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FUNCTIONAL BOUNDARY VALUE PROBLEMS FOR SECOND ORDER FUNCTIONAL DIFFERENTIAL EQUATIONS OF THE NEUTRAL TYPE

Svatoslav Staněk
Palacký University, Czech Republic

Abstract. The functional differential equation $(x'(t) + L(x')(t))' = F(x)(t)$ together with functional boundary conditions is considered. Existence results are proved by the Leray-Schauder degree and the Borsuk theorem for α-condensing operators. We demonstrate on examples that our existence assumptions are optimal.

1. Introduction, notation

Let $J = [0, T]$ be a compact interval and $\mathcal{A}$ be the set of all functionals $\varphi : C^0(J) \to \mathbb{R}$ which are

(1) continuous, $\varphi(0) = 0$, and
(2) increasing (i.e., $x, y \in C^0(J)$, $x(t) < y(t)$ for $t \in J \Rightarrow \varphi(x) < \varphi(y)$).

Example 1. Let $k \in C^0(\mathbb{R})$ be an increasing function, $k(0) = 0$, $0 \le a < b \le T$, $0 \le t_1 < t_2 < \cdots < t_n \le T$ and $a_j > 0$ $(j = 1, 2, \cdots, n)$ be positive constants. Then the following functionals

$$k\Big(\int_a^b x(s)\, ds\Big), \quad \int_a^b k(x(s))\, ds, \quad \int_a^b \int_a^s k(x(\nu))\, d\nu\, ds,$$

$$\max\{k(x(t)) : a \le t \le b\}, \quad \min\{k(x(t)) : a \le t \le b\}, \quad \sum_{j=1}^n a_j k(x(t_j))$$

and their linear combinations with positive coefficients belong to the set $\mathcal{A}$.

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Remark 1. The assumption $\varphi(0) = 0$ for $\varphi \in \mathcal{A}$ is not an essential restriction. Indeed, if $\psi : C^0(J) \to \mathbb{R}$ is a continuous increasing functional and if we define $\varphi(x) = \psi(x) - \psi(0)$ for $x \in C^0(J)$, then $\varphi \in \mathcal{A}$.

For any $x \in C^0(J)$, $y \in L_1(J)$ and $(x, a, b) \in C^0(J) \times \mathbb{R}^2$ we set

$$\|x\| = \max\{|x(t)| : t \in J\},\ \|y\|_{L_1} = \int_0^T |y(t)|\,dt,\ \|(x, a, b)\|_* = \|x\| + |a| + |b|.$$

Consider the functional differential equation of the neutral type

$$(x'(t) + L(x')(t))' = F(x)(t). \tag{1}$$

Here $L : C^0(J) \to C^0(J)$ and $F : C^1(J) \to L_1(J)$ are continuous operators.

Together with (1) consider the boundary conditions

$$\varphi(x) = 0, \quad \psi(x') = 0, \tag{2}$$

where φ, $\psi \in \mathcal{A}$.

We say that $x \in C^1(J)$ is a *solution of the boundary value problem* (BVP for short) (1), (2) if the function $x'(t) + L(x')(t)$ is absolutely continuous on J, x satisfies the boundary conditions (2) and (1) is satisfied for a.e. $t \in J$.

In this paper we use the following assumptions:

(H_1) There exists $k \in [0, 1/2)$ such that

$$\|L(x) - L(y)\| \le k\|x - y\|, \quad x, y \in C^0(J);$$

(H_2) There exist non-negative functions A, B, $C \in L_1(J)$ and ε_1, $\varepsilon_2 \in [0, 1)$ such that

$$|F(x)(t)| \le A(t) + B(t)\|x\|^{\varepsilon_1} + C(t)\|x'\|^{\varepsilon_2}$$

for a.e. $t \in J$ and each $x \in C^1(J)$.

