_Y \\ C^2(X) & \xrightarrow{c^2(f)} & C^2(Y) \end{array} \tag{1.1}$$

The diagram (1.1) commutes if and only if

$$f(\alpha_X(x)) = \alpha_Y(f(x)).$$

From the continuity of f it follows the following relation.

$$\{c(f)(A) : A \in \alpha_X(x)\} \subseteq \{B : B \in \alpha_Y(f(x))\}.$$

LEMMA 1.12. *Diagram* (1.1) *commutes if and only if, for every* $B \in \alpha_Y(f(x))$*, there exists* $A \in \alpha_X(x)$ *such that* $c(f)(A) = B$.

LEMMA 1.13. [15, Theorem 4.2]. *Diagram* (1.1) *commutes if and only if* $f : X \to Y$ *is a confluent mapping.*

THEOREM 1.14. [3, Theorem 2.]. *Let* $\boldsymbol{X} = \{X_n, p_{nm}, \mathbb{N}\}$ *be a usual inverse sequence of continua and confluent bonding mappings* p_{nm}*. If each* X_n *has the property of Kelley, then* $X = \lim \boldsymbol{X}$ *has the property of Kelley.*

We first establish the following theorem.

THEOREM 1.15. *Let* $\boldsymbol{X} = \{X_n, p_{nm}, \mathbb{N}\}$ *be an approximate inverse sequence of continua and confluent projections* $p_n : \lim \boldsymbol{X} \to X_n$*. If each* X_n *has the property of Kelley, then* $X = \lim \boldsymbol{X}$ *has the property of Kelley.*

PROOF. By virtue of [8, Corollary 2.11] $X = \lim \mathbf{X}$ is a continuum. Let x be any point of X and K any subcontinuum of X such that $x \in K$. Let U be a sufficiently small basis neighborhood of x and let $V = \langle V_1, V_2, ..., V_n\rangle$ be any basis neighborhood of K in $C(X)$. It remains to prove that for each point $y \in U$ there exists a continuum L such that $y \in L$ and $L \subset V \in C(X)$. The proof is broken into several steps.

Step 1. There exists a $n \in \mathbb{N}$ and an open set U_n containing $p_n(x) = x_n$ such that $x \in p_n^{-1}(U_n) \subset U$. This follows from the definition of the basis of X (Lemma 3.3).

Step 2. For the neighborhood $V = \langle V_1, V_2, ..., V_n\rangle$ of K (in $C(X)$) there exists a $n \in \mathbb{N}$ such that for each $m \in \mathbb{N}, m \geq n$, there exist open subsets $V_1(m), ..., V_n(m)$ of X_m such that $V(m) = \langle V_1(m), ..., V_n(m)\rangle$ is an open set in $C(X_m)$ containing $p_m(K)$ and $p_m^{-1}(V(m)) \subseteq V_1 \bigcup ...V_n$. This follows from Lemma 3.4 of Appendix.

Step 3. The space $X = \lim X$ has the property of Kelley. By virtue of Steps 1 - 2 one can obtain a $m \in \mathbb{N}$ and open sets $U_m, V_i(m), i = 1, ..., n$, such that $x \in p_m^{-1}(U_m) \subset U$ and $K \subset \cup\{p_m^{-1}(V_i(m)) : i = 1, ..., n\} = V$. If y is any point of $p_m^{-1}(U_m)$ then $p_m(y)$ is in U_m. There exists a subcontinuum K_m of X_m such that $p_m(y)$ is in K_m and $K_m \subset \cup\{V_i(m) : i = 1, ..., n\}$ since X_m has the property of Kelley. Let Q be any component of $p_m^{-1}(K_m)$. By Theorem 1.1 it follows $p_m(Q) = K_m$. Let us prove that $Q \in V$ (in $C(X)$). Suppose that $Q \notin V$ (in $C(X)$). Then there exists $p_m^{-1}(V_i(m))$ in $\{p_m^{-1}(V_i(m)) : i = 1, ..., n\}$ such that $p_m^{-1}(V_i(m)) \bigcap Q = \emptyset$. Hence $V_i(m) \bigcap p_m(Q) = \emptyset$. This is impossible since $p_m(Q) = K_m$ and $K_m \subset \cup\{V_i(m) : i = 1, ..., n\}$. ◻

Now we shall prove the main theorem of this paper.

