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SOME NEW TRIPLANES OF ORDER TWELVE

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Abstract. Up to isomorphism there are 72 symmetric (71,15,3) designs admitting an automorphism of order 6 acting with one fixed point. Full automorphism groups of those designs are isomorphic to one of the following groups: $S_3 \times E_4$, $A_4 \times Z_2$, $E_4 \times A_4$, $E_8 : F_{21}$, $(E_8 : F_{21}) \times Z_2$.

1. Introduction and preliminaries

Let $\mathcal{P}$ and $\mathcal{B}$ be disjoint sets and $I \subseteq \mathcal{P} \times \mathcal{B}$ where elements of $\mathcal{P}$ are called points and elements of $\mathcal{B}$ are called blocks or lines. Symmetric (v, k, λ) design is a finite incidence structure $(\mathcal{P}, \mathcal{B}, I)$ having v blocks and v points, where each block is incident with k points and each pair of points is incident with λ blocks. The difference $n = k - \lambda$ is called the order of design.

Triplane is a symmetric $(v, k, 3)$ design. The smallest nontrivial triplane is a complement of unique biplane (11,5,2). Known nontrivial triplanes have parameters (11,6,3), (15,7,3), (25,9,3), (31,10,3), (45,12,3) or (71,15,3). According to [6], classifications of triplanes of orders three, four, six and seven are already completed and there are 2893 triplanes of order nine known. Triplanes of order twelve are the biggest known triplanes. There are 11 symmetric (71,15,3) designs known (see [3], [4], [6]). The orders of their full automorphism groups are 48, 168 and 336.

Let $\mathcal{D} = (\mathcal{P}, \mathcal{B}, I)$ be a symmetric (v, k, λ) design and $G \leq Aut\mathcal{D}$. Group G has the same number of point and block orbits. Let us denote the number of $G-$orbits by t, point orbits by $\mathcal{P}_1, \ldots, \mathcal{P}_t$, block orbits by $\mathcal{B}_1, \ldots, \mathcal{B}_t$, and put $|\mathcal{P}_r| = \omega_r$, $|\mathcal{B}_i| = \Omega_i$. Further, denote by γ_{ir} the number of points of $\mathcal{P}_r$ which are incident with the representative of the block orbit $\mathcal{B}_i$. For these

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numbers the following equalities hold:

$$\sum_{r=1}^{t} \gamma_{ir} = k\,, \tag{1.1}$$

$$\sum_{r=1}^{t} \frac{\Omega_j}{\omega_r} \gamma_{ir}\gamma_{jr} = \lambda\Omega_j + \delta_{ij} \cdot (k-\lambda)\,. \tag{1.2}$$

The $(t \times t)$-matrix (γ_{ir}) with entries satisfying properties (1) and (2) is called the orbit structure for parameters (v,k,λ) and orbit distribution $(\omega_1,\ldots,\omega_t)$, $(\Omega_1,\ldots,\Omega_t)$.

2. Z_6 ACTING ON A TRIPLANE (71,15,3)

From now on we shall denote by G an abelian group isomorphic to a cyclic group of order 6.

The first step in the construction of all symmetric designs admitting an action of G is to determine all possible orbit distributions and to find all possible orbit structures related to them. The following facts are useful (see [5]):

LEMMA 2.1. *Let $\rho \neq \langle 1 \rangle$ be an automorphism of a symmetric (v,k,λ) design and let F be a number of fixed points of the automorphism ρ. Then equalities $F \leq v-2n$ and $F \leq \frac{\lambda}{k-\sqrt{n}}$ hold.*

LEMMA 2.2. *Let $\rho \neq \langle 1 \rangle$ be an involution of a symmetric (v,k,λ) design and let F be a number of fixed points of the automorphism ρ. Then*

$$F \geq \begin{cases} 1+\frac{k}{\lambda}, & for\; k,\lambda\; even, \\ 1+\frac{k-1}{\lambda}, & else. \end{cases}$$

We also know that number of point orbits of an automorphism acting on symmetric (71,15,3) design must be odd, because the order of (71,15,3) is not a square of an integer, and that the number of fixed points of an automorphism of prime order p acting on some design is congruent ($modulo\, p$) to number of points of that design. Action of G is semistandard, so to determine all possible orbit distributions it is sufficient to determine point orbit distributions $(\omega_1,...,\omega_t)$.

LEMMA 2.3. *Let σ be an automorphism of a symmetric $(71,15,3)$ design. If $|\sigma| = 3$, then number F_3 of fixed points of the automorphism σ satisfy condition $F_3 \in \{2,5,8,11\}$.*

PROOF. From lemma 1 and condition $F_3 \equiv 71\,(mod\,3)$ follows $F_3 \in \{2,5,8,11,14,17\}$. Automorphism of symmetric design fixes the same number of points and blocks. One can not construct required number of fixed blocks for $F_3 \in \{14,17\}$. ∎

LEMMA 2.4. *Let τ be an automorphism of a symmetric* $(71, 15, 3)$ *design. If* $|\tau| = 2$ *, then number* F_2 *of fixed points of the automorphism* τ *satisfy condition* $F_2 \in \{7, 11, 15\}$.

PROOF. From lemma 1 and condition $F_2 \equiv 71 \,(mod\, 2)$ follows $F_2 \in \{1, 3, 7, 9, 11, 13, 15, 17\}$. From lemma 2 we obtain $F_2 \geq 5$. Cases $F_2 = 5$, $F_2 = 9$, $F_2 = 13$ and $F_2 = 17$ can be eliminated because the number of orbits of automorphism τ must be odd. Therefore, $F_2 \in \{7, 11, 15\}$. $\square$

LEMMA 2.5. *Let ρ be an automorphism of a symmetric* $(71, 15, 3)$ *design and* $|\rho| = 6$. *Furthermore, let* $F_3 = 11$ *be a number of fixed points of the automorphism* ρ^2. *Then number* F *of fixed points of the automorphism* ρ *satisfy condition* $F \in \{1, 3\}$.

PROOF. Let d_i $(i = 1, 2, 3, 6)$ be a number of orbits of length i of the automorphism ρ. From $F_3 = d_1 + 2d_2 = F + 2d_2$ and $F_3 = 11$ follows $F \in \{1, 3, 5, 7, 9, 11\}$.

Fixed blocks are made of full point orbits. Therefore, if $F > 3$, fixed blocks of the automorphism ρ contain at most one point orbit of length six. If $F \leq 3$, then fixed blocks of ρ contain at most two point orbits of length six.

If $F = 11$ and $F_3 = 11$, then $d_2 = 0$, $d_3 = 0$ and $d_6 = 10$. So, every fixed block must be made of nine fixed points and one orbit of length six. That led to existence of eleven different point orbits of length six, i.e. to $d_6 = 11$. It follows $F \neq 11$.