Remark 2. From (H_1) we see that

$$\|L(x)\| \le k\|x\| + \|L(0)\|, \quad x \in C^0(J).$$

A special case of the operator L satisfying (H_1) and the operator F satisfying (H_2) is the operator

$$L(x)(t) = w(t)x(z(t)) + w_1(t), \quad x \in C^0(J),$$

where $w, w_1, z \in C^0(J)$, $z : J \to J$, $\|w\| < 1/2$ and the Nemytskii operator

$$F(x)(t) = f(t, (Ux)(t), V(x')(t)), \quad x \in C^1(J),$$

where f satisfies the local Carathéodory conditions on $J \times \mathbb{R}^2$, $|f(t, u, v)| \le A(t) + B(t)|u|^{\varepsilon_1} + C(t)|v|^{\varepsilon_2}$, U, $V : C^0(J) \to C^0(J)$ are continuous operators and $\|U(x)\| \le r\|x\|$, $\|V(x)\| \le r\|x\|$ with a positive constant r, respectively.

Special cases of boundary conditions (2) (with $\varphi(x) = x(T)$ and $\psi(x) = x(0)$; $\varphi(x) = x(0)$, $\psi(x) = \int_0^T x(s)\,ds$) are the mixed boundary conditions

$$x(T) = 0, \quad x'(0) = 0 \tag{3}$$

or the Dirichlet conditions

$$x(0) = 0, \quad x(T) = 0. \tag{4}$$

We observe that many papers and monographs have been devoted to existence results for the differential equation

$$x'' = f(t, x, x')$$

and boundary conditions (3) and (4). Here f is either continuous or satisfies the Carathéodory conditions. We refer for example to [1], [5], [8]-[12], [15], [16] and the references given therein. The proofs of the existence results are mostly based upon the Schauder fixed point theorem, a priori estimates, the shooting procedure, topological degree and the technique of lower and upper solutions. In [13] equation (1) was considered with $L = 0$, a special type of the operator F and a linear functional φ and $\psi(x) = x(0)$ in (2). The boundary conditions (2) are the special case of those considered by Brykalov ([2]-[4]) for BVP with equation (1) where $L = 0$.

In this paper we will show that under assumptions (H_1) and (H_2) BVP $(1), (2)$ is solvable. The results are proved by the Leray-Schauder degree and the Borsuk theorem for α-condensing operators (see [6]) which have in our case the form $G + Q$ where G is a strict contraction and Q is a completely continuous operator. Examples 2–5 demonstrate that our assumptions are optimal.

Throughout the paper we will make use of the continuous operators

$$\Pi : C^0(J) \times \mathbb{R} \to C^1(J), \quad H : C^0(J) \times \mathbb{R} \to L_1(J)$$

given by

$$\Pi(x, a)(t) = \int_0^t x(s)\, ds + a \tag{5}$$

and

$$H(x, a)(t) = F(\Pi(x, a))(t). \tag{6}$$

Here F is the operator on the right-hand side of (1).

2. Lemmas

Lemma 1. [14, Lemma 3]. *Let $\varphi \in \mathcal{A}$ and let the equality*

$$\varphi(x) = \varphi(y)$$

be satisfied for some $x, y \in C^0(J)$. Then there exists $\xi \in J$ such that

$$x(\xi) = y(\xi).$$

Lemma 2. *Let* $\varphi, \psi \in \mathcal{A}$, h, l *and* m *be positive constants and set*

$$\Omega_1 = \Big\{(x,a,b) : \ (x,a,b) \in C^0(J) \times \mathbb{R}^2, \ \|x\| < h, \ |a| < l, \ |b| < m\Big\}. \tag{7}$$

Let $\Gamma : \bar{\Omega}_1 \to C^0(J) \times \mathbb{R}^2$ *be defined by*

$$\Gamma(x,a,b) = \left(a, a + \varphi\Big(\int_0^t x(s)\,ds + b\Big), b + \psi(x)\right). \tag{8}$$

Then

$$\mathrm{D}(I - \Gamma, \Omega_1, 0) \neq 0 \tag{9}$$

where "D" *stands for the Leray-Schauder degree and* I *is the identity operator on* $C^0(J) \times \mathbb{R}^2$.

