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Example[section]

FUZZIFICATIONS OF IDEALS IN BCC-ALGEBRAS

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Abstract. In this paper we consider the fuzzification of ideals in the sense of W. A. Dudek in BCC-algebras. We discuss the relations among fuzzy BCK-ideal, fuzzy BCC-ideal and fuzzy g-ideal. We state fuzzy characteristic g-ideals, and also discuss fuzzy relations on BCC-algebras.

1. Introduction

In 1966, Y. Imai and K. Iséki ([12]) defined a class of algebras of type (2,0) called *BCK-algebras* which generalizes on one hand the notion of algebra of sets with the set subtraction as the only fundamental non-nullary operation, on the other hand the notion of implication algebra ([14]). The class of all BCK-algebras is a quasivariety. K. Iséki posed an interesting problem (solved by A. Wroński [17]) whether the class of BCK-algebras is a variety. In connection with this problem, Y. Komori ([15]) introduced a notion of BCC-algebras, and W. A. Dudek ([3, 4]) redefined the notion of BCC-algebras by using a dual form of the ordinary definition in the sense of Y. Komori. In [9], W. A. Dudek and X. H. Zhang introduced a notion of BCC-ideals in BCC-algebras and described connections between such ideals and congruences. W. A. Dudek and Y. B. Jun ([6]) considered the fuzzification of BCC-ideals in BCC-algebras. They showed that every fuzzy BCC-ideal of a BCC-algebra is a fuzzy BCK-ideal, and showed that the converse is not true by providing an example. They also proved that in a BCC-algebra every fuzzy BCK-ideal is a fuzzy BCC-subalgebra, and in a BCK-algebra the notion of a fuzzy BCK-ideal and a fuzzy BCC-ideal coincide. W. A. Dudek, Y. B. Jun and Z. Stojaković ([7]) described several properties of fuzzy BCC-ideals in BCC-algebras, and discussed an extension of fuzzy BCC-ideals. In [5], W. A. Dudek introduced a new notion of ideals in BCC-algebras, and gave its characterizations.

In this paper we consider the fuzzification of ideals in the sense of W. A. Dudek in BCC-algebras. We discuss the relations among fuzzy BCK-ideal, fuzzy BCC-ideal and fuzzy g-ideal. We state fuzzy characteristic g-ideals, and also discuss fuzzy relations on BCC-algebras.

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2. Preliminaries

By a *BCK-algebra* we mean an algebra $(G, *, 0)$ of type (2,0) satisfying the following axioms:

(I) $((x*y)*(x*z))*(z*y) = 0$,
(II) $(x*(x*y))*y = 0$,
(III) $x*x = 0$,
(IV) $0*x = 0$,
(V) $x*y = 0$ and $y*x = 0$ imply $x = y$,

for all $x, y, z \in G$.

In what follows, a binary multiplication will be denoted by juxtaposition. Dots we use only to avoid repetitions of brackets. For example, the formula $((xy)(zy))(xz) = 0$ will be written as $(xy \cdot zy) \cdot xz = 0$.

Definition 2.1. *A non-empty set G with a constant 0 and a binary operation denoted by juxtaposition is called a BCC-algebra if for all $x, y, z \in G$ the following axioms hold:*

(1) $(xy \cdot zy) \cdot xz = 0$,
(2) $xx = 0$,
(3) $0x = 0$,
(4) $x0 = x$,
(5) $xy = 0$ *and* $yx = 0$ *imply* $x = y$.

Any BCK-algebra is a BCC-algebra, but there are BCC-algebras which are not BCK-algebras (cf. [4]). Note that a BCC-algebra is a BCK-algebra if and only if it satisfies: (6) $xy \cdot z = xz \cdot y$.

On any BCC-algebra (similarly as in the case of BCK-algebras) one can define the natural order "$\leq$" by putting (7) $x \leq y \Longleftrightarrow xy = 0$.

It is not difficult to verify that this order is partial and 0 is its smallest element. Moreover, in any BCC-algebra (also in BCK-algebra), the following are true:

(8) $xy \cdot zy \leq xz$,
(9) $x \leq y$ *implies* $xz \leq yz$ *and* $zy \leq zx$.

A non-empty subset A of a BCK-algebra G is called an *ideal* if $0 \in A$ and $y, xy \in A$ imply $x \in A$. In the sequel this ideal will be called a *BCK-ideal* and will be considered also in BCC-algebras.