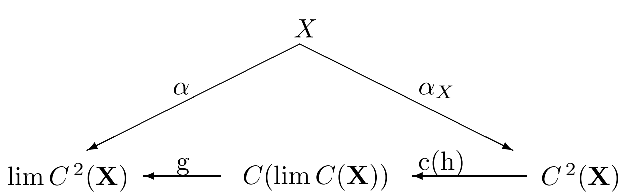
THEOREM 1.16. *Let $\boldsymbol{X} = \{X_n, p_{nm}, \mathbb{N}\}$ be an approximate inverse sequence of continua and confluent bonding mappings. If each X_n has the property of Kelley, then $X = \lim \boldsymbol{X}$ has the property of Kelley. Moreover, each projection $p_n : X \to X_n$ is confluent.*

PROOF. By virtue of [2, Proposition 8] there exist: **a)** a cofinal subset $M = \{m_i, i \in \mathbb{N}\}$ of $\mathbb{N}$, **b)** a usual inverse sequence $\mathbf{Y} = \{Y_i, q_{ij}, M\}$ such that $Y_i = X_{m_i}$ and $q_{ij} = p_{m_i m_{i+1}} p_{m_{i+1} m_{i+2}} ... p_{m_{j-1} m_j}$ for each $i, j \in \mathbb{N}$, **c)** a homeomorphism $H : \lim \mathbf{X} \to \lim \mathbf{Y}$. Using Theorem 1.14 we infer that Y has the property of Kelley since each q_{ij} is confluent. This means that X has the property of Kelley since there exists a homeomorphism $H : \lim \mathbf{X} \to \lim \mathbf{Y}$.

Let us prove that the projections $p_a, a \in A$, are confluent. Let $\alpha_n : X_n \to C^2(X_n), n \in \mathbb{N}$, be the mapping $\alpha_{X_n} : X_n \to C^2(X_n), n \in \mathbb{N}$. The collection $C^2\mathbf{X} = \{C^2(X_n), c^2(p_{nm}), \mathbb{N}\}$ is an approximate inverse sequence (see Appendix, p. 59). The collection $\{\alpha_n : n \in \mathbb{N}\}$ is a mapping between inverse systems $\mathbf{X} = \{X_n, p_{nm}, \mathbb{N}\}$ and $C^2(\mathbf{X}) = \{C^2(X_n), c^2(p_{nm}), \mathbb{N}\}$ as shows the following diagram.

$$\begin{array}{ccccccc} X_n & \stackrel{p_{nm}}{\longleftarrow} & X_m & \longleftarrow & ... & & \lim \mathbf{X} \\ \downarrow \alpha_n & & \downarrow \alpha_m & & ... & & \downarrow \lim \alpha_n = \alpha \\ C^2(X_n) & \stackrel{c^2(p_{nm})}{\longleftarrow} & C^2(X_m) & \longleftarrow & ... & & \lim C^2(\mathbf{X}) \end{array} \tag{1.2}$$

The limit mapping $\alpha = \lim \alpha_n : \lim X \to \lim C^2(\mathbf{X})$ is continuous since each α_n is continuous (Theorem 1.9). It remains to prove that the following diagram commutes, where g and $c(h)$ are homeomorphisms described on pp. 59 - 60.



Let us prove the relation

$$\alpha = g \circ c(h) \circ \alpha_X. \tag{1.3}$$

Let x be any point of X. Then $\alpha_X(x)$ is a point of $C^2(X)$. This means that $K = c(h)(\alpha_X(x))$ is a continuum of $C(\lim C\mathbf{X})$. On the other hand, $\alpha(x)$ is a thread in $C^2(\mathbf{X})$. This means that $L = g^{-1}(\alpha(x))$ is a continuum in $C(\lim C(\mathbf{X}))$ such that $c(p_n)(L) = \alpha_n(x) \subseteq C(X_n), n \in \mathbb{N}$. Let us prove that $K = L$.