Let $x_1x_2x_3x_6$ be a fixed block containing x_i $(i = 1, 2, 3, 6)$ point orbits of length i. For example, fixed block 9020 contains nine fixed points and two full point orbits of length three.

If $F = 9$ and $F_3 = 11$, then $d_2 = 1$, $d_3 = 2$ and $d_6 = 9$. The possible types of fixed blocks are the following: 9020, 9001, 7120, 7101, 6011, 4111, 3021 and 1121.

If $F = 7$ and $F_3 = 11$, then $d_2 = 2$, $d_3 = 0$ and $d_6 = 10$. The possible types of fixed blocks are the following: 7101 and 5201.

If $F = 5$ and $F_3 = 11$, then $d_2 = 3$, $d_3 = 2$ and $d_6 = 9$. The possible types of fixed blocks are the following: 5220, 5201, 4111, 3320, 3301, 3021, 2211, 1121 and 0311.

Investigating intersection of fixed blocks one can obtain that required number of fixed blocks can not be constructed for $F \in \{5, 7, 9\}$. $\square$

In the similar way we prove the following

LEMMA 2.6. *Let ρ be an automorphism of a symmetric* $(71, 15, 3)$ *design and* $|\rho| = 6$. *Furthermore, let* $F_3 = 8$ *be a number of fixed points of the automorphism* ρ^2. *Then number* F *of fixed points of the automorphism* ρ *satisfy condition* $F \in \{0, 2\}$.

Lemma 2.7. *Let ρ be an automorphism of a symmetric* $(71, 15, 3)$ *design and* $|\rho| = 6$. *Then number* F *of fixed points of the automorphism* ρ *satisfy condition* $F \leq 3$.

Proof. Let d_i $(i = 1, 2, 3, 6)$ be a number of orbits of length i. From lemmas 3, 4, 5 and 6 follows that only remaining orbit distribution for $F > 3$ is $d_1 = 5$, $d_2 = 0$, $d_3 = 2$ and $d_6 = 10$. All fixed blocks in this case should be made of three fixed points, two point orbits of length three and one point orbit of length six. Fixed block of that type can occur only once and that proves the lemma. ◻

Lemma 2.8. *Symmetric* $(71, 15, 3)$ *design admitting an action of* G *has one of the following orbit distributions:*

1. $(1, 1, 1, 2, 3, 3, 3, 3, 6, 6, 6, 6, 6, 6, 6, 6)$,
2. $(1, 2, 2, 3, 3, 6, 6, 6, 6, 6, 6, 6, 6, 6)$,
3. $(2, 3, 3, 3, 3, 3, 6, 6, 6, 6, 6, 6, 6, 6)$.

Proof. From previous lemmas follows that required number of fixed blocks of such a design can be constructed for the following orbit distributions: (1, 1, 1, 2, 2, 2, 2, 3, 3, 3, 3, 6, 6, 6, 6, 6, 6, 6), (1, 1, 1, 2, 3, 3, 3, 3, 6, 6, 6, 6, 6, 6, 6, 6), (1, 1, 2, 2, 2, 3, 3, 3, 6, 6, 6, 6, 6, 6, 6, 6), (1, 1, 3, 3, 3, 6, 6, 6, 6, 6, 6, 6, 6, 6), (1, 2, 2, 2, 2, 2, 3, 3, 6, 6, 6, 6, 6, 6, 6, 6), (1, 2, 2, 3, 3, 6, 6, 6, 6, 6, 6, 6, 6, 6), (2, 2, 2, 2, 3, 3, 3, 3, 3, 6, 6, 6, 6, 6, 6, 6, 6), (2, 3, 3, 3, 3, 3, 6, 6, 6, 6, 6, 6, 6, 6). For that remaining cases we should construct orbital structures satisfying equalities (1) and (2). Because of the large number of possibilities, it was necessary to involve a computer in this final step of the construction of orbital structures. With the help of the computer program by V. Ćepulić we obtained that complete orbit structures can be constructed only in the cases from the statement of the lemma. ◻

3. Construction of designs

We shall consider the construction of designs admitting an action of G which corresponds to the second type from lemma 8. For that case we are able to prove the following

Theorem 3.1. *There are* 72 *mutually nonisomorphic symmetric* $(71, 15, 3)$ *designs admitting an action of the cyclic group of order 6 acting with one fixed point. Between them, there are* 6 *self-dual designs and* 33 *pairs of dually isomorphic designs. Exactly* 37 *designs have full automorphism groups of order* 24*;* 16 *of these groups are isomorphic to the group* $S_3 \times E_4$ *and* 21 *of them are isomorphic to the group* $A_4 \times Z_2$*. Furtermore,* 26 *designs have full automorphism groups of order* 48 *isomorphic to the group* $E_4 \times A_4$*,* 3 *designs have full automorphism groups of order* 168 *isomorphic to the group* $E_8 : F_{21}$

and 6 *designs have full automorphism groups of order* 336 *isomorphic to the group* $(E_8 : F_{21}) \times Z_2$.

Proof. First, we present representatives of possible types of block orbits related to the orbit distribution (1, 2, 2, 3, 3, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6). In first column is number of type. It is followed with length of corresponding block orbit and representative of that type.

