GLASNIK MATEMATIČKI
Vol. 36(56)(2001), 95 – 103

A STRONGER LIMIT THEOREM IN EXTENSION THEORY

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Abstract. This work contains an improvement to a limit theorem which has been proved by the author and P. J. Schapiro. In that result it was shown that for a given simplicial complex K, if an inverse sequence of metrizable spaces X_i each has the property that $X_i\tau|K|$, then it is true that $X\tau|K|$, where X is the limit of the sequence. The property that $X\tau|K|$ means that for each closed subset A of X and each map $f : A \to |K|$, there exists a map $F : X \to |K|$ which is an extension of f. This is the fundamental notion of extension theory.

The version put forth herein is stronger in that it places a requirement only on the bonding maps, but one which is necessarily true in case each $X_i\tau|K|$.

1. Introduction

The notion of extension theory is a generalization of dimension theories such as covering and cohomological dimensions; good sources for extension theory can be found in [DD] and [Sh]. Under the light shed by extension theory, it is frequently possible to obtain theorems which apply to dimension theory, but which are much more general. The limit theorem ([RS]) for inverse sequences of metrizable spaces in extension theory is such an example. We are going to prove here a stronger version of that limit theorem.

Recall that if K is a CW-complex and X is a space, then $X\tau K$ means that for each closed subset A of X and map $f : A \to K$, there exists a map

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2000 *Mathematics Subject Classification.* 54F45, 55M15.

Key words and phrases. covering dimension, cohomological dimension, extension, limit, inverse sequence, metrizable space.

This work was completed while the author was a visitor at the University of Zagreb, Croatia, as a Fulbright Lecturer during the fall semester of 1999 while on sabbatical leave from the Department of Mathematics of the University of Oklahoma, Norman, OK, USA..

$F : X \to K$ which is an extension of F. This is the fundamental notion of extension theory.

For information about inverse sequences and their limits, one may consult [Du]. When K below is a simplicial complex, then $|K|$ will be given the weak topology determined by K.

The result in [RS], Theorem 3.1, goes this way:

Theorem 1.1. *Let K be a simplicial complex and $X = \lim \mathbf{X}$, where $\mathbf{X} = (X_i, p_{i\,i+1}, \mathbb{N})$ is an inverse sequence of metrizable spaces X_i and $X_i \tau |K|$ for all $i \in \mathbb{N}$. Then $X \tau |K|$.*

In order to state the improved version, let us first give a definition taken from a notion previously introduced by A. Dranishnikov.

Definition 1.2. *Let $\mathbf{X} = (X_i, p_{i\,i+1}, \mathbb{N})$ be an inverse sequence and K be a CW-complex. We shall write that $\mathbf{X}\tau K$ if for each $i \in \mathbb{N}$, closed subset A of X_i, and map $f : A \to K$, there exists $j \geq i$ and a map $g : X_j \to K$ such that $g(x) = f \circ p_{i\,j}(x)$ for every $x \in p_{i\,j}^{-1}(A)$.*

The next lemma is easy to prove.

Lemma 1.3. *Let $\mathbf{X} = (X_i, p_{i\,i+1}, \mathbb{N})$ be an inverse sequence and K be a CW-complex. Then $\mathbf{X}\tau K$ if and only if for each $i \in \mathbb{N}$, closed subset A of X_i, and map $f : A \to K$, there exists $j \geq i$ such that for all $k \geq j$, there is a map $g : X_k \to K$ such that $g(x) = f \circ p_{i\,k}(x)$ for every $x \in p_{i\,k}^{-1}(A)$.*

We shall prove the following.