$\mathbf{K} \subseteq \mathbf{L}$. Each element M of K is a continuum of X which induces the thread $\{c(p_n)(M) : n \in \mathbb{N}\}$ in $C(\mathbf{X})$ such that $c(p_n)(M) \in \alpha_n(x)$. We infer that this thread induces an element of $g^{-1}(\alpha(x)) = L$.

$\mathbf{K} \supseteq \mathbf{L}$. Conversely, if $M \in L = g^{-1}(\alpha(x))$, then $c(p_n)(M) \in \alpha_n(x)$ for each $n \in \mathbb{N}$. We have the thread which induces a continuum N containing $x \in X$. This means that $N \in K$. Finally, we have $K = L$.

From diagram (1.2) it follows, that for each $n \in \mathbb{N}$, the diagram

$$\begin{array}{ccc} X_n & \xleftarrow{p_n} & \lim \mathbf{X} \\ \downarrow \alpha_n & & \downarrow \lim \alpha_n = \alpha \\ C^2(X_n) & \xleftarrow{c^2(p_n)} & \lim C^2(\mathbf{X}) \end{array} \tag{1.4}$$

commutes. The commutativity of this diagram and the relation (1.3) imply that the diagram

$$\begin{array}{ccc} X_n & \xleftarrow{p_n} & \lim \mathbf{X} \\ \downarrow \alpha_n & & \downarrow \alpha_X \\ C^2(X_n) & \xleftarrow{c^2(p_n)} & C^2(\lim \mathbf{X}) \end{array} \tag{1.5}$$

commutes. By Lemma 1.13, we infer that p_n is confluent. □

THEOREM 1.17. *Let $\mathbf{X} = \{X_n, p_{nm}, \mathbb{N}\}$ be an approximate inverse sequence of dendrites X_n and confluent bonding mappings p_{mn}. Then $X = \lim \mathbf{X}$ is a dendrite if and only if it is arcwise connected.*

PROOF. Each dendrite X_n is locally connected and hereditarily unicoherent [16, p. 88]. We infer that X is a continuum [8, Corollary 2.11]. Moreover, X is hereditarily unicoherent [8, Corollary 4.3]. By virtue of Remark 1.5 and Theorem 1.16 X has the property of Kelley. If X is arcwise connected, then X is a dendroid. By virtue of [5] it follows that if a dendroid (see Remark 1.7) has the property of Kelley, then it is smooth and, consequently, locally connected. We infer that X is locally connected and hereditarily unicoherent. Thus, X is a dendrite [16, p. 88]. Conversely, if X is a dendrite, then X is arcwise connected [16, p. 89]. □

A mapping $f : X \to Y$ is said to be *weakly confluent* [13, p. 22] provided, for every subcontinuum K of Y, there exists a component A of $f^{-1}(K)$ such that $f(A) = K$. Each confluent mapping is weakly confluent.

THEOREM 1.18. [13, Theorem 0.49.1]. *Let $f : X \to Y$ be a surjective mapping. Then the mapping $c(f) : C(X) \to C(Y)$ is a surjection if and only if f is weakly confluent.*

THEOREM 1.19. *Let $\boldsymbol{X} = \{X_a, p_{ab}, A\}$ be an approximate inverse system of continua and weakly confluent bonding mappings. Then the projections $p_a : \lim \boldsymbol{X} \to X_a, a \in A$, are weakly confluent.*

PROOF. Consider the inverse system $C(\mathbf{X}) = \{C(X_a), c(p_{ab}), A\}$. The bonding mappings $c(p_{ab})$ are surjective since p_{ab} are weakly confluent. By virtue of [11, Corollary 4.5], it follows that the projections $c(p_a)$ are surjective. This means that p_a are weakly confluent. □

2. Monotone bonding mappings

A continuous mapping $f : X \to Y$ is said to be *monotone relative to a point* $p \in X$ if for each subcontinuum Q of Y such that $f(p) \in Q$ the inverse image $f^{-1}(Q)$ is connected [3, p. 184].