Proof. Set $P : [0,1] \times \bar{\Omega}_1 \to C^0(J) \times \mathbb{R}^2$,

$$P(\lambda,x,a,b) = \left(a, a + \varphi\Big(\int_0^t x(s)\,ds + b\Big) - (1-\lambda)\varphi\Big(-\int_0^t x(s)\,ds - b\Big), b + \psi(x) - (1-\lambda)\psi(-x)\right).$$

To prove (9) it suffices to show, by the homotopy theory and the Borsuk theorem, that

(a) $P(0,\cdot)$ is an odd operator on $\bar{\Omega}_1$,
(b) P is a compact operator, and
(c) $P(\lambda,x,a,b) \neq (x,a,b)$ for $(\lambda,x,a,b) \in [0,1] \times \partial\Omega_1$.

For $(x,a,b) \in \bar{\Omega}_1$ we have

$$\begin{aligned} P(0,-x,-a,-b) = &\left(-a, -a + \varphi\Big(-\int_0^t x(s)\,ds - b\Big) - \varphi\Big(\int_0^t x(s)\,ds + b\Big), -b + \psi(-x) - \psi(x)\right) \\ = &- P(0,x,a,b). \end{aligned}$$

Consequently, $P(0,\cdot)$ is an odd operator.

The continuity of P follows from that of φ and ψ. Since $\bar{\Omega}_1$ is bounded and φ, ψ map any bounded subset of $C^0(J)$ into a bounded subset of $\mathbb{R}$, the set $P([0,1] \times \bar{\Omega}_1)$ is relatively compact by the Bolzano-Weierstrass theorem. Hence P is a compact operator.

Assume, on the contrary, that

$$P(\lambda_0, x_0, a_0, b_0) = (x_0, a_0, b_0)$$

for some $(\lambda_0, x_0, a_0, b_0) \in [0,1] \times \partial\Omega_1$. Then

$$x_0(t) = a_0, \quad t \in J$$

and

$$\begin{aligned} \varphi(a_0 t + b_0) &= (1-\lambda_0)\varphi(-a_0 t - b_0), \\ \psi(a_0) &= (1-\lambda_0)\psi(-a_0). \end{aligned} \tag{10}$$

If $a_0 \neq 0$ then $\psi(a_0)\psi(-a_0) < 0$, which contradicts (cf. (10)) $\psi(a_0)\psi(-a_0) = (1-\lambda_0)(\psi(-a_0))^2 \geq 0$. Therefore $a_0 = 0$, and so (cf. (10))

$$\varphi(b_0) = (1-\lambda_0)\varphi(-b_0).$$

We can now proceed analogously to prove that $b_0 = 0$. Hence $(x_0, a_0, b_0) \notin \partial\Omega_1$, a contradiction. ∎

Consider the BVP (cf. (6))

$$x(t) = a + \lambda\Big(-L(x)(t) + \int_0^t H(x,b)(s)\,ds\Big), \tag*{$(11)_{(\lambda,a,b)}$}$$

$$\varphi\Big(\int_0^t x(s)\,ds + b\Big) = 0, \tag*{$(12)_b$}$$

$$\psi(x) = 0 \tag{13}$$

depending on the parameters λ, a, b, $(\lambda, a, b) \in [0,1] \times \mathbb{R}^2$. Here $\varphi, \psi \in \mathcal{A}$.

We say that $x \in C^0(J)$ is a *solution of BVP* $(11)_{(\lambda,a,b)}$, $(12)_b$, (13) for some $(\lambda, a, b) \in [0,1] \times \mathbb{R}^2$ if $(11)_{(\lambda,a,b)}$ is satisfied for $t \in J$ and x satisfies the boundary conditions $(12)_b$, (13).

Lemma 3. *Let assumptions* (H_1) *and* (H_2) *be satisfied. Let* $x(t)$ *be a solution of BVP* $(11)_{(\lambda,a,b)}$, $(12)_b$, (13) *for some* $(\lambda, a, b) \in [0,1] \times \mathbb{R}^2$. *Then*

$$\|x\| \leq S, \quad |a| \leq S, \quad |b| \leq ST, \tag{14}$$

where S *is a positive constant such that*

$$\frac{2\|L(0)\| + \|A\|_{L_1}}{S} + \frac{\|B\|_{L_1} T^{\varepsilon_1}}{S^{1-\varepsilon_1}} + \frac{\|C\|_{L_1}}{S^{1-\varepsilon_2}} \leq 1 - 2k. \tag{15}$$