Theorem 1.4. *Let $\mathbf{X} = (X_i, p_{i\,i+1}, \mathbb{N})$ be an inverse sequence of metrizable spaces, K be a CW-complex such that $\mathbf{X}\tau K$, and $X = \lim \mathbf{X}$. Then $X \tau K$.*

Since every CW-complex K is homotopy equivalent to $|K_0|$, for some simplicial complex K_0, and every CW-complex is an absolute neighborhood extensor for metrizable spaces, then Theorem 1.4 is equivalent to our main result,

Theorem 1.5. *Let $\mathbf{X} = (X_i, p_{i\,i+1}, \mathbb{N})$ be an inverse sequence of metrizable spaces X_i, K be a simplicial complex such that $\mathbf{X}\tau |K|$, and $X = \lim \mathbf{X}$. Then $X \tau |K|$.*

Surely Theorem 1.5 implies Theorem 1.1.

In section 3, we shall prove the following (seemingly weaker) theorem.

Theorem 1.6. *Let $\mathbf{X} = (X_i, p_{i\,i+1}, \mathbb{N})$ be an inverse sequence of metrizable spaces X_i with surjective bonding maps $p_{i\,i+1}$, K be a simplicial complex such that $\mathbf{X}\tau |K|$, and $X = \lim \mathbf{X}$. Then $X \tau |K|$.*

Proposition 1.7. *Theorem 1.6 implies Theorem 1.5.*

PROOF. Proof Let $\mathbf{X}$ be an inverse sequence as in Theorem 1.5. We begin a recursive process. Let $X_1^* = X_1$ and put $Y_1 = X_1^* \backslash p_{1\,2}(X_2)$. There exists a metrizable space $X_2^* = X_2 \cup Z_2$ where X_2 is an open and closed subspace of X_2^*, and Z_2 is a discrete subspace of X_2^* having the same cardinality as Y_1.

Define $p_{1\,2}^* : X_2^* \to X_1^*$ so that $p_{1\,2}^*|X_2 = p_{1\,2}$ and $p_{1\,2}^*(Z_2) = Y_1$. Such a procedure may be continued recursively resulting in an inverse sequence $\mathbf{X}^* = (X_i^*, p_{i\,i+1}^*, \mathbb{N})$ of metrizable spaces so that for each $i \in \mathbb{N}$,

(1) $p_{i\,i+1}^*$ is surjective,
(2) X_i is an open and closed subspace of X_i^*,
(3) $p_{i\,i+1}^*|X_{i+1} = p_{i\,i+1} : X_{i+1} \to X_i$, and,
(4) $X_i^* \backslash X_i$ is a discrete subspace of X_i^*.

Using (2)–(4), along with the information $\mathbf{X}\tau|K|$, the reader will easily check that $\mathbf{X}^*\tau|K|$. Let $X^* = \lim \mathbf{X}^*$. By Theorem 1.6, $X^*\tau|K|$. Of course X^* is a metrizable space; one sees from (2) and (3) that X embeds as a closed subspace of X^*. So $X\tau|K|$.

We would like to express our thanks to Professor Ivan Ivanšić of the University of Zagreb for his careful reading of this paper and for his many suggestions which helped the author to make a more clear presentation of this work. We are extremely grateful to the referee of this paper who pointed out to us several incorrect steps in the original version. Without that thorough and careful reading, we would not have been aware of the problems, and hence would not have been able to make the appropriate repairs.

2. Preliminaries

For the convenience of the current reader, we shall list items 2.1, 2.4–2.7 of [RS]. More details, of course, are given in the latter.

LEMMA 2.1. *Let X be a space satisfying the first countability axiom, K be a simplicial complex, and $f : X \to |K|$ be a map. Then the (indexed) collection $\{Q_v = f^{-1}(\mathrm{st}(v, K)) \,|\, v \in K^{(0)}\}$ is a locally finite open cover of X.*

If $\mathcal{Q} = \{Q_v \,|\, v \in \Gamma\}$ is a collection of sets, then $N(\mathcal{Q})$ will denote its nerve.