THEOREM 2.1. [4, Theorem 1.]. *Let $\boldsymbol{X} = \{X^\lambda, f^{\lambda\mu}, \Lambda\}$ be an inverse system with limit X. If there exists a thread $p = \{p^\lambda\}$ such that, for each $\lambda \in \Lambda$ with $\alpha \leq \lambda$, the bonding mapping $f^{\alpha\lambda}$ is monotone relative to p^λ, then the projection π^α is monotone relative to p.*

REMARK 2.2. Theorem 2.1 is not true if $\mathbf{X}$ is an approximate inverse sequence as shows Example 1.2. In the sequel we shall prove that smoothness of the spaces X_n is a sufficient condition for the monotonicity of the projections.

We say that a metric continuum X is *smooth at the point* $p \in X$ [9, p. 81] if for each convergent sequence $x_1, x_2,\ldots$ of points of X and for each subcontinuum K of X such that $p, x \in K$, where $x = \lim_{n\to\infty} x_n$, there exists a sequence $K_1, K_2,\ldots$ of subcontinua of X such that $p, x_n \in K_n$ for each $n = 1, 2, \ldots$, and $Lim_{n\to\infty} K_n = K$.

THEOREM 2.3. [9, Theorem (3.1)]. *Let p be an arbitrary point of a continuum X.The following statements are equivalent:*

(i) *X is smooth at p,*

(ii) *for each open set G such that $p \in G$, the set $C(G, p) = \{x \in X$, there exists a continuum $K \subseteq G$ such that $x, p \in K\}$ is open,*

(iii) *for each subcontinuum N of X such that $p \in N$ and for each open set V which contains N, there exists an open connected set U such that $N \subseteq U \subseteq V$,*

(iv) *for each subcontinuum N of X such that $p \in N$ and for each open set V which contains N there exists a continuum K such that $N \subseteq IntK \subseteq K \subseteq V$.*

A continuum X is locally connected at the point p if X is smooth at the point p.

Lemma 2.4. [9, Corollary (3.4)]. *A continuum X is locally connected if and only if it is smooth at each of its points.*

Remark 2.5. The continuum X in Example 1.2 is smooth at no point $p \in I_\infty \setminus \{-1, 0\}$ because it is not locally connected at $p \in I_\infty \setminus \{-1, 0\}$.

The proof of the following theorem is similar to the proof of Theorem 1.15.

Theorem 2.6. *Let $\boldsymbol{X} = \{X_n, p_{nm}, \mathbb{N}\}$ be an approximate inverse sequence of continua and monotone projections $p_n : \lim \boldsymbol{X} \to X_n$. If each X_n is smooth, then $X = \lim \boldsymbol{X}$ is smooth.*

Proof. A straightforward modification of the proof of Theorem 1.15 using (iv) of Theorem 2.3 instead of the definition of the property of Kelley. $\square$

Let p be a fixed point of a continuum X. For each point $x \in X$ consider the family $\{K : K \in C(X), p, x \in K\}$ of all subcontinua K of X containing both p and x [3, p. 185]. We define $F[X,p](x) = \{K : K \in C(X), \quad p, x \in K\}$. For each $x \in X, F[X,p](x)$ is compact and it is an arcwise connected subset of $C(X)$, i.e., $F[X,p](x)$ is an element of $C^2(X)$ [3, p. 185]. Thus, we have a mapping $F[X,p] : X \to C^2(X)$.

Lemma 2.7. [3, Proposition 2.]. *The mapping $F[X,p]$ is continuous if and only if the continuum X is smooth at the point p.*

Lemma 2.8. [3, Proposition 3.]. *Let a continuous surjection $f : X \to Y$ and points $p \in X$ and $q \in Y$ with $q = f(p)$ be given. If $F_1 = F[X,p]$ and $F_2 = F[Y,q]$, then the diagram*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow F_1 & & \downarrow F_2 \\ C^2(X) & \xrightarrow{c^2(f)} & C^2(Y) \end{array}$$

commutes if and only if f is monotone relative to p.