Proof. From (13) and Lemma 1 it follows: $x(\xi) = 0$, $\xi \in J$. Then (cf. $(11)_{(\lambda,a,b)}$) $a = \lambda(L(x)(\xi) - \int_0^\xi H(x,b)(s)\,ds)$, and so

$$x(t) = \lambda\Big(L(x)(\xi) - L(x)(t) + \int_\xi^t H(x,b)(s)\,ds\Big), \quad t \in J.$$

By (H_1), (H_2), Remark 2, the definition (6) of H and the equality $\int_0^\nu x(s)\,ds + b = 0$ for some $\nu \in J$ which follows from $(12)_b$ and Lemma 1,

$$\begin{aligned} |x(t)| &\leq 2k\|x\| + 2\|L(0)\| \\ &+ \int_0^T \Big(A(s) + B(s)\Big\|\int_0^t x(v)\,dv + b\Big\|^{\varepsilon_1} + C(s)\|x\|^{\varepsilon_2}\Big)\,ds \end{aligned}$$

$$= 2k\|x\| + 2\|L(0)\| + \|A\|_{L_1} + \|B\|_{L_1}\Big\| \int_\nu^t x(v)\,dv\Big\|^{\varepsilon_1} + \|C\|_{L_1}\|x\|^{\varepsilon_2}$$

for $t \in J$. Consequently,

$$\|x\| \le \frac{1}{1-2k}\Big(2\|L(0)\| + \|A\|_{L_1} + \|B\|_{L_1}(T\|x\|)^{\varepsilon_1} + \|C\|_{L_1}\|x\|^{\varepsilon_2}\Big). \tag{16}$$

Set $p(u) = (2\|L(0)\| + \|A\|_{L_1})/u + (\|B\|_{L_1}T^{\varepsilon_1})/u^{1-\varepsilon_1} + \|C\|_{L_1}/u^{1-\varepsilon_2}$ for $u \in (0,\infty)$. Then p is decreasing and $\lim_{u\to\infty} p(u) = 0$. Hence there exists $S > 0$ such that $p(u) \le 1 - 2k$ for all $u \ge S$, and so (cf. (16)) $\|x\| \le S$. Then

$$|a| = \Big|\lambda\Big(L(x)(\xi) - \int_0^\xi H(x,b)(s)\,ds\Big)\Big|$$

$$\le kS + \|L(0)\| + \|A\|_{L_1} + \|B\|_{L_1}(TS)^{\varepsilon_1} + \|C\|_{L_1}S^{\varepsilon_2}$$

$$\le kS + (1-2k)S \le S$$

and $|b| = |\int_0^\nu x(s)\,ds| \le ST$. □

Under assumptions (H_1) and (H_2) Lemma 3 gives a priori bounds for solutions of BVP $(11)_{(\lambda,a,b)}$, $(12)_b, (13)$ which are very important in proofs of existence results for BVP (1), (2) (see the proof of Theorem 1). If the operator L in (1) satisfies some another assumptions and contingently boundary conditions (2) are more specified, (H_1) can be relaxed. The next Remark 3 (resp. Remark 4) shows that we can assume $k \in [0,1)$ in (H_1) if $\sup\{|L(x)(0)| : x \in C^0(J)\} < \infty$ and $\psi(x) = x(0)$ in (13) (resp. L is a bounded linear operator).

Remark 3. Let assumption (H_2) be satisfied,

$$\sup\{|L(x)(0)| : x \in C^0(J)\} = m < \infty$$

and there exists $k_1 \in [0,1)$ such that $\|L(x) - L(y)\| \le k_1\|x-y\|$ for $x, y \in C^0(J)$. Let $x(t)$ be a solution of $(11)_{(\lambda,a,b)}$ for some $(\lambda,a,b) \in [0,1]\times\mathbb{R}^2$ satisfying the boundary condition $(12)_b$ and $x(0) = 0$. Then $|a| \le m$, $b = -\int_0^\nu x(s)\,ds$ with some $\nu \in J$ and $\Pi(x,b)(t) = \int_\nu^t x(s)\,ds$. Thus

$$x(t) = a + \lambda\Big(-L(x)(t) + \int_0^t F(\tilde{x})(s)\,ds\Big), \quad t \in J,$$

where $\tilde{x}(t) = \int_\nu^t x(s)\,ds$. We can check that $\|x\| \le S_1$, and consequently $|b| \le S_1T$, where S_1 is a positive constant such that

$$\frac{\|L(0)\| + \|A\|_{L_1} + m}{S_1} + \frac{\|B\|_{L_1}T^{\varepsilon_1}}{S_1^{1-\varepsilon_1}} + \frac{\|C\|_{L_1}}{S_1^{1-\varepsilon_2}} \le 1 - k_1. \tag{17}$$