LEMMA 2.2. *Let P be a closed subset of a metrizable space B and $\mathcal{U} = \{U_v \,|\, v \in \Gamma\}$ be an open cover of B. Put $\mathcal{E} = \{E_v = U_v \cap P \,|\, v \in \Gamma\}$ and let $f : P \to |N(\mathcal{E})|$ be an $\mathcal{E}$-canonical map. Let $\theta : N(\mathcal{E}) \to N(\mathcal{U})$ be the simplicial injection determined by the vertex map $E_v \mapsto U_v$. Then there is a $\mathcal{U}$-canonical map $g : B \to |N(\mathcal{U})|$ such that $g(P) \subset \theta(|N(\mathcal{E})|)$ and for all $x \in P$, $\theta^{-1}(g(x)) = f(x)$ (thus, $\theta(f(x)) = g(x)$).*

LEMMA 2.3. *Let X be a metrizable space, K be a simplicial complex, and $f,\, g : X \to |K|$ be maps such that for each $x \in X$, there is a simplex σ of K such that $f(x),\, g(x) \in \sigma$. Then $f \simeq g$.*

Definition 2.4. *Let $f : X \to Y$ be a map and W be an open subset of X. Then* $\mathrm{resp}(W, f)$ *is the maximal open subset U of Y such that $f^{-1}(U) \subset W$. We call U the W-***response** *to f. Suppose that $\mathcal{W} = \{W_v \mid v \in \Gamma\}$ is an indexed collection of open subsets of X. Then by* $\mathrm{resp}(\mathcal{W}, f)$ *we mean the (indexed) collection,* $\{U_v = \mathrm{resp}(W_v, f) \mid v \in \Gamma\}$.

Lemma 2.5. *Let $W \subset W'$ be open subsets of a space X, and let $f : X \to Y$, $g : Y \to Z$ be maps. We write $h = gf : X \to Z$. Then* $g^{-1}(\mathrm{resp}(W, h)) \subset \mathrm{resp}(W, f) \subset \mathrm{resp}(W', f)$.

Considering this fact and the nature of basic open subsets of an inverse limit, we leave a proof of the following to the reader.

Lemma 2.6. *Let $\mathbf{X} = (X_i, p_{i\,i+1}, \mathbb{N})$ be an inverse sequence with coordinate projections $p_i : X = \lim \mathbf{X} \to X_i$. If H is an open subset of X and we define $H_i = \mathrm{resp}(H, p_i)$ for each $i \in \mathbb{N}$, then*

(a) *H_i is open in X_i,*
(b) *$p_i^{-1}(H_i) \subset H$,*
(c) *$H = \bigcup\{p_i^{-1}(H_i) \mid i \in \mathbb{N}\}$, and*
(d) *$p_{i\,j}^{-1}(H_i) \subset H_j$ whenever $i \leq j$.*

A few additional items also will help below. Let us state the homotopy extension theorem for metrizable spaces. It follows easily from III.10.4 of [Hu] and the standard proof of the (Borsuk) homotopy extension theorem.

Theorem 2.7. *Let K be a simplicial complex, X be a metrizable space, A be a closed subset of X, and f, $g : A \to |K|$ be homotopic maps. Then if f extends to a map of X to $|K|$, so does g.*

Here is a routine fact from general topology.

Lemma 2.8. *Let W be an open subset of a space X and $H = X \setminus \overline{W}$. Then $\partial H = \overline{H} \cap \partial W = \overline{H} \cap \overline{W}$.*

Using the preceding lemma and the homotopy extension theorem, one attains the next lemma.

Lemma 2.9. *Let W be an open subset of a metrizable space X, K be a simplicial complex, and $H = X \setminus \overline{W}$. Suppose that $f : \overline{W} \to |K|$ and $g : \overline{H} \to |K|$ are maps such that $g|\partial H \simeq f|\partial H$. Then f extends to a map of X to $|K|$.*

3. Proof of Limit Theorem

Although the proof we give here resembles the one in [RS], it is sufficiently different that one cannot attain it through minor adjustments. We shall try, however, to maintain a parallel notation.