The following theorem for usual inverse sequences was proved in the paper [3].

Theorem 2.9. [3, Theorem 1.]. *Let $\{X^i, f^i\}_{i=1}^\infty$ be an inverse sequence such that for each $i = 1, 2, \dots$ (a) the continuum X^i is smooth at a point p^i; (b) $f^i(p^{i+1}) = p^i$; (c) f^i is monotone relative to p^{i+1}. Then the inverse limit continuum $X = \lim\{X^i, f^i\}$ is smooth at the thread $p = \{p^i\}_{i=1}^\infty$.*

An approximate version of Theorem 2.9 is the following theorem.

Theorem 2.10. *Let $\boldsymbol{X} = \{X_n, f_{mn}, \mathbb{N}\}$ be an inverse sequence such that for each $n \in N$:*

1. *The continuum X_n is smooth at a point p_n;*
2. *$p = (p_n : n \in N)$ is an approximate thread;*
3. *f_{mn} is monotone relative to p_m.*

Then the approximate inverse limit $X = \lim \boldsymbol{X}$ is smooth at the thread p. Moreover, the projections $p_m, m \in \mathbb{N}$, are monotone at p_m.

Proof. The proof of the theorem is a straightforward modification of the proof of Theorem 1.16. In order to prove the theorem it is enough to replace in the proof of Theorem 1.16 the mapping α_n by $F[X_n, p_n]$, for $n = 1, 2, \ldots$ and use Lemmas 2.7 and 2.8 instead of Theorem 1.9 and Lemma 1.13. □

3. Appendix

In this Appendix we give some basic definitions and facts concerning approximate inverse systems in the sense of S. Mardešić [10]. $Cov(X)$ is the set of all normal coverings of a topological space X. For other details see [1]. If $\mathcal{U}, \mathcal{V} \in Cov(X)$ and $\mathcal{V}$ refines $\mathcal{U}$, we write $\mathcal{V} < \mathcal{U}$. If $f, g : Y \to X$ are $\mathcal{U}$-near mappings, i.e. for any $y \in Y$ there exists a $U \in \mathcal{U}$ with $f(y), g(y) \in U$, we write $(f, g) < \mathcal{U}$.

Definition 3.1. *An approximate inverse system is a collection $\mathbf{X} = \{X_a, p_{ab}, A\}$, where $(A, \leq)$ is a directed preordered set, $X_a, a \in A$, is a topological space and $p_{ab} : X_b \to X_a, a \leq b$, are mappings such that $p_{aa} = id$ and the following condition* (A2) *is satisfied:*

(A2) *For each $a \in A$ and each normal cover $\mathcal{U} \in Cov(X_a)$ there is an index $b \geq a$ such that $(p_{ac}p_{cd}, p_{ad}) < \mathcal{U}$, whenever $a \leq b \leq c \leq d$.*

An *approximate map* [11] $\mathbf{p} = \{p_a : a \in A\} : X \to \mathbf{X}$ into an approximate system $\mathbf{X}= \{X_a, p_{ab}, A\}$ is a collection of maps $p_a : X \to X_a, a \in A$, such that the following condition holds

(AS) For any $a \in A$ and any $\mathcal{U} \in Cov(X_a)$ there is $b \geq a$ such that $(p_{ac}p_c, p_a) < \mathcal{U}$, for each $c \geq b$.

Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be an approximate inverse system and let $\mathbf{p} =\{p_a : a \in A\} : X \to \mathbf{X}$ be an approximate map. We say that $\mathbf{p}$ is a *limit* of $\mathbf{X}$provided it has the following universal property:

(UL) For any approximate map $\mathbf{q}= \{q_a : a \in A\} : Y \to \mathbf{X}$ of a space Y there exists a unique map $g : Y \to X$ such that $p_a g = q_a$.

Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be an approximate system. A point $x = (x_a) \in \prod\{X_a : a \in A\}$ is called a *thread* of $\mathbf{X}$provided it satisfies the following condition:

(L) $(\forall a \in A)(\forall \mathcal{U} \in Cov(X_a))(\exists b \geq a)(\forall c \geq b) p_{ac}(x_c) \in st(x_a, \mathcal{U})$.