REMARK 4. Let assumption (H_2) be satisfied. Assume that L is a bounded linear operator. By [7, p. 517],

$$L(x)(t) = \int_0^T x(s)\, dg(t,s) \quad \text{for } x \in C^0(J),\ t \in J$$

where $g : J \times J \to \mathbb{R}$ satisfies the following conditions:

(j) for each $t \in J$ the function $g(t,\cdot)$ is a normalized function of bounded variation on J,

(jj) $g(t,T)$ and $\displaystyle\int_0^r g(t,s)\, ds$ are continuous in t for each $r \in J$,

(jjj) $\sup\Big\{ \operatorname*{var}_{0\le s\le T} g(t,s) : t \in J \Big\} < \infty$.

Suppose that there exists $k_2 \in [0,1)$ such that

(jv) $\operatorname*{var}_{0\le s\le T}(g(t_1,s) - g(t_2,s)) \le k_2$ for each $t_1,\ t_2 \in J$.

Then

$$\begin{aligned} |(Lx)(t) - (Lx)(\xi)| &= \Big| \int_0^T x(s)\, d(g(t,s) - g(\xi,s)) \Big| \\ &\le \|x\| \operatorname*{var}_{0\le s\le T}(g(t,s) - g(\xi,s)) \le k_2\|x\| \end{aligned}$$

for each $x \in C^0(J)$ and $t, \xi \in J$. If $x(t)$ is a solution of BVP $(11)_{(\lambda,a,b)}$, $(12)_b$, (13) for some $(\lambda,a,b) \in [0,1] \times \mathbb{R}^2$ then it is easy to check that

$$\|x\| \le S_2, \quad |a| \le (\|L\| + 1 - k_2)S_2, \quad |b| \le S_2 T$$

where S_2 is a positive constant such that

$$\frac{\|A\|_{L_1}}{S_2} + \frac{\|B\|_{L_1} T^{\varepsilon_1}}{S_2^{1-\varepsilon_1}} + \frac{\|C\|_{L_1}}{S_2^{1-\varepsilon_2}} \le 1 - k_2.$$

3. Existence results, examples

THEOREM 1. *Let assumptions (H_1) and (H_2) be satisfied. Then for each $\varphi, \psi \in \mathcal{A}$, BVP* (1), (2) *has a solution $x(t)$ such that $\|x\| \le 2ST$ and $\|x'\| \le S$ where S is a positive constant satisfying* (15).

PROOF. Fix $\varphi, \psi \in \mathcal{A}$ and set

$$\begin{aligned} \Omega = \Big\{ (x,a,b) :\ & (x,a,b) \in C^0(J) \times \mathbb{R}^2,\ \|x\| < S+1, \\ & |a| < S+1,\ |b| < ST+1 \Big\}. \end{aligned} \tag{18}$$

Let the operators $\mathcal{C}$, $\mathcal{K} : \bar{\Omega} \to C^0(J) \times \mathbb{R}^2$ be defined by

$$\mathcal{C}(x,a,b) = \bigg(a + \int_0^t H(x,b)(s)\, ds,\ a + \varphi\Big(\int_0^t x(s)\, ds + b \Big),\ b + \psi(x) \bigg),$$

$$\mathcal{K}(x,a,b) = (-L(x),0,0)$$

and let $U,\,V : [0,1]\times\bar{\Omega} \to C^0(J)\times\mathbb{R}^2$,

$$U(\lambda,x,a,b) = \left(a+\lambda\int_0^t H(x,b)(s)\,ds,\ a+\varphi\Big(\int_0^t x(s)\,ds+b\Big),\ b+\psi(x)\right),$$