1	1	1	2	0	3	3	6	0	0	0	0	0	0	0	0	0
2	1	1	2	0	0	0	6	6	0	0	0	0	0	0	0	0
3	1	0	0	0	3	0	6	6	0	0	0	0	0	0	0	0
4	2	1	2	0	0	0	3	3	3	3	0	0	0	0	0	0
5	2	0	0	0	3	0	3	3	3	3	0	0	0	0	0	0
6	3	1	2	0	1	1	2	2	2	2	2	0	0	0	0	0
7	3	1	2	0	0	0	2	2	2	2	2	2	0	0	0	0
8	3	1	0	0	2	2	2	2	2	2	2	0	0	0	0	0
9	3	1	0	0	1	1	4	2	2	2	2	0	0	0	0	0
10	3	1	0	0	0	0	4	2	2	2	2	2	0	0	0	0
11	3	0	2	0	2	1	2	2	2	2	2	0	0	0	0	0
12	3	0	0	0	3	0	2	2	2	2	2	2	0	0	0	0
13	3	0	0	0	2	1	4	2	2	2	2	0	0	0	0	0
14	6	1	1	1	2	0	1	1	1	1	1	1	1	1	1	1
15	6	1	1	1	1	1	2	2	1	1	1	1	1	1	0	0
16	6	1	1	1	0	0	3	1	1	1	1	1	1	1	1	1
17	6	1	1	1	0	0	2	2	2	1	1	1	1	1	1	0
18	6	1	1	0	2	0	2	1	1	1	1	1	1	1	1	1
19	6	1	1	0	1	1	3	1	1	1	1	1	1	1	1	0
20	6	1	1	0	1	1	2	2	2	1	1	1	1	1	0	0
21	6	1	1	0	0	0	3	2	1	1	1	1	1	1	1	1
22	6	1	1	0	0	0	2	2	2	2	1	1	1	1	1	0
23	6	1	0	0	2	0	2	2	1	1	1	1	1	1	1	1
24	6	1	0	0	1	1	3	2	1	1	1	1	1	1	1	0
25	6	1	0	0	1	1	2	2	2	2	1	1	1	1	0	0
26	6	1	0	0	0	0	3	2	2	1	1	1	1	1	1	1
27	6	1	0	0	0	0	2	2	2	2	2	1	1	1	1	0
28	6	0	2	1	1	0	2	1	1	1	1	1	1	1	1	1
29	6	0	2	0	1	0	2	2	1	1	1	1	1	1	1	1
30	6	0	1	1	2	1	2	2	1	1	1	1	1	1	0	0
31	6	0	1	1	1	0	3	2	2	1	1	1	1	1	0	0
32	6	0	1	1	1	0	2	2	2	2	2	1	1	0	0	0
33	6	0	1	0	2	1	3	1	1	1	1	1	1	1	1	0
34	6	0	1	0	2	1	2	2	2	1	1	1	1	1	0	0
35	6	0	1	0	1	0	4	1	1	1	1	1	1	1	1	1
36	6	0	1	0	1	0	3	3	1	1	1	1	1	1	1	0
37	6	0	1	0	1	0	3	2	2	2	1	1	1	1	0	0
38	6	0	1	0	1	0	2	2	2	2	2	2	1	0	0	0
39	6	0	0	0	2	1	3	2	1	1	1	1	1	1	1	0
40	6	0	0	0	2	1	2	2	2	2	1	1	1	1	0	0
41	6	0	0	0	1	0	4	2	1	1	1	1	1	1	1	1
42	6	0	0	0	1	0	3	3	2	1	1	1	1	1	1	0
43	6	0	0	0	1	0	3	2	2	2	2	1	1	1	0	0
44	6	0	0	0	1	0	2	2	2	2	2	2	2	0	0	0

From given types we shall construct orbit structures satisfying equations (1) and (2). With the help of the computer program by V. Ćepulić we obtained that there are 1464 orbit structures for the group $G \cong \langle \rho \mid \rho^6 = 1 \rangle$ related to the orbit distribution (1, 2, 2, 3, 3, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6).

Let us present just one of constructed orbital structures. It is ninety-third obtained orbital structure and first orbital structure that will lead to (71,15,3) designs. (Ordering of obtained orbital structures is described in [2].) This orbit structure is obtained from type representatives as follows: 1, 4, 5, 6, 9, 27, 31, 31, 37, 37, 37, 37, 40, 43, 43.

OS_{93}	1	2	2	3	3	6	6	6	6	6	6	6	6	6	6
1	1	2	0	3	3	6	0	0	0	0	0	0	0	0	0
2	1	2	0	0	0	0	3	3	3	3	0	0	0	0	0
2	0	0	0	3	0	0	3	3	0	0	3	3	0	0	0
3	1	0	2	1	1	0	2	2	0	0	0	0	2	2	2
3	1	0	0	1	1	0	0	0	2	2	2	2	4	0	0
6	1	0	0	0	0	2	1	1	1	1	2	2	0	2	2
6	0	1	1	0	1	1	2	0	2	0	3	1	1	1	1
6	0	1	1	0	1	1	0	2	0	2	1	3	1	1	1
6	0	1	0	1	0	1	2	0	0	2	1	1	2	3	1
6	0	1	0	1	0	1	0	2	2	0	1	1	2	1	3
6	0	0	1	1	0	2	1	1	3	1	0	2	1	2	0
6	0	0	1	1	0	2	1	1	1	3	2	0	1	0	2
6	0	0	0	1	2	0	1	1	2	2	1	1	0	2	2
6	0	0	0	0	1	2	3	1	1	1	0	2	2	0	2
6	0	0	0	0	1	2	1	3	1	1	2	0	2	2	0

For final step of the construction, called indexing, we developed computer programs. During construction of symmetric designs we shall use elements of a normalizer of an automorphism group in the group $S = S(\mathcal{P}) \times S(\mathcal{B})$ to avoid construction of mutually isomorphic designs (see [1], [2]). We shall check all possibilities corresponding to obtained orbital structures. There is a very large number of possibilities. For example, smallest number of possibilities for some orbit of length six of presented ninety-third structure is $52\,488\,000 \approx 16\,000 \cdot 2^{15}$ and biggest number of possibilities for some orbit of length six is $1\,594\,323\,000\,000 \approx 48\,650\,000 \cdot 2^{15}$. This process of indexing will take too much time, even if we involve computers. Therefore, we proceed by "lifting" obtained orbital structures for the group $G \cong \langle \rho \mid \rho^6 = 1 \rangle$ to orbital structures for the cyclic group $\langle \rho^3 \rangle$ with the assumption that they admit ρ^2 as an automorphism. As a result we got that 248 orbit structures can be lifted to orbit structures for the cyclic group $\langle \rho^3 \rangle$ with the assumption that they admit $\langle \rho^2 \rangle$ as an automorphism. Because of their large number, we present just ordinal numbers of that structures corresponding to the ordering described in [2].

3 117 166 232 311 390 481 577 624 680 774 970 1308
4 119 167 233 321 391 482 579 625 682 783 979 1327
31 120 168 234 323 392 484 581 627 684 799 981 1382
33 121 171 236 327 393 485 582 632 687 830 983 1404
35 122 174 240 328 401 486 583 633 691 839 991 1425
61 124 177 243 330 405 487 584 637 694 856 993 1428
64 127 180 246 331 407 497 589 639 697 864 996 1452
66 129 194 255 342 421 498 591 643 700 884 1012 1463
91 130 196 261 343 439 508 592 646 702 885 1070

93 132 197 270 346 449 513 593 649 703 888 1076
96 141 202 290 362 450 524 594 650 708 908 1106
97 142 205 291 363 453 555 595 654 709 910 1124
100 147 206 292 366 456 558 597 655 719 911 1138
102 148 207 293 368 457 559 598 656 720 932 1150
104 151 208 294 369 461 563 600 661 722 933 1168
105 152 211 296 373 463 566 611 664 726 936 1227
107 159 217 297 374 464 568 612 665 730 955 1237
110 161 226 299 385 467 569 613 667 732 956 1261
111 164 229 303 386 468 571 618 671 742 967 1282
114 165 231 309 388 474 574 622 678 747 968 1292

For each orbit structure that "pass" in this first step of indexing we obtain more than five and less than sixty-one orbit structures for the group $\langle \rho^3 \rangle$. In final step of indexing we deal with 3270 orbit structures for the group $\langle \rho^3 \rangle$. In this case, the number of possibilities for each row of orbit structure is less than 2^{15}.