If X_a is a $T_{3.5}$-space, then the sets $st(x_a, \mathcal{U}), \mathcal{U} \in Cov(X_a)$, form a basis of the topology at the point x_a. Therefore, for an approximate system of

Tychonoff spaces condition (L) is equivalent to the following condition:
(L)* $(\forall a \in A)\ \lim\{p_{ac}(x_c) : c \geq a\} = x_a$.

The existence of the limit of any approximate system was proved in [11, (1.14) Theorem].

THEOREM 3.2. *Let* $\boldsymbol{X} = \{X_a, p_{ab}, A\}$ *be an approximate system. Let* $X \subseteq \Pi\{X_a : a \in A\}$ *be the set of all threads of* $\boldsymbol{X}$*and let* $p_a : X \to X_a$ *be the restriction* $p_a = \pi_a|X$ *of the projection* $\pi_a : \Pi X_a \to X_a, a \in A$. *Then* $\mathbf{p}= \{p_a : a \in A\} : X \to \mathbf{X}$ *is a limit of* $\boldsymbol{X}$.

We call this limit the *canonical limit* of $\mathbf{X} = \{X_a, p_{ab}, A\}$. In the sequel limit means the canonical limit.

LEMMA 3.3. [12, (2.13) Lemma]. *Let* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *be an approximate system of Tychonoff spaces, let* X *be the canonical limit of* $\mathbf{X}$*and let* $B \subseteq A$ *be a cofinal subset of* A. *Then the collection* $\mathcal{B}$ *of all sets of the form* $p_b^{-1}(U_b)$, *where* $b \in B$ *and* $V_b \subseteq X_b$ *is open, is a basis of the topology for* X.

Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be an approximate inverse system of compact spaces. The collection $\{p_a : \lim \mathbf{X} \to X_a, a \in A\}$ is an approximate map [11, Definition 1.9]. Moreover, this collection is a resolution and a limit of $\mathbf{X}$[11, Definition 1.10, Theorem 1.14, Theorem 4.2].

For any topological space X the set of all closed subsets of X is denoted by 2^X. The Vietoris topology on 2^X is the topology with a base $\langle U_1, U_2, \ldots, U_n \rangle = \{F : F \in 2^X,\ F \subset \bigcup_{i=1}^n U_i\ , F \bigcap U_i \neq \emptyset, i = 1, \ldots, n\ \}$ where $U_1, \ldots, U_n$ are open subsets of X [6, p. 162]. If $f : X \to Y$ is a continuous mapping, then we define a mapping $2^f : 2^X \to 2^Y$ by $2^f(F) = Cl_Y(f(F))$, $F \in 2^X$. If X is compact, then $2^f(F) = f(F), F \in 2^X$. For a continuum X, $C(X)$ will denote the subspace of 2^X of all subcontinua of X. If $f : X \to Y$ is a continuous mapping, then $c(f)$ will denote the restriction $2^f|C(X) : C(X) \to C(Y)$. Similarly, $C^2(X)$ will denote $C(C(X))$ and $c^2(f)$ will denote $c(c(f))$ for a mapping $f : X \to Y$.

If $\mathbf{X} = \{X_a, p_{ab}, A\}$ is an approximate system of compact spaces, then $C(\mathbf{X}) = \{C(X_a), c(p_{ab}), A\}$ is an approximate inverse system [14, Lemma 9.4]. Moreover, if $\mathbf{p}= \{p_a : a \in A\} : X \to \mathbf{X}$ is an approximate map, then so is $c(\mathbf{p}) = \{c(p_a) : a \in A\} : C(X) \to C(\mathbf{X})$ [14, Lemma 9.5]. This means that the collection $\{c(\mathbf{p}) = \{c(p_a) : a \in A\}$ is an approximate resolution [14, Lemma 9.9] and a limit [11, Theorem 4.2]. Hence, we have a homeomorphism $h : C(\lim \mathbf{X}) \to \lim C(\mathbf{X})$ defined by

$$h(K) = \{c(p_a)(K) : a \in A\},$$

where K is a subcontinuum of $X = \lim \mathbf{X}$, $c(p_a)(K) = p_a(K)$ and $\{c(p_a)(K) : a \in A\}$ is a thread in $C(\mathbf{X})$. From this and Lemma 3.3 it follows the following lemma.