$$V(\lambda,x,a,b) = \lambda\mathcal{K}(x,a,b).$$

Then $U(0,\cdot)+V(0,\cdot)=\Gamma(\cdot)$ and $U(1,\cdot)+V(1,\cdot)=\mathcal{C}(\cdot)+\mathcal{K}(\cdot)$, where Γ is defined on $\bar{\Omega}$ by (8). Hence $\mathrm{D}(I-U(0,\cdot)-V(0,\cdot),\Omega,0)\neq 0$ by Lemma 2, and to prove that

$$\mathrm{D}(I-\mathcal{C}-\mathcal{K},\Omega,0)\neq 0 \tag{19}$$

it suffices to verify, by the theory of homotopy for α-condensing operators, that

(i) U is a compact operator,

(ii) there exists $m\in[0,1)$ such that for $(\lambda,x,a_1,b_1),\ (\lambda,y,a_2,b_2)\in[0,1]\times\bar{\Omega}$,

$$\|V(\lambda,x,a_1,b_1)-V(\lambda,y,a_2,b_2)\|_*\le m\|(x,a_1,b_1)-(y,a_2,b_2)\|_*,$$

(iii) $U(\lambda,x,a,b)+V(\lambda,x,a,b)\neq(x,a,b)$ for $(\lambda,x,a,b)\in[0,1]\times\partial\Omega$.

Continuity of U follows from that of H, φ and ψ. Let $\{(\lambda_n,x_n,a_n,b_n)\}\subset[0,1]\times\bar{\Omega}$. Then (cf. (H_2) and (18))

$$\begin{aligned}
&\Big|a_n+\lambda_n\int_0^t H(x_n,b_n)(s)\,ds\Big|\\
&\le|a_n|+\int_0^T\Big(A(t)+B(t)\Big\|\int_0^s x_n(\nu)\,d\nu+b_n\Big\|^{\varepsilon_1}+C(t)\|x_n\|^{\varepsilon_2}\Big)\,dt\\
&\le S+1+\|A\|_{L_1}+(2ST+T+1)^{\varepsilon_1}\|B\|_{L_1}+(S+1)^{\varepsilon_2}\|C\|_{L_1},
\end{aligned}$$

$$\begin{aligned}
\Big|\int_{t_1}^{t_2}H(x_n,b_n)(s)\,ds\Big| &\le \Big|\int_{t_1}^{t_2}A(s)\,ds\Big|\\
&+(2ST+T+1)^{\varepsilon_1}\Big|\int_{t_1}^{t_2}B(s)\,ds\Big|+(S+1)^{\varepsilon_2}\Big|\int_{t_1}^{t_2}C(s)\,ds\Big|,
\end{aligned}$$

$$\Big|a_n+\varphi\Big(\int_0^t x_n(s)\,ds+b_n\Big)\Big|\le S+1+\max\{\varphi(2ST+T+1),\,-\varphi(-2ST-T-1)\},$$

$$|b_n+\psi(x_n)|\le ST+1+\max\{\psi(S+1),\,-\psi(-S-1)\}$$

for $t,t_1,t_2\in J$ and $n\in\mathbb{N}$. By the Arzelà-Ascoli theorem and the Bolzano-Weierstrass theorem, there is a convergent subsequence of $\{U(\lambda_n,x_n,a_n,b_n)\}$. Hence U is a compact operator.

For $(\lambda,x,a_1,b_1),(\lambda,y,a_2,b_2)\in[0,1]\times\bar{\Omega}$ we have

$$\begin{aligned}
\|V(\lambda,x,a_1,b_1)-V(\lambda,y,a_2,b_2)\|_* &= \lambda\|L(x)-L(y)\|\le\lambda k\|x-y\|\\
&\le k\|(x,a_1,b_1)-(y,a_2,b_2)\|_*,
\end{aligned}$$

and consequently (ii) is satisfied with $m = k$.