PROOF OF 1.6. Noting that X is a metrizable space, let $A \subset X$ be closed and $f : A \to |K|$ be a map. We may as well assume that $A \neq \emptyset$ and $A \neq X$. Extend f to a map $f_0 : W_0 \to |K|$ where W_0 is an open neighborhood of A in X. Let W_0^* be an open neighborhood of A in X whose closure $\overline{W}_0^*$ (relative to X) is contained in W_0.

Put $H = X \setminus \overline{W}_0^*$. We are going to show that there is a map $G : \overline{H} \to |K|$ such that for each $x \in \partial H$, $f_0(x)$ and $G(x)$ lie in a simplex of K. Then an application of Lemmas 2.3 and 2.9 (since f_0 is an extension of f) will complete our proof.

For each $v \in K^{(0)}$, let $W_v = f_0^{-1}(\mathrm{st}(v, K))$. We shall denote by Γ the subset of $K^{(0)}$ consisting of those v such that $W_v \cap \partial H \neq \emptyset$. Let $\mathcal{W} = \{W_v \mid v \in \Gamma\}$ and $W_1 = \bigcup \mathcal{W}$. Applying Lemma 2.1, we see that $\mathcal{W}$ is a locally finite open cover of the open subset W_1 of W_0 in terms of the indexing set Γ. Also one sees that W_1 is a neighborhood of ∂H.

We define

$$f_1 = f_0 | W_1 : W_1 \to |K|.$$

The inclusion $\Gamma \hookrightarrow K^{(0)}$ induces a simplicial injection of nerves, $\eta_1 : N(\mathcal{W}) \to K$ so that $\eta_1(W_v) = v$.

Using p_i to denote the ith coordinate projection of X to X_i, put

$$H_i = \mathrm{resp}(H, p_i).$$

Then (a)–(d) of Lemma 2.6 hold true.

Let us fix some more notation. For each $v \in \Gamma$, define

$$U_{i,v} = \mathrm{resp}(W_v, p_i).$$

We thus have a certain indexed open collection in X_i: $\mathcal{U}_i = \{U_{i,v} \mid v \in \Gamma\}$. This gives rise to an open subset of X_i, namely, $U_i = \bigcup \mathcal{U}_i$. Since $p_i^{-1}(U_{i,v}) \subset W_v \in \mathcal{W}$, the identity function $\Gamma \to \Gamma$ induces a simplicial injection $\beta_i : N(\mathcal{U}_i) \to N(\mathcal{W})$ where $\beta_i(U_{i,v}) = W_v$. Taking into account Lemma 2.5, one deduces that for all $k \in \mathbb{N}$,

(1) $p_{i\,i+k}^{-1}(U_{i,v}) \subset U_{i+k,v}$, and, moreover,
(2) $p_{i\,i+k}^{-1}(U_i) \subset U_{i+k}$.

Here now are certain closed sets. Put

$$Z_i = \mathrm{cl}_{X_i}(p_i(\partial H)).$$

It follows from 2.6(a) and (b) that,

$$Z_i \cap H_i = \emptyset.$$

Consider the open subset

$$K_i = U_i \cap Z_i$$

of Z_i. By a recursive process using (2), choose for each $i \in \mathbb{N}$, a sequence $(K_i^j)_{j=1}^{\infty}$ of closed subsets of X_i such that

(3) $K_i^j \subset K_i^{j+1} \subset K_i$ for each $j \in \mathbb{N}$
(4) $\bigcup\{K_i^j \mid j \in \mathbb{N}\} = K_i$, and
(5) $p_{i\,k+1}^{-1}(K_i^k) \cap Z_{k+1} \subset K_{k+1}^1$ whenever $1 \leq i \leq k$.

One sees from this definition and the preceding that,

(6) $K_i^j \cap H_i = \emptyset$ for each i and j.

We want to describe an inductive procedure. To begin this simply, we make the assumption, without losing generality, that X_1 is a singleton.