Checking all possibilities we obtained that only 30 of 1464 orbit structures for cyclic group of order six can be lifted to the design with parameters (71,15,3). The ordinal numbers of those structures are 93, 100, 114, 119, 122, 124, 132, 152, 159, 164, 168, 177, 206, 243, 468, 484, 485, 568, 571, 577, 589, 594, 643, 649, 665, 691, 702, 885, 910 and 996.

After eliminating isomorphic copies (using program by V. Ćepulić) we got exactly 72 non-isomorphic designs. Between them, there are six self-dual designs and thirty-three pairs of dually isomorphic designs. For example, from presented ninety-third orbital structure we obtain eight non-isomorphic designs 1-8 presented below. Duals of those eight designs are obtained from eight hundred and eighty-fifth structure.

We present six self-dual designs and one design for each obtained pair of dually isomorphic designs. For brevity, representatives of common line orbits of length one, two or three for more designs are written only once. In those cases, first number in each row, namely l, is the length of line orbit. It is followed by the line that is representative of line orbit and others can be obtained by applying $l-1$ times automorphism

$\rho = (1,2)(3,4)\,(5,6,7)\,(8,9,10)\,(11,16,12,14,13,15)\,(17,22,18,20,19,21)\,(23,28,24,26,25,27)\,(29,34,30,32,31,33)\,(35,40,36,38,37,39)\,(41,46,42,44,43,45)\,(47,52,48,50,49,51)\,(53,58,54,56,55,57)\,(59,64,60,62,61,63)\,(65,70,66,68,67,69)$.

Representatives for block orbits of length six are given for each design separately and other blocks from these orbits can be obtain by applying five times given automorphism ρ.

common lines for designs 1-39:
1 0 1 2 5 6 7 8 9 10 11 12 13 14 15 16
2 0 1 2 17 18 19 23 24 25 29 30 31 35 36 37
2 5 6 7 17 18 19 26 27 28 41 42 43 47 48 49

common lines for designs 1-24:
3 0 3 4 5 8 17 20 23 26 53 56 59 62 65 68
3 0 5 8 29 32 35 38 41 44 47 50 54 55 57 58

design 1:

0 11 14 17 23 32 38 42 46 49 51 60 61 66 67
1 3 8 11 18 22 30 34 41 42 44 52 53 60 70
1 4 8 11 25 27 37 39 46 47 48 50 56 61 69
1 5 14 18 22 37 39 45 49 54 55 59 62 64 66
1 5 14 25 27 30 34 43 51 57 58 63 65 67 68
3 5 13 15 22 24 29 30 32 40 48 49 56 61 63
3 5 12 16 18 28 34 35 36 38 45 46 56 67 69
5 9 10 17 26 30 31 39 40 44 50 60 64 67 69
8 12 13 17 20 22 27 31 36 48 52 55 57 66 67
8 12 13 18 23 25 26 33 40 42 46 54 58 63 64

design 2:

0 11 14 17 23 32 38 42 46 49 51 60 61 66 67
1 3 8 11 18 22 30 34 41 42 44 52 53 60 70
1 4 8 11 25 27 37 39 46 47 48 50 56 61 69
1 5 14 19 21 37 39 42 52 57 58 59 62 64 66
1 5 14 24 28 30 34 46 48 54 55 63 65 67 68
3 5 12 16 22 24 29 32 33 37 48 49 56 60 64
3 5 13 15 18 28 31 35 38 39 45 46 56 66 70
5 9 10 17 26 33 34 36 37 44 50 61 63 66 70
8 12 13 17 20 22 27 31 36 48 52 55 57 66 67
8 12 13 18 23 25 26 33 40 42 46 54 58 63 64

design 3:

0 11 14 17 23 32 38 42 46 49 51 60 61 66 67
1 3 8 11 18 22 30 34 41 44 45 49 53 63 67
1 4 8 11 24 28 36 40 46 47 48 50 56 61 69
1 5 14 18 22 36 40 42 52 57 58 59 62 64 66
1 5 14 24 28 30 34 43 51 54 55 60 65 68 70
3 5 12 16 22 27 29 30 32 37 48 52 56 60 61

3 5 13 15 18 25 34 35 38 39 42 46 56 69 70
5 9 10 17 26 30 31 39 40 44 50 60 64 67 69
8 12 13 17 19 20 27 34 36 51 52 55 57 67 69
8 12 13 21 23 25 26 30 40 42 43 54 58 61 63

design 4:

0 11 14 17 23 32 38 42 46 49 51 60 61 66 67
1 3 8 11 18 22 31 33 41 44 45 49 53 60 70
1 4 8 11 24 28 37 39 46 47 48 50 56 64 66
1 5 14 18 22 36 40 42 52 54 55 59 62 64 66
1 5 14 24 28 30 34 43 51 57 58 60 65 68 70
3 5 13 15 19 24 29 32 33 40 49 51 56 63 64
3 5 12 16 21 28 31 35 36 38 43 45 56 66 67
5 9 10 17 26 33 34 36 37 44 50 61 63 66 70
8 12 13 17 19 20 27 31 39 51 52 55 57 66 70
8 12 13 21 23 25 26 33 37 42 43 54 58 60 64

design 5:

0 11 14 17 23 32 38 42 46 49 51 60 61 66 67
1 3 8 11 18 22 30 34 41 42 44 52 53 64 66
1 4 8 11 25 27 37 39 46 47 48 50 56 63 67
1 5 14 18 22 36 40 46 48 54 55 59 61 62 69
1 5 14 25 27 31 33 42 52 57 58 60 65 68 70
3 5 13 15 22 24 29 31 32 39 48 49 56 60 64
3 5 12 16 18 28 33 35 37 38 45 46 56 66 70
5 9 10 17 26 30 31 39 40 44 50 61 63 66 70
8 12 13 17 18 20 25 34 39 49 51 54 58 69 70
8 12 13 22 23 26 27 30 37 43 45 55 57 60 61

design 6:

0 11 14 17 23 32 38 42 46 49 51 60 61 66 67
1 3 8 11 18 22 30 34 41 44 45 49 53 61 69
1 4 8 11 24 28 36 40 46 47 48 50 56 63 67
1 5 14 19 21 36 40 43 51 57 58 59 61 62 69
1 5 14 25 27 30 34 42 52 54 55 63 65 67 68
3 5 13 15 19 24 29 32 34 39 49 51 56 63 64
3 5 12 16 21 28 30 35 37 38 43 45 56 66 67
5 9 10 17 26 30 31 39 40 44 50 61 63 66 70
8 12 13 17 18 20 28 34 36 51 52 55 57 66 70
8 12 13 22 23 24 26 30 40 42 43 54 58 60 64

design 7:

0 11 14 17 23 32 38 43 45 48 52 60 61 66 67
1 3 8 11 18 22 30 34 41 42 44 52 53 61 69
1 4 8 11 24 28 36 40 43 47 50 51 56 63 67
1 5 14 18 22 36 40 46 48 54 55 59 62 64 66
1 5 14 24 28 30 34 45 49 57 58 60 65 68 70
3 5 13 15 19 24 29 32 34 39 48 52 56 63 64
3 5 12 16 21 28 30 35 37 38 42 46 56 66 67
5 9 10 17 26 33 34 36 37 44 50 61 63 66 70
8 12 13 17 20 21 25 34 36 48 49 54 58 67 69
8 12 13 19 23 26 27 30 40 45 46 55 57 61 63

design 8:

0 11 14 17 23 32 38 43 45 48 52 60 61 66 67
1 3 8 11 18 22 31 33 41 42 44 52 53 64 66
1 4 8 11 25 27 36 40 46 47 48 50 56 63 67
1 5 14 18 22 36 40 43 51 57 58 59 61 62 69
1 5 14 25 27 31 33 45 49 54 55 60 65 68 70
3 5 13 15 22 24 29 32 34 36 48 49 56 60 64
3 5 12 16 18 28 30 35 38 40 45 46 56 66 70
5 9 10 17 26 33 34 36 37 44 50 61 63 66 70
8 12 13 17 18 20 25 34 39 48 52 55 57 69 70
8 12 13 22 23 26 27 30 37 42 46 54 58 60 61

design 9:

0 11 14 17 23 32 38 42 46 49 51 60 61 66 67
1 3 8 11 18 22 30 40 41 42 44 52 53 60 64
1 4 8 11 24 28 34 36 43 47 50 51 56 66 70
1 5 14 18 28 30 34 45 46 54 58 63 65 67 68
1 5 14 22 24 36 40 48 49 55 57 59 61 62 69
3 5 12 13 18 22 33 35 37 38 46 48 56 66 70
3 5 15 16 25 27 29 32 34 36 42 52 56 61 63
5 9 10 17 26 30 34 37 39 44 50 60 61 69 70
8 12 16 17 18 20 25 33 34 49 51 54 55 64 69
8 12 16 19 23 24 26 39 40 42 46 57 58 63 70

design 10:

0 11 14 17 23 32 38 42 46 49 51 60 61 66 67
1 3 8 11 18 22 33 37 41 44 45 49 53 60 64
1 4 8 11 25 27 34 36 43 47 50 51 56 67 69
1 5 14 21 28 30 31 43 45 54 58 60 65 67 68
1 5 14 19 24 39 40 49 51 55 57 59 62 64 69
3 5 15 16 19 21 33 35 37 38 43 51 56 66 70
3 5 12 13 25 27 29 31 32 39 42 52 56 60 64
5 9 10 17 26 30 34 37 39 44 50 60 61 69 70
8 12 16 17 20 21 25 31 33 48 49 57 58 61 69
8 12 16 22 23 24 26 36 40 42 43 54 55 60 70

design 11:

0 11 14 17 23 32 38 43 45 48 52 60 61 66 67
1 3 8 11 18 22 30 40 41 42 44 52 53 61 63
1 4 8 11 24 28 34 36 43 47 50 51 56 67 69
1 5 14 18 28 30 34 45 46 54 58 60 65 68 70
1 5 14 22 24 36 40 48 49 55 57 59 62 64 66
3 5 15 16 18 22 33 35 37 38 43 51 56 66 70
3 5 12 13 25 27 29 32 34 36 45 49 56 61 63
5 9 10 17 26 30 34 37 39 44 50 63 64 66 67
8 12 16 17 20 21 28 30 31 49 51 57 58 61 66
8 12 16 22 23 26 27 36 37 42 46 54 55 60 67

design 12:

0 11 14 17 23 32 38 43 45 48 52 60 61 66 67
1 3 8 11 18 22 33 37 41 42 44 52 53 60 64
1 4 8 11 25 27 34 36 46 47 48 50 56 67 69
1 5 14 21 28 30 31 42 46 54 58 60 65 67 68
1 5 14 19 24 39 40 48 52 55 57 59 62 64 69
3 5 12 13 18 22 30 35 38 40 46 48 56 66 70
3 5 15 16 24 28 29 32 34 36 45 49 56 60 64
5 9 10 17 26 30 34 37 39 44 50 63 64 66 67
8 12 16 17 20 21 25 31 33 48 49 54 55 64 66
8 12 16 22 23 24 26 36 40 42 43 57 58 63 67

design 13:

0 11 14 17 23 32 38 43 45 48 52 60 61 66 67
1 3 8 11 18 22 30 40 41 43 44 51 53 61 63
1 4 8 11 25 27 31 39 45 47 49 50 56 66 70
1 5 14 19 24 33 34 43 45 55 57 63 65 68 70

1 5 14 21 28 36 37 49 51 54 58 59 61 62 66
3 5 12 13 18 22 34 35 36 38 46 48 56 66 70
3 5 15 16 24 28 29 30 32 40 45 49 56 60 64
5 9 10 17 26 31 33 36 40 44 50 63 64 66 67
8 12 16 17 20 21 25 30 34 48 49 54 55 63 67
8 12 16 22 23 24 26 37 39 42 43 57 58 64 66

design 14:

0 11 14 17 23 32 38 43 45 48 52 60 61 66 67
1 3 8 11 18 22 30 40 41 44 46 48 53 60 64
1 4 8 11 24 28 34 36 45 47 49 50 56 66 70
1 5 14 22 24 30 34 42 43 54 58 63 65 67 68
1 5 14 18 28 36 40 51 52 55 57 59 61 62 69
3 5 15 16 19 21 34 35 36 38 46 48 56 67 69
3 5 12 13 24 28 29 32 33 37 42 52 56 60 64
5 9 10 17 26 31 33 36 40 44 50 63 64 66 67
8 12 16 17 18 20 25 33 34 48 52 54 55 63 70
8 12 16 19 23 24 26 39 40 43 45 57 58 64 69

design 15:

0 11 14 17 23 32 38 43 45 48 52 60 61 66 67
1 3 8 11 18 22 30 40 41 44 46 48 53 60 64
1 4 8 11 24 28 34 36 45 47 49 50 56 66 70
1 5 14 22 24 31 33 42 43 55 57 60 65 68 70
1 5 14 18 28 37 39 51 52 54 58 59 62 64 66
3 5 12 13 18 22 34 35 36 38 43 51 56 67 69
3 5 15 16 25 27 29 32 33 37 45 49 56 60 64
5 9 10 17 26 30 34 37 39 44 50 60 61 69 70
8 12 16 17 18 20 25 33 34 48 52 54 55 63 70
8 12 16 19 23 24 26 39 40 43 45 57 58 64 69

design 16:

0 11 14 17 23 32 38 43 45 48 52 60 61 66 67
1 3 8 11 18 22 33 37 41 43 44 51 53 60 64
1 4 8 11 25 27 34 36 45 47 49 50 56 67 69
1 5 14 22 27 30 31 42 46 54 58 60 65 67 68
1 5 14 18 25 39 40 48 52 55 57 59 62 64 69
3 5 15 16 18 22 34 35 36 38 46 48 56 66 70
3 5 12 13 24 28 29 30 32 40 45 49 56 60 64
5 9 10 17 26 30 34 37 39 44 50 60 61 69 70

8 12 16 17 20 21 25 31 33 48 49 54 55 60 70
8 12 16 22 23 24 26 36 40 42 43 57 58 61 69

design 17:

0 11 14 17 23 32 38 42 43 51 52 60 64 66 70
1 3 8 11 18 22 30 34 41 44 48 52 53 63 64
1 4 8 11 24 28 36 40 43 45 47 50 56 69 70
1 5 14 18 22 36 40 42 46 54 55 60 61 65 68
1 5 14 24 28 30 34 49 51 57 58 59 62 66 67
3 5 15 16 18 22 29 32 37 39 49 51 56 69 70
3 5 12 13 25 27 30 34 35 38 43 45 56 60 61
5 9 10 17 26 30 31 39 40 44 50 61 63 66 70
8 12 16 21 22 23 26 30 31 42 43 55 57 67 69
8 12 16 17 20 27 28 36 37 51 52 54 58 61 63

design 18:

0 11 14 17 23 32 38 42 43 51 52 60 64 66 70
1 3 8 11 18 22 30 34 41 44 48 52 53 63 64
1 4 8 11 25 27 37 39 42 46 47 50 56 66 67
1 5 14 18 22 36 40 42 46 54 55 60 61 65 68
1 5 14 25 27 31 33 48 52 57 58 59 62 69 70
3 5 15 16 19 21 29 32 36 40 48 52 56 66 67
3 5 12 13 25 27 30 34 35 38 43 45 56 60 61
5 9 10 17 26 30 31 39 40 44 50 61 63 66 70
8 12 16 18 19 23 26 33 34 45 46 55 57 66 70
8 12 16 17 20 27 28 36 37 51 52 54 58 61 63

design 19:

0 11 14 17 23 32 38 42 43 51 52 60 64 66 70
1 3 8 11 18 22 30 34 41 44 48 52 53 63 64
1 4 8 11 25 27 37 39 42 46 47 50 56 66 67
1 5 14 18 22 36 40 42 46 54 55 60 61 65 68
1 5 14 25 27 31 33 48 52 57 58 59 62 69 70
3 5 15 16 19 21 29 32 36 40 48 52 56 66 67
3 5 12 13 25 27 30 34 35 38 43 45 56 60 61
5 9 10 17 26 30 31 39 40 44 50 61 63 66 70
8 12 16 18 19 23 26 33 34 45 46 55 57 66 70
8 12 16 17 20 27 28 36 37 51 52 54 58 61 63

design 20:

0 11 14 17 23 32 38 42 43 51 52 60 64 66 70
1 3 8 11 18 22 30 34 41 44 49 51 53 60 61
1 4 8 11 25 27 37 39 43 45 47 50 56 69 70
1 5 14 18 22 37 39 42 46 54 55 63 64 65 68
1 5 14 25 27 30 34 48 52 57 58 59 62 66 67
3 5 15 16 18 22 29 32 36 40 48 52 56 69 70
3 5 12 13 24 28 30 34 35 38 43 45 56 63 64
5 9 10 17 26 30 31 39 40 44 50 61 63 66 70
8 12 16 21 22 23 26 30 31 42 43 55 57 67 69
8 12 16 17 20 24 25 39 40 48 49 54 58 60 64

design 21:

0 11 14 17 23 32 38 45 46 48 49 60 64 66 70
1 3 8 11 18 22 30 34 42 46 47 50 53 63 64
1 4 8 11 24 28 36 40 41 44 49 51 56 69 70
1 5 14 18 22 36 40 43 45 57 58 59 62 66 67
1 5 14 24 28 30 34 48 52 54 55 60 61 65 68
3 5 15 16 18 22 31 33 35 38 48 52 56 69 70
3 5 12 13 25 27 29 32 36 40 42 46 56 60 61
5 9 10 17 26 33 34 36 37 44 50 60 64 67 69
8 12 16 17 20 27 28 30 31 45 46 55 57 67 69
8 12 16 21 22 23 26 36 37 48 49 54 58 61 63

design 22:

0 11 14 17 23 32 38 45 46 48 49 60 64 66 70
1 3 8 11 18 22 30 34 42 46 47 50 53 63 64
1 4 8 11 25 27 37 39 41 44 48 52 56 66 67
1 5 14 19 21 37 39 42 46 57 58 59 62 69 70
1 5 14 24 28 30 34 48 52 54 55 60 61 65 68
3 5 15 16 19 21 30 34 35 38 49 51 56 66 67
3 5 12 13 25 27 29 32 36 40 42 46 56 60 61
5 9 10 17 26 33 34 36 37 44 50 60 64 67 69
8 12 16 17 20 24 25 33 34 42 43 55 57 66 70
8 12 16 21 22 23 26 36 37 48 49 54 58 61 63

design 23:

0 11 14 17 23 32 38 45 46 48 49 60 64 66 70
1 3 8 11 18 22 31 33 43 45 47 50 53 63 64
1 4 8 11 24 28 37 39 41 44 48 52 56 69 70
1 5 14 19 21 37 39 43 45 57 58 59 62 66 67
1 5 14 25 27 31 33 48 52 54 55 60 61 65 68
3 5 15 16 19 21 31 33 35 38 49 51 56 69 70
3 5 12 13 24 28 29 32 36 40 43 45 56 60 61
5 9 10 17 26 33 34 36 37 44 50 60 64 67 69
8 12 16 17 20 27 28 30 31 45 46 55 57 67 69
8 12 16 21 22 23 26 36 37 48 49 54 58 61 63

design 24:

0 11 14 17 23 32 38 45 46 48 49 60 64 66 70
1 3 8 11 18 22 31 33 43 45 47 50 53 63 64
1 4 8 11 25 27 36 40 41 44 49 51 56 66 67
1 5 14 18 22 36 40 42 46 57 58 59 62 69 70
1 5 14 25 27 31 33 48 52 54 55 60 61 65 68
3 5 15 16 18 22 30 34 35 38 48 52 56 66 67
3 5 12 13 24 28 29 32 36 40 43 45 56 60 61
5 9 10 17 26 33 34 36 37 44 50 60 64 67 69
8 12 16 17 20 24 25 33 34 42 43 55 57 66 70
8 12 16 21 22 23 26 36 37 48 49 54 58 61 63

common lines for designs 25-39:
3 0 3 4 5 8 17 20 29 32 41 44 53 56 59 62
3 0 5 8 23 26 35 38 47 50 54 55 57 58 65 68

design 25:

0 11 14 17 26 29 38 45 46 48 52 60 61 66 70
1 3 5 11 18 22 24 40 47 54 61 62 66 67 69
1 4 8 14 22 30 36 40 41 46 48 49 51 57 65
1 5 11 28 30 34 36 42 44 45 55 56 64 68 70
1 8 14 18 24 28 34 43 50 52 53 58 59 60 63
3 5 15 16 24 26 29 31 34 39 49 51 57 63 70
3 8 13 15 20 25 27 28 35 36 43 45 57 61 66
5 9 10 19 28 30 32 35 39 46 52 62 63 65 66
5 12 13 17 21 30 37 38 40 43 51 56 58 61 63
8 12 16 18 19 22 23 32 33 45 49 56 58 66 70

design 26:

0 11 14 17 26 29 38 45 46 49 51 60 61 66 70
1 3 5 11 18 22 24 40 49 54 59 64 66 68 69
1 4 8 14 22 30 36 40 43 44 47 48 51 57 70
1 5 11 28 30 34 36 41 50 52 55 56 60 61 63
1 8 14 18 24 28 34 42 45 46 53 58 62 65 67
3 5 15 16 23 27 32 36 37 40 42 46 58 60 70
3 8 12 16 21 25 26 28 29 30 48 52 58 64 66
5 9 10 19 28 30 32 35 39 45 51 62 64 68 70
5 12 13 18 20 31 33 34 35 46 48 56 57 66 70
8 12 16 19 20 22 27 35 36 45 49 53 54 61 63

design 27:

0 11 14 17 23 33 39 42 46 49 51 59 61 65 67
1 3 5 12 18 22 25 39 48 55 59 63 67 68 70
1 4 8 15 25 26 31 33 44 48 49 51 54 64 66
1 5 12 27 32 35 36 42 46 52 54 56 60 61 64
1 8 15 17 22 30 40 42 43 45 50 56 58 67 69
3 5 14 16 21 30 32 35 37 40 43 49 51 55 66
3 8 11 16 19 24 26 27 36 40 44 46 57 63 67
5 9 10 17 23 33 34 36 40 41 50 63 64 66 70
5 11 13 22 25 27 30 31 34 42 53 57 62 65 66
8 11 16 17 18 21 28 31 39 50 52 53 55 60 64

design 28:

0 11 14 17 23 32 38 42 46 49 51 60 61 66 67
1 3 5 12 18 22 25 38 48 55 60 62 67 69 70
1 4 8 15 23 28 36 40 41 42 45 52 55 61 69
1 5 12 27 29 31 34 42 46 52 54 56 63 65 66
1 8 15 19 20 33 37 46 47 48 51 56 57 60 64
3 5 14 16 21 29 33 36 37 40 43 49 51 55 65
3 8 13 14 17 24 27 28 31 33 50 52 58 60 70
5 9 10 17 23 33 34 37 39 41 50 63 64 67 69
5 11 13 22 25 27 33 35 36 42 53 57 59 61 64
8 11 16 18 21 22 23 31 39 41 43 56 57 66 70

design 29:

0 11 14 17 26 29 38 45 46 48 52 60 61 66 70
1 3 5 11 18 22 33 37 47 51 55 59 66 67 70
1 4 8 14 25 27 34 36 45 47 49 52 58 59 63
1 5 11 19 24 39 40 41 45 54 56 61 63 64 68

1 8 14 21 28 30 31 41 43 46 51 53 57 66 68
3 5 15 16 25 31 32 36 38 39 43 45 48 57 67
3 8 13 15 19 20 24 26 27 37 46 57 63 69 70
5 9 10 18 27 31 32 35 40 46 52 62 63 65 66
5 12 13 20 25 26 30 31 33 49 56 58 60 64 66
8 12 16 18 19 21 32 37 38 42 51 52 56 58 61

design 30:

0 11 14 17 26 29 38 45 46 49 51 60 61 66 70
1 3 5 12 18 22 31 38 49 50 53 63 66 67 69
1 4 8 12 25 33 35 40 43 45 47 48 51 53 61
1 5 16 24 28 29 33 42 52 57 58 59 61 64 66
1 8 16 17 21 27 37 43 44 46 54 58 63 69 70
3 5 13 14 21 23 28 36 37 39 43 53 60 61 68
3 8 11 16 19 23 24 27 33 34 45 49 56 67 68
5 9 10 19 24 30 32 38 40 43 51 59 63 68 70
5 11 12 20 30 31 34 37 39 41 45 48 57 58 70
8 11 12 18 19 22 28 32 36 51 52 54 58 60 62

design 31:

0 11 14 17 26 29 38 45 46 48 52 60 61 66 70
1 3 5 11 18 24 28 34 50 54 59 64 66 69 70
1 4 8 14 28 30 34 36 43 44 48 51 52 57 68
1 5 11 22 30 36 40 41 42 45 55 56 61 65 67
1 8 14 18 22 24 40 46 47 49 53 58 60 62 63
3 5 15 16 24 26 29 37 39 40 43 51 57 61 63
3 8 13 15 19 20 22 27 35 36 45 49 57 66 70
5 9 10 19 28 30 32 35 39 46 52 62 63 65 66
5 12 13 17 21 30 31 34 38 49 51 56 58 63 70
8 12 16 18 23 25 28 32 33 43 45 56 58 61 66

design 32:

0 11 14 17 23 32 38 42 46 49 51 60 61 66 67
1 3 5 12 19 24 28 38 45 55 60 62 67 69 70
1 4 8 15 25 26 33 37 44 48 49 51 55 60 64
1 5 12 21 29 31 34 43 49 51 54 56 63 65 66
1 8 15 17 22 36 40 42 43 45 50 56 57 61 69
3 5 14 16 27 29 33 36 37 40 42 46 52 55 65
3 8 13 14 18 20 21 25 34 36 47 49 58 67 69
5 9 10 17 23 33 34 37 39 41 50 63 64 67 69

5 11 13 19 21 28 33 35 36 51 53 57 59 61 64
8 11 16 19 24 26 27 30 34 44 46 56 57 63 67

design 33:

0 11 14 17 26 29 38 45 46 48 52 60 61 66 70
1 3 5 12 18 34 35 40 43 45 50 53 66 69 70
1 4 8 12 25 26 31 33 46 48 49 51 53 63 65
1 5 15 28 29 37 39 41 45 51 57 58 61 63 64
1 8 15 19 20 22 24 42 49 55 57 59 60 66 70
3 5 13 14 22 24 26 27 30 37 53 60 64 65 67
3 8 11 16 18 22 25 36 38 39 45 47 49 56 64
5 9 10 19 25 30 32 38 39 43 52 59 63 65 66
5 11 13 18 20 31 33 34 36 48 52 57 58 59 67
8 11 13 21 25 27 29 30 40 41 43 46 55 57 69

design 34:

0 11 14 17 23 33 39 42 46 49 51 59 61 65 67
1 3 5 12 24 28 30 32 46 48 55 63 65 67 70
1 4 8 15 17 22 26 37 48 51 52 54 61 63 66
1 5 12 21 25 36 38 42 43 54 56 59 61 64 69
1 8 15 19 30 34 39 42 44 45 49 50 56 58 70
3 5 14 16 19 21 29 37 39 40 42 52 55 66 68
3 8 11 16 24 27 28 33 36 37 43 44 50 57 61
5 9 10 17 23 33 34 36 40 41 50 63 64 66 70
5 11 13 19 27 30 31 34 38 51 52 53 57 59 63
8 11 16 17 18 21 25 26 34 46 53 55 60 69 70

design 35:

0 11 14 17 26 29 38 45 46 49 51 60 61 66 70
1 3 5 12 23 28 31 33 46 47 53 63 66 69 70
1 4 8 12 18 34 37 38 43 45 48 51 52 53 65
1 5 16 17 21 30 40 48 52 57 58 61 62 64 66
1 8 16 20 25 27 36 42 43 46 54 58 59 60 70
3 5 13 14 19 21 25 36 38 39 42 53 61 65 67
3 8 11 16 19 24 26 27 33 37 50 52 56 61 63
5 9 10 19 24 30 32 38 40 43 51 59 63 68 70
5 11 12 24 31 32 34 37 39 42 44 49 57 58 60
8 11 12 19 21 22 28 29 30 41 51 54 58 67 69

design 36:

0 11 14 17 26 29 38 45 46 49 51 60 61 66 70
1 3 5 11 24 28 33 37 43 44 54 61 66 68 69
1 4 8 14 19 21 34 36 44 48 49 51 57 67 68
1 5 11 22 27 30 31 49 50 55 56 60 62 63 67
1 8 14 18 25 39 40 42 43 45 50 53 58 61 62
3 5 15 16 18 20 26 36 37 40 51 58 60 64 67
3 8 12 16 24 25 28 30 32 38 46 48 49 58 63
5 9 10 19 28 30 32 35 39 45 51 62 64 68 70
5 12 13 18 31 32 34 36 38 43 45 52 56 57 66
8 12 16 17 19 22 23 27 33 45 53 54 64 66 67

design 37:

0 11 14 17 26 29 38 45 46 48 52 60 61 66 70
1 3 5 11 24 30 34 40 42 44 47 55 66 67 70
1 4 8 14 22 24 28 30 48 49 52 58 59 63 65
1 5 11 18 22 28 36 45 54 56 61 62 64 68 69
1 8 14 18 34 36 40 41 43 46 50 51 53 57 60
3 5 15 16 17 22 31 36 38 39 49 51 57 63 70
3 8 13 15 20 24 27 28 35 37 43 45 57 61 66
5 9 10 18 27 31 32 35 40 46 52 62 63 65 66
5 12 13 25 26 29 30 33 40 43 51 56 58 61 63
8 12 16 18 19 21 23 32 34 45 49 56 58 66 70

design 38:

0 11 14 17 26 29 38 45 46 49 51 60 61 66 70
1 3 5 11 24 30 34 40 47 49 52 55 60 62 69
1 4 8 14 22 24 28 30 41 45 48 58 67 68 70
1 5 11 18 22 28 36 44 54 56 61 63 64 65 66
1 8 14 18 34 36 40 42 43 46 50 51 53 57 59
3 5 15 16 25 26 29 36 39 40 43 45 58 63 67
3 8 12 16 18 20 21 28 35 37 49 51 58 61 69
5 9 10 18 27 31 32 35 40 45 51 62 64 68 70
5 12 13 17 22 30 31 33 38 43 51 56 57 67 69
8 12 16 22 24 26 27 29 31 42 52 53 54 60 64

design 39:

0 11 14 17 23 32 38 42 46 49 51 60 61 66 67
1 3 5 12 24 37 38 40 43 45 49 55 60 62 69
1 4 8 15 23 28 31 33 41 48 51 52 55 60 70
1 5 12 19 21 28 29 51 54 56 61 63 64 65 66
1 8 15 19 20 34 36 42 45 46 47 56 57 67 69
3 5 14 16 22 25 27 29 33 36 42 55 65 67 70
3 8 13 14 17 18 21 28 36 40 50 52 58 61 69
5 9 10 17 23 33 34 37 39 41 50 63 64 67 69
5 11 13 21 31 33 34 35 36 43 49 51 53 57 59
8 11 16 22 23 24 27 30 40 41 43 56 57 61 63

Designs 25, 26, 29, 36, 37 and 38 are self-dual.

Program by V.Ćepulić gave us generators of full automorphism groups for constructed designs. We determine the full automorphism groups using GAP (see [7]). Full automorphism groups of designs 1, 2, 3, 4, 6, 7, 8, 10, 11, 12, 14, 15 and 16 are isomorphic to group $E_4 \times A_4$ and full automorphism groups of designs 5, 9 and 13 are isomorphic to group $(E_8 : F_{21}) \times Z_2$. Furthermore, full automorphism groups of designs 17, 18, 19, 20, 21, 22, 23 and 24 are isomorphic to group $S_3 \times E_4$ and full automorphism groups of designs 25, 26, 27, 28, 29, 30, 31, 32, 33, 35, 37, 38 and 39 are isomorphic to group $A_4 \times Z_2$. Finally, full automorphism groups of designs 34 and 36 are isomorphic to group $E_8 : F_{21}$. □

Remark 3.2. Triplanes from theorem 1 having full automorphism groups of orders 168 and 336 are isomorphic to symmetric (71,15,3) designs described in [3]. Constructed triplanes also include all (71,15,3) designs described in [4]. Precisely, design 3 was also known. So, triplanes from theorem 1 include all 11 known triplanes with parameters (71,15,3). We constructed sixty-one new mutually non-isomorphic symmetric (71,15,3) designs.

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