Assume that

$$U(\lambda_0, x_0, a_0, b_0) + V(\lambda_0, x_0, a_0, b_0) = (x_0, a_0, b_0)$$

for some $(\lambda_0, x_0, a_0, b_0) \in [0, 1] \times \partial\Omega$. Then

$$x_0(t) = a_0 + \lambda_0\Big(-L(x_0)(t) + \int_0^t H(x_0, b_0)(s)\, ds\Big), \quad t \in J,$$

$$\varphi\Big(\int_0^t x_0(s)\, ds + b_0\Big) = 0, \quad \psi(x_0) = 0.$$

Hence $x_0(t)$ is a solution of BVP $(11)_{(\lambda_0, a_0, b_0)}$, $(12)_{b_0}$, (13) and then, by Lemma 3,

$$\|x_0\| \le S, \quad |a_0| \le S, \quad |b_0| \le ST,$$

which contradicts $(x_0, a_0, b_0) \in \partial\Omega$. We have proved (19). Therefore there exists a fixed point (u, a, b) of the operator $\mathcal{C} + \mathcal{K}$. Then

$$u(t) = a - L(u)(t) + \int_0^t H(u, b)(s)\, ds, \quad t \in J, \tag{20}$$

$$\varphi\Big(\int_0^t u(s)\, ds + b\Big) = 0, \quad \psi(u) = 0.$$

Setting $x(t) = \int_0^t u(s)\, ds + b$ for $t \in J$, we see that

$$x'(t) + L(x')(t) = a + \int_0^t F(x)(s)\, ds, \quad t \in J,$$

$$\varphi(x) = 0, \quad \psi(x') = 0,$$

and consequently $x(t)$ is a solution of BVP (1), (2). Moreover (cf. Lemma 3), $\|x\| \le T\|u\| + |b| \le 2ST$, $\|x'\| \le S$. □

Remark 5. It is easily seen that the results of our paper are true if we consider instead of $J = [0, T]$ a compact interval $[a, b]$ with $b - a = T$.

Remark 6. Consider the boundary conditions

$$\varphi(x) = a, \quad \psi(x') = b \tag{21}$$

where $\varphi, \psi \in \mathcal{A}$ and $a \in \mathrm{Im}(\varphi)$, $b \in \mathrm{Im}(\psi)$. Here $\mathrm{Im}(\varrho)$ denotes the range of $\varrho \in \mathcal{A}$. Under the assumptions of Theorem 1 we can even prove that, for each $\varphi, \psi \in \mathcal{A}$ and $a \in \mathrm{Im}(\varphi)$, $b \in \mathrm{Im}(\psi)$, BVP (1), (21) has a solution

Remark 7. The solvability of BVP (1), (2) has been proved in Theorem 1 under the assumption that the operator L satisfies

$$\|L(x) - L(y)\| \le k\|x - y\|, \quad x, y \in C^0(J) \tag{22}$$

with some $k \in [0, 1/2)$. If we consider, for example, the boundary conditions

$$\varphi(x) = 0, \quad x'(0) = 0, \tag{23}$$

which are a special case of (2), we can even proof the following result:

> *Let* $\sup\{|L(x)(0)| : x \in C^0(J)\} < \infty$ *and* (22) *be fulfilled with some* $k \in [0, 1)$. *Then under assumption* (H_2), *for each* $\varphi \in \mathcal{A}$, *BVP* (1), (23) *has a solution.*

The proof of the above result is based on Remark 3 and uses the procedure of the proof of Theorem 1.

Remark 8. Let L in (1) be a linear bounded operator. Using Remark 4 and applying the procedure of the proof of Theorem 1, we can prove the followig assertion:

> *Let* $(Lx)(t) = \int_0^T x(s)\, dg(t, s)$ *for* $x \in C^0(J)$ *and* $t \in J$ *with* $g : J \times J \to \mathbb{R}$ *satisfying conditions* $(j)-(jv)$ *in Remark* 4 *with* $k_2 \in [0, 1)$. *Suppose also that assumption* (H_2) *be satisfied. Then BVP* (1), (2) *has a solution for each* $\varphi, \psi \in \mathcal{A}$.

In the next example, Example 2, we will show that the conditions $k \in [0, 1/2)$ in assumption (H_1) is optimal for BVP (1), (2) and cannot be replaced by $k \in [0, 1/2]$. Analogously, Example 3 shows that the condition $k \in [0, 1)$ in (22) is optimal for BVP (1), (23) (with $\sup\{|L(x)(0)| : x \in C^0(J)\} < \infty$).