Surely H_1, U_1, $K_1 = \emptyset$. Define $D_1 = \emptyset$ and for each $j \in \mathbb{N}$, put $H_1^j = \emptyset$.

Let $k \in \mathbb{N}$. We assume inductively that for each $1 \leq i \leq k$ we have chosen a positive integer l_i, so that $1 = l_1 < \cdots < l_k$, a sequence $(H_i^j)_{j=1}^{\infty}$ of closed subsets of X_{l_i} and a closed neighborhood D_i of $K_{l_i}^1$ in X_{l_i}. The sets H_i^j, D_i are to satisfy,

(7) $H_i^j \subset H_i^{j+1} \subset H_{l_i}$ for each $j \in \mathbb{N}$,
(8) $\bigcup\{\operatorname{int} H_i^j \mid j \in \mathbb{N}\} = H_{l_i}$,
(9) $p_{l_u\,l_s}^{-1}(H_u^s) \subset H_s^1$ whenever $1 \leq u < s \leq k$,
(10) $D_i \cap H_i^1 = \emptyset$,
(11) $p_{l_i\,l_{i+1}}^{-1}(D_i) \subset D_{i+1}$ whenever $i < k$, and
(12) $D_i \subset U_{l_i}$.

Put

(13) $\mathcal{E}_i = \{E_{i,v} = U_{l_i,v} \cap D_i \mid v \in \Gamma\}$, an indexed open cover of D_i,
(14) $\tau_i : N(\mathcal{E}_i) \to N(\mathcal{U}_{l_i})$ the simplicial injection determined by the vertex map $E_{i,v} \mapsto U_{l_i,v}$, and
(15) $\alpha_i = \eta_1 \beta_{l_i} \tau_i : N(\mathcal{E}_i) \to K$, noting that α_i is a simplicial injection.

In addition, assume we have selected

(16) an $\mathcal{E}_i$-canonical map $g_i : D_i \to |N(\mathcal{E}_i)|$.

Put

$$T_i = D_i \cup H_i^1.$$

We further require that we have chosen a map $g_i^* : T_i \to |K|$ which is an extension of $\alpha_i g_i : D_i \to |K|$ in such a manner that,

(17) $g_i^*(x) = g_{i-1}^* p_{l_{i-1}\,l_i}(x)$ whenever $1 < i \leq k$ and $x \in p_{l_{i-1}\,l_i}^{-1}(T_{i-1})$.

Suppose that $1 \leq i < k$ and that $\{E_{i,v_1}, \ldots, E_{i,v_s}\}$ is the vertex set of a simplex of $N(\mathcal{E}_i)$. Then (13) shows that $\{U_{l_i,v_1}, \ldots, U_{l_i,v_s}\}$ is the vertex set of a simplex of $N(\mathcal{U}_{l_i})$. This, the surjectivity of $p_{l_i\,l_{i+1}}$, (1), (11), and (13) show that the vertex maps $E_{i,v} \mapsto E_{i+1,v}$ and $U_{l_i,v} \mapsto U_{l_{i+1},v}$ respectively determine simplicial injections $\theta_i : N(\mathcal{E}_i) \to N(\mathcal{E}_{i+1})$ and $\theta_i^* : N(\mathcal{U}_{l_i}) \to N(\mathcal{U}_{l_{i+1}})$. One can see from the definitions that,

(18) $\theta_i^* \tau_i = \tau_{i+1} \theta_i$ and $\beta_{l_{i+1}} \theta_i^* = \beta_{l_i}$.

Choose $l_{k+1} > l_k$ by applying $\mathbf{X}\tau|K|$, to the closed subset T_k of X_{l_k}. Let

$$R = p_{l_k\,l_{k+1}}^{-1}(T_k).$$

Next select a sequence $(H^j_{k+1})_{j=1}^{\infty}$ of closed subsets of $X_{l_{k+1}}$ so that (7)–(9) are true when the index k is increased to $k+1$. One is assured of being able to obtain (9) because of 2.6(d).