LEMMA 3.4. *Let $\boldsymbol{X} = \{X_a, p_{ab}, A\}$ be an approximate inverse system with limit X and let K be a subcontinuum of X. For each neighborhood $U = \langle U_1, \ldots, U_n \rangle$ of K in $C(X)$ there exists an $a \in A$ and an open set $U_a \supseteq p_a(K)$ in $C(X_a)$ such that $U_a = \langle U_{a_1}, \ldots, U_{a_n} \rangle$ and $p_a^{-1}(U_a) \subseteq U$ in $C(X)$.*

The homeomorphism $h : C(\lim \mathbf{X}) \to \lim C(\mathbf{X})$ induces the homeomorphism $c(h) : C(C(\lim \mathbf{X})) \to C(\lim C(\mathbf{X}))$, i.e. the homeomorphism $c(h) : C^2(\lim \mathbf{X}) \to C(\lim C(\mathbf{X}))$.

Similarly, from the fact that the collection $\{c(p_a) : a \in A\}$ is an approximate resolution of the approximate inverse system $C(\mathbf{X}) = \{C(X_a), c(p_{ab}), A\}$ it follows that there exists a homeomorphism $g : C(\lim C(\mathbf{X})) \to \lim C(C(\mathbf{X}))$ $= \lim C^2(\mathbf{X})$ defined by

$$g(K) = \{c^2(p_a)(K) : a \in A\},$$

where K is a subcontinuum of $C(\lim C(\mathbf{X}))$ and $\{c^2(p_a)(K) : a \in A\}$ is a thread in $C^2(\mathbf{X})$.

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REFERENCES

[1] R.A. Aló and H.L. Shapiro, *Normal topological spaces*, Cambridge Univ. Press, 1974.
[2] M.G. Charalambous, *Approximate inverse systems of uniform spaces and an application of inverse systems*, Comment. Math. Univ. Carolinae, 32, 33(1991), 551-565.
[3] W.J. Charatonik, *Inverse limits of smooth continua,* Commentationes Math. Univ. Carolinae 23 (1982), 183-191.
[4] J.J Charatonik and W.J. Charatonik, *On projections and limiting mappings of inverse systems of compact spaces*, Topology and its Applications 16 (1983), 1-9.
[5] S.T. Czuba, *On dendroids with Kelley's property*, Proc. Amer. Math. Soc. 102(1988), 728 - 730.
[6] R. Engelking, *General Topology*, PWN, Warszawa, 1977.
[7] I. Lončar, *A note on approximate limits,* Zbornik radova FOI Varaždin 19 (1995), 1 - 21.
[8] I. Lončar, *Set convergence and local connectedness of approximate limits*, Acta Mathematica Hungarica 77 (3) (1997), 193 - 213.
[9] T. Maćkowiak, *On smooth continua,* Fund. Math. 85(1974), 79-95
[10] S. Mardešić, *On approximate inverse systems and resolutions*, Fund. Math. 142(1993),241-255.
[11] S. Mardešić and T. Watanabe, *Approximate resolutions of spaces and mappings*, Glasnik Mat. 24(3)(1989), 587-637.
[12] V. Matijević, *A note on limit and resolution of (nongauged) approximate inverse systems*, Glasnik Matematički 28 (48) (1993), 111 - 122.
[13] S.B. Nadler, *Hyperspaces of sets*, Marcel Dekker, Inc., New York, 1978.
[14] J. Segal, T. Watanabe, *Cosmic approximate limits and fixed points*, Trans. Amer. Math. Soc. 333(1992), 1-61.
[15] R.W. Wardle, *On a property of J.L. Kelley*, Houston J. Math. 3(1977), 291 - 299.

[16] G.T. Whyburn, *Analytic Topology*, Amer. Math. Soc. 28(1971).

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