A non-empty subset A of a BCC-algebra G is called a *BCC-ideal* if $0 \in A$ and $y, xy \cdot z \in A$ imply $xz \in A$.

Definition 2.2. *A fuzzy set μ in a BCK-algebra G is called a fuzzy BCK-ideal of G if*

(FK1) $\mu(0) \geq \mu(x), \quad \forall x \in G$,
(FK2) $\mu(x) \geq \min\{\mu(xy), \mu(y)\}, \quad \forall x, y \in G$.

DEFINITION 2.3. ([6]). *A fuzzy set μ in a BCC-algebra G is called a fuzzy BCC-ideal of G if*

(FK1) $\mu(0) \geq \mu(x), \quad \forall x \in G,$
(FC1) $\mu(xy) \geq \min\{\mu(xa \cdot y),\, \mu(a)\}, \quad \forall a, x, y \in G.$

3. FUZZY g-IDEALS IN BCC-ALGEBRAS

DEFINITION 3.1. ([5]). *A subset A of a BCC-algebra G is called an ideal if it satisfies*
(I1) $0 \in A,$
(I2) $ab \in A$ *for* $a \in A$ *and* $b \in G,$
(I3) $b(ba_1 \cdot a_2) \in A$ *for* $a_1, a_2 \in A$ *and* $b \in G.$

Here we call this ideal A a *g-ideal* to avoid the confusion. We begin with the fuzzification of the above g-ideal.

DEFINITION 3.2. *A fuzzy set μ in a BCC-algebra G is called a fuzzy g-ideal if it satisfies*
(FK1) $\mu(0) \geq \mu(a), \quad \forall a \in G,$
(FI1) $\mu(ab) \geq \mu(a), \quad \forall a, b \in G,$
(FI2) $\mu(b(ba_1 \cdot a_2)) \geq \min\{\mu(a_1),\, \mu(a_2)\}, \quad \forall b, a_1, a_2 \in G.$

Observe that (FK1) follows from (FI1) and (2). Using (FI1) we know that every fuzzy g-ideal is a fuzzy subalgebra. Moreover, putting $a_1 = a$ and $a_2 = 0$ in (FI2) we obtain the following proposition.

PROPOSITION 3.3. If μ is a fuzzy g-ideal of a BCC-algebra G, then

$$\mu(b \cdot ba) \geq \mu(a), \quad \forall a, b \in G.$$

COROLLARY 3.4. Every fuzzy g-ideal μ of a BCC-algebra G is order reversing, i.e., if $x \leq a$ then $\mu(x) \geq \mu(a)$ for all $a, x \in G$.

PROOF. If $x, a \in G$ are such that $x \leq a$, then $\mu(x) = \mu(x0) = \mu(x \cdot xa) \geq \mu(a)$, which completes the proof. $\square$

THEOREM 3.5. A fuzzy set μ in a BCC-algebra G is a fuzzy g-ideal if and only if it is a fuzzy BCC-ideal.

PROOF. Let μ be a fuzzy g-ideal and let $a, x, y \in G$. Then

$$\begin{aligned}\mu(xy) &= \mu(xy \cdot 0)\\ &= \mu(xy \cdot ((xy \cdot (xa \cdot y))(x \cdot xa)))\\ &\geq \min\{\mu(xa \cdot y), \mu(x \cdot xa)\}\\ &\geq \min\{\mu(xa \cdot y), \mu(a)\},\end{aligned}$$

which shows that μ satisfies (FC1). Hence μ is a fuzzy BCC-ideal.

Conversely, let μ be a fuzzy BCC-ideal. Then $\mu(y) \le \mu(x)$ for all $x \le y$. Indeed,

$$\begin{aligned}\mu(x) &= \mu(x0) = \mu(x \cdot xy) \ge \min\{\mu(xy \cdot xy),\, \mu(y)\} \\ &= \min\{\mu(0),\, \mu(y)\} = \mu(y).\end{aligned}$$

Moreover, for all $a, x \in G$, we have

$$\begin{aligned}\mu(ax) &\ge \min\{\mu(aa \cdot x), \mu(a)\} \\ &= \min\{\mu(0x), \mu(a)\} \\ &= \min\{\mu(0), \mu(a)\} = \mu(a),\end{aligned}$$

which proves (FI1). To prove (FI2), let $x, a_1, a_2 \in G$. Note that

$$\begin{aligned}\mu(x \cdot xa_1) &\ge \min\{\mu(xa_1 \cdot xa_1), \mu(a_1)\} \\ &= \min\{\mu(0), \mu(a_1)\} = \mu(a_1).\end{aligned}$$