Select a closed neighborhood D_{k+1} of $K^1_{l_{k+1}}$ in $X_{l_{k+1}}$ so that (10)–(12) are true for k replaced by $k+1$. This may be accomplished because of (6), (2) and the fact that always $K_i \subset U_i$. Then pick $\mathcal{E}_{k+1}$, τ_{k+1}, and α_{k+1} in analogy with (13)–(15). We are not yet ready for g_{k+1}, but the reader easily can see that there are maps θ_k and θ^*_k like those above satisfying (18).

Let

$$P = p^{-1}_{l_k\,l_{k+1}}(D_k),$$

and put $\mathcal{E} = \{E_v = E_{k+1,v} \cap P \mid v \in \Gamma\}$. Then $\mathcal{E}$ is an open cover of P because of (11) and (13). For each vertex $E_{k,v}$ of $N(\mathcal{E}_k)$, we know from (1) and (11) that, $p^{-1}_{l_k\,l_{k+1}}(E_{k,v}) = p^{-1}_{l_k\,l_{k+1}}(U_{l_k,v} \cap D_k) = p^{-1}_{l_k\,l_{k+1}}(U_{l_k,v}) \cap p^{-1}_{l_k\,l_{k+1}}(D_k) \subset U_{l_{k+1},v} \cap P = E_v$, i.e.,

(19) $p^{-1}_{l_k\,l_{k+1}}(E_{k,v}) \subset E_v$.

So, again using the surjectivity of the bonding maps, the vertex map $E_{k,v} \mapsto E_v$ determines a simplicial injection $\phi : N(\mathcal{E}_k) \to N(\mathcal{E})$. Define $\hat{f} : P \to |N(\mathcal{E})|$ by

$$\hat{f}(x) = \phi g_k p_{l_k\,l_{k+1}}(x),\ x \in P.$$

We wish to show that

(20) $\hat{f}$ is an $\mathcal{E}$-canonical map.

Surely $\phi^{-1}(\mathrm{st}(E_v, N(\mathcal{E}))) \subset \mathrm{st}(E_{k,v}, N(\mathcal{E}_k))$ for each vertex E_v of $N(\mathcal{E})$. From (16) we get that $g_k^{-1}(\mathrm{st}(E_{k,v}, N(\mathcal{E}_k))) \subset E_{k,v}$. We conclude from this, the definition of $\hat{f}$, and (19), that (20) is true.

Next define $\theta : N(\mathcal{E}) \to N(\mathcal{E}_{k+1})$ to be the simplicial injection determined by the vertex map $E_v \mapsto E_{k+1,v}$. With $B = D_{k+1}$, $\hat{f}$ in place of f, and $\mathcal{E}_{k+1}$ in place of $\mathcal{U}$, we apply Lemma 2.2. This yields an $\mathcal{E}_{k+1}$-canonical map g_{k+1} as requested in (16), but which enjoys the property that for $x \in p^{-1}_{l_k\,l_{k+1}}(D_k)$, $\theta\hat{f}(x) = g_{k+1}(x)$. The definition of $\hat{f}$ thus shows that $g_{k+1}(x) = \theta\phi g_k p_{l_k\,l_{k+1}}(x)$, $x \in P$. One readily checks that $\theta\phi = \theta_k$, so,

(21) $g_{k+1}(x) = \theta_k g_k p_{l_k\,l_{k+1}}(x)$, $x \in P$.