Example 2. Let $J = [0, 1]$ and consider the BVP

$$(x'(t) + w(t)x'(1))' = 1, \tag{24}$$

$$\varphi(x) = 0, \quad \min\{x'(t) : t \in J\} = 0, \tag{25}$$

where $w \in C^0(J)$, $\|w\| = 1/2$, $w(0) = 1/2$ and $w(1) = -1/2$. Assume that $u(t)$ is a solution of BVP (24), (25). Then (cf. (25))

$$u'(t) \ge 0, \quad t \in J \tag{26}$$

and there exists $\nu \in J$ such that $u'(\nu) = 0$. Therefore

$$u'(t) = t - \nu + (w(\nu) - w(t))u'(1), \quad t \in J. \tag{27}$$

If $\nu = 0$ then

$$u'(1) = 1 + (w(0) - w(1))u'(1) = 1 + u'(1),$$

which is impossible. If $\nu = 1$ then

$$u'(t) = t - 1 + \Big(-\frac{1}{2} - w(t)\Big)u'(1) \le t - 1 < 0, \quad t \in [0, 1),$$

which contradicts (26). Hence $\nu \in (0, 1)$ and from (27) we have

$$u'(0) = -\nu + \Big(w(\nu) - \frac{1}{2}\Big)u'(1) \le -\nu,$$

contrary to (26). We have proved that BVP (24), (25) is not solvable.

EXAMPLE 3. Let $J = [0,1]$ and consider the BVP

$$(x'(t) - x'(t^2))' = 1, \quad (23) \tag{28}$$

where $\varphi \in \mathcal{A}$ in (23). Assume that $u(t)$ is a solution of BVP (28). Since $u'(0) = 0$ we have

$$u'(t) - u'(t^2) = t, \quad t \in J,$$

and so $u'(1) - u'(1) = 1$, which is impossible.

The next two examples illustrate that the constants $\varepsilon_1, \varepsilon_2 \in [0,1)$ in assumption (H_2) are optimal for BVP (1), (2) that it, if either $\varepsilon_1 = 1$ or $\varepsilon_2 = 1$ then there exists an unsolvable BVP of the type (1), (2).

EXAMPLE 4. Consider the BVP (for $\varphi \in \mathcal{A}$)

$$x''(t) = 1 + \|x'\|, \tag{29}$$

$$\varphi(x) = 0, \quad \min\{x'(t) : t \in J\} = 0 \tag{30}$$

on the interval $J = [0,1]$. Assume that there exists a solution $u(t)$ of BVP (29), (30). From (30) it follows that $u'(t) \geq 0$ on J and there is a $\nu \in J$ such that $u'(\nu) = 0$. Since $u'(t)$ is increasing on J, $\nu = 0$, $\|u'\| = u'(1)$, and consequently

$$u'(t) = (1 + u'(1))t, \quad t \in J.$$

Therefore

$$u'(1) = 1 + u'(1),$$

which is impossible.

EXAMPLE 5. Consider the BVP

$$x''(t) = 1 + 2\|x\|, \tag{31}$$

$$\min\{x(t) : t \in J\} = 0, \quad \min\{x'(t) : t \in J\} = 0 \tag{32}$$

on the interval $J = [0,1]$. Suppose that $u(t)$ is a solution of BVP (31), (32). Then $u'(t)$ is increasing on J and from (32) we deduce that $u(t) \geq 0$, $u'(t) \geq 0$ for $t \in J$ and $u(0) = 0$, $u'(0) = 0$, $\|u\| = u(1)$. Then from the equality

$$u''(t) = 1 + 2u(1), \quad t \in J,$$

we obtain

$$u(t) = \frac{t^2}{2}\Big(1 + 2u(1)\Big), \quad t \in J.$$

Hence $u(1) = 1/2 + u(1)$, which is impossible.

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Department of Mathematical Analysis,
Faculty of Science,
Palacký University,
Tomkova 40, 779 00 Olomouc, Czech Republic
E-mail: `stanek@risc.upol.cz`

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