Since $xa_2 \cdot (xa_1 \cdot a_2) \le x \cdot xa_1$ by (8), then

$$\mu(xa_2 \cdot (xa_1 \cdot a_2)) \ge \mu(x \cdot xa_1) \ge \mu(a_1).$$

By using (FC1), we see that

$$\begin{aligned}\mu(x(xa_1 \cdot a_2) &\ge \min\{\mu(xa_2 \cdot (xa_1 \cdot a_2)), \mu(a_2)\} \\ &\ge \min\{\mu(a_1), \mu(a_2)\},\end{aligned}$$

which proves (FI2). Hence μ is a fuzzy g-ideal. □

THEOREM 3.6. Let μ be a fuzzy set in a BCK-algebra G. Then μ is a fuzzy g-ideal if and only if μ is a fuzzy BCK-ideal.

PROOF. Since every BCK-algebra is a BCC-algebra, every fuzzy g-ideal is a fuzzy BCC-ideal (see Theorem 3.5) and hence a fuzzy BCK-ideal. Let μ be a fuzzy BCK-ideal. Then

$$\begin{aligned}\mu(ax) &\ge \min\{\mu(ax \cdot a), \mu(a)\} \\ &= \min\{\mu(aa \cdot x), \mu(a)\} \\ &= \min\{\mu(0x), \mu(a)\} \\ &= \min\{\mu(0), \mu(a)\} \\ &= \mu(a),\end{aligned}$$

which shows (FI1). Now let $x, a_1, a_2 \in G$. Using (6), (8) and (II), we have

$$x(xa_1 \cdot a_2) \cdot a_2 = xa_2 \cdot (xa_1 \cdot a_2) \le x \cdot xa_1 \le a_1.$$

Since every fuzzy BCK-ideal of a BCK-algebra is order reversing, it follows that $\mu(x(xa_1 \cdot a_2) \cdot a_2) \ge \mu(a_1)$, and hence using (FK2) we obtain

$$\begin{aligned}\mu(x(xa_1 \cdot a_2)) &\ge \min\{\mu(x(xa_1 \cdot a_2) \cdot a_2), \mu(a_2)\} \\ &\ge \min\{\mu(a_1), \mu(a_2)\},\end{aligned}$$

which proves that μ satisfies (FI2). This completes the proof. □

The following example shows that a fuzzy BCK-ideal of a BCC-algebra may not be a fuzzy g-ideal.

EXAMPLE 3.7. Consider a BCC-algebra $G = \{0, a, b, c, d\}$ with Cayley table as follows (cf. [9]):

$\cdot$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	0	0	0
b	b	b	0	0	0
c	c	c	a	0	0
d	d	c	d	c	0

Let μ be a fuzzy set in G defined by

$$\mu(x) := \begin{cases} t_1 & \text{if } x \in \{0, a\}, \\ t_2 & \text{otherwise,} \end{cases}$$

where $t_1 > t_2$ in $[0, 1]$. It is easy to verify that μ is a fuzzy BCK-ideal of G, but it is not a fuzzy g-ideal since

$$\mu(d(da \cdot a)) = t_2 < t_1 = \min\{\mu(a), \mu(a)\}.$$

PROPOSITION 3.8. Let A be a non-empty subset of a BCC-algebra G and let μ be a fuzzy set in G defined by

$$\mu(a) := \begin{cases} t_1 & \text{if } a \in A, \\ t_2 & \text{otherwise,} \end{cases}$$

where $t_1 > t_2$ in $[0, 1]$. Then μ is a fuzzy g-ideal of G if and only if A is a g-ideal of G.

PROOF. Assume that μ is a fuzzy g-ideal of G. Since $\mu(0) \geq \mu(a)$ for all $a \in G$, we have $\mu(0) = t_1$ and so $0 \in A$. Let $a \in A$ and $b \in G$. Then $\mu(ab) \geq \mu(a) = t_1$ and thus $\mu(ab) = t_1$. Hence $ab \in A$. For any $a_1, a_2 \in A$ and $b \in G$, we get $\mu(b(ba_1 \cdot a_2)) \geq \min\{\mu(a_1), \mu(a_2)\} = t_1$ which implies that $\mu(b(ba_1 \cdot a_2)) = t_1$. It follows that $b(ba_1 \cdot a_2) \in A$. Therefore A is a g-ideal of G.