For such x, $\alpha_{k+1} g_{k+1}(x) \in |K|$ and by the definition of α_{k+1} and (18), $\alpha_{k+1}\theta_k = \eta_1\beta_{l_{k+1}}\tau_{k+1}\theta_k = \eta_1\beta_{l_{k+1}}\theta^*_k\tau_k = \eta_1\beta_{l_k}\tau_k$. From this and (21),

$$\alpha_{k+1} g_{k+1}(x) = \eta_1\beta_{l_k}\tau_k g_k p_{l_k\,l_{k+1}}(x) = \alpha_k g_k p_{l_k\,l_{k+1}}(x).$$

Since $p_{l_k\,l_{k+1}}(x) \in D_k$, and g^*_k is an extension of $\alpha_k g_k$, we see that $\alpha_{k+1} g_{k+1}(x) = g^*_k p_{l_k\,l_{k+1}}(x)$. Therefore we may extend $\alpha_{k+1} g_{k+1} : P \to |K|$ to a map $\hat{g}_{k+1} : R \to |K|$ by setting

(22) $\hat{g}_{k+1}(x) = g^*_k p_{l_k\,l_{k+1}}(x)$, $x \in R$.

Our choice of l_{k+1} guarantees that we may extend $\hat{g}_{k+1}$ to a map $\tilde{g}_{k+1} : S \to |K|$, where

$$S = R \cup H^1_{k+1}.$$

From (10), $D_{k+1} \cap S \subset D_{k+1} \cap R$. Moreover, (9) shows that $p^{-1}_{l_k\, l_{k+1}}(H^1_k) \subset H^1_{k+1}$, so $C = D_{k+1} \cap S \subset P$. On C, the map $\tilde{g}_{k+1}$ is defined by $\tilde{g}_{k+1}(x) = \hat{g}_{k+1}(x) = \alpha_{k+1} g_{k+1}(x)$. We therefore extend $\tilde{g}_{k+1}$ to a map $g^*_{k+1} : T_{k+1} \to |K|$ by setting $g^*_{k+1}(x) = \alpha_{k+1} g_{k+1}(x)$, $x \in D_{k+1}$.

It is clear from the construction that g^*_{k+1} is an extension of $\alpha_{k+1} g_{k+1}$ on D_{k+1}. We have to check (17), so let $x \in R$. By (22) we need only show that $g^*_{k+1}(x) = \hat{g}_{k+1}(x)$. But for such x, $\hat{g}_{k+1}(x) = \tilde{g}_{k+1}(x)$, and g^*_{k+1} is an extension of $\tilde{g}_{k+1}$.

This concludes the inductive construction.

Now we shall use the preceding to show the existence of a map $G : \overline{H} \to |K|$ such that for each $x \in \partial H$, $f_0(x)$ and $G(x)$ lie in a simplex of K. As we mentioned at the outset, this will conclude our proof.

Let $x \in \overline{H}$. We are going to show that there exists $k \in \mathbb{N}$ and a neighborhood Q of x_{l_k} in X_{l_k} which lies in the domain T_k of g^*_k. We shall show, moreover, that Q may be chosen so that if $n > k$, then

(23) $p^{-1}_{l_k\, l_n}(Q) \subset T_n$, and
(24) for any $z \in p^{-1}_{l_k\, l_n}(Q)$, $g^*_n(z) = g^*_k(p_{l_k\, l_n}(z))$.

Assuming this for the moment, let $k = k(x)$ be the minimal element of $\mathbb{N}$ which admits such a Q. We then define $G(x) = g^*_k(x_{l_k})$. Property (24) shows that if $n > k$, then $g^*_k(x_{l_k}) = g^*_n(x_{l_n})$. That G is continuous at x can be seen as follows. We know that $M = p^{-1}_{l_k}(Q) \cap \overline{H}$ is a neighborhood of x in $\overline{H}$. For any $y \in M$, it is clear that $k(y) \leq k(x)$. Hence $g^*_{k(y)}(y_{l_{k(y)}}) = G(y) = g^*_{k(x)}(y_{l_{k(x)}})$. This implies that $G|M = g^*_{k(x)} \circ p_{l_{k(x)}}|M$.

To prove the statement above about $x \in \overline{H}$ and get (23) and (24), we shall consider the two cases, $x \in H$ and $x \in \partial H$.

First suppose that $x \in H$. An application of 2.6(c) shows that there is an i such that $x \in p^{-1}_{l_i}(H_{l_i})$, so $x_{l_i} \in H_{l_i}$. From (8), there is j with $x_{l_j} \in \operatorname{int} H^j_i$. Using (7) and (9), one finds $k \geq i$ such that $x_{l_k} \in p^{-1}_{l_i\, l_k}(\operatorname{int} H^j_i) \subset Q = \operatorname{int} H^1_k \subset T_k$. A recursive application of (9) and (17) shows that (23) and (24) are satisfied.

The other possibility is that $x \in \partial H$. There exists a neighborhood V_x of x in W_1 and a finite subset $\mathcal{F}_x \subset \Gamma$ such that $V_x \cap W_v \neq \emptyset$ precisely for $v \in \mathcal{F}_x$. This of course shows that $\mathcal{F}_x$ is the vertex set of a simplex of K, and that $f_0(x)$ lies in that simplex–we shall shortly need this information. We may as well assume that $V_x \subset W_v$ when $v \in \mathcal{F}_x$. There exists i and a neighborhood V^i of x_{l_i} in X_{l_i} such that $x \in p^{-1}_{l_i}(V^i) \subset V_x \subset W_v$. Then for each $v \in \mathcal{F}_x$, $x_{l_i} \in V^i \subset U_{l_i,v} = \operatorname{resp}(W_v, p_{l_i}) \subset U_{l_i}$.

Since $x_{l_i} \in U_{l_i} \cap p_{l_i}(\partial H) \subset U_{l_i} \cap Z_{l_i} = K_{l_i}$, (4) shows that for some j, $x_{l_i} \in K^j_{l_i}$. Applying (5) there exists $k \geq i$ with $x_{l_k} \in K^1_{l_k}$. Recall that D_k is a neighborhood of $K^1_{l_k}$ in X_{l_k}. Put $Q = \operatorname{int} D_k$. Then $Q \subset T_k$. One may use (11) and (17) to get (23) and (24).

The final step is to consider $x \in \partial H$ and show that $f_0(x)$, $G(x)$ lie in a simplex of K. To this end, let us maintain the notation we just produced for such x. Then $x_{l_k} \in D_k$ and since g^*_k is an extension of $\alpha_k g_k$ on D_k, we see that $G(x) = \alpha_k g_k(x_{l_k})$.

Let us observe that if $x_{l_k} \in U_{l_k,v}$, then it has to be true that $v \in \mathcal{F}_x$. To see this, note that $x_{l_k} \in p^{-1}_{l_i\,l_k}(V^i)$, and hence $x \in p^{-1}_{l_k} p^{-1}_{l_i\,l_k}(V^i) = p^{-1}_{l_i}(V^i) \subset V_x$. Moreover, $x_{l_k} \in p^{-1}_{l_i\,l_k}(V^i) \cap U_{l_k,v}$ and since $U_{l_k,v} = \operatorname{resp}(W_v, p_{l_k})$, then $x \in p^{-1}_{l_k}(U_{l_{k(x)},v}) \subset W_v$. Therefore $x \in V_x \cap W_v$, so $v \in \mathcal{F}_x$ as stated.

By (16), g_k is an $\mathcal{E}_k$-canonical map. So for some subset $\tilde{\mathcal{F}} \subset \mathcal{F}_x$, $g_k(x_{l_k})$ lies in the simplex whose vertices are $\{E_{k(x),v} \mid v \in \tilde{\mathcal{F}}\}$. The map α_k (see its definition) sends $g_k(x_{l_k})$ into the simplex of K having vertex set $\tilde{\mathcal{F}} \subset \mathcal{F}_x$.

But we have observed already that the map f_0 sends x into the simplex of K having $\mathcal{F}_x$ as its set of vertices. Our proof is complete. ∎

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Received: 08.12.1999.
Revised: 28.03.2000.