Conversely suppose that A is a g-ideal of G. Since $0 \in A$, it follows that $\mu(0) = t_1 \geq \mu(a)$ for all $a \in G$. Let $a, b \in G$. If $a \in A$, then $ab \in A$ and so $\mu(ab) = t_1 = \mu(a)$. If $a \in G \setminus A$, then $\mu(a) = t_2$ and hence $\mu(ab) \geq t_2 = \mu(a)$. Finally let $a_1, a_2, b \in G$. If $a_1 \in G \setminus A$ or $a_2 \in G \setminus A$, then $\mu(a_1) = t_2$ or $\mu(a_2) = t_2$. It follows that

$$\mu(b(ba_1 \cdot a_2)) \geq t_2 = \min\{\mu(a_1), \mu(a_2)\}.$$

Assume that $a_1, a_2 \in A$. Then $b(ba_1 \cdot a_2) \in A$ and thus

$$\mu(b(ba_1 \cdot a_2)) = t_1 = \min\{\mu(a_1), \mu(a_2)\}.$$

Hence μ is a fuzzy g-ideal of G. $\square$

LEMMA 3.9. ([5]). An initial segment $[0, c] := \{x \in G : 0 \leq x \leq c\}$ of a BCC-algebra G is a g-ideal if and only if the inequality $x(xc \cdot c) \leq c$ holds for all $x \in G$.

If we combine Proposition 3.8 with Lemma 3.9, then we have the following theorem.

THEOREM 3.10. Let μ be a fuzzy set in a BCC-algebra G defined by

$$\mu(x) := \begin{cases} t_1 & \text{if } x \in [0, c], \\ t_2 & \text{otherwise,} \end{cases}$$

where $t_1 > t_2$ in $[0, 1]$. Then μ is a fuzzy g-ideal if and only if the inequality $x(xc \cdot c) \leq c$ holds for all $x \in G$.

As a simple consequence of the above Theorem and [10, Proposition 2.7] we obtain

COROLLARY 3.11. Let μ be as in Theorem 3.10. Then
(i) μ is a fuzzy g-ideal if and only if $xc \cdot y \leq c$ implies $xy \leq c$ for all $x, y \in G$.
(ii) μ is a fuzzy g-ideal if and only if $xc \leq c$ implies $x \leq c$ for all $x \in G$.

4. FUZZY CHARACTERISTIC g-IDEALS

For an endomorphism f of a BCC-algebra G and a fuzzy set μ in G, we define a new fuzzy set μ^f in G by $\mu^f(x) = \mu(f(x))$ for all $x \in G$.

PROPOSITION 4.1. Let f be an endomorphism of a BCC-algebra G. If μ is a fuzzy g-ideal of G, then so is μ^f.

PROOF. We first have that $\mu^f(x) = \mu(f(x)) \leq \mu(0) = \mu(f(0)) = \mu^f(0)$ for all $x \in G$. Let $a, b \in G$. Then

$$\mu^f(ab) = \mu(f(ab)) = \mu(f(a)f(b)) \geq \mu(f(a)) = \mu^f(a),$$

proving the condition (FI1). Finally for any $b, a_1, a_2 \in G$ we get

$$\begin{aligned} \mu^f(b(ba_1 \cdot a_2)) &= \mu(f(b(ba_1 \cdot a_2))) \\ &= \mu(f(b)(f(b)f(a_1) \cdot f(a_2))) \\ &\geq \min\{\mu(f(a_1)), \mu(f(a_2))\} \\ &= \min\{\mu^f(a_1), \mu^f(a_2)\}, \end{aligned}$$

ending the proof. □

DEFINITION 4.2. *A g-ideal A of a BCC-algebra G is said to be* characteristic *if $f(A) = A$ for all $f \in \mathrm{Aut}(G)$, where Aut(G) is the set of all automorphisms of G.*

DEFINITION 4.3. *A fuzzy g-ideal μ of a BCC-algebra G is said to be* fuzzy characteristic *if $\mu^f(x) = \mu(x)$ for all $x \in G$ and $f \in \mathrm{Aut}(G)$.*

LEMMA 4.4. Let μ be a fuzzy set in a BCC-algebra G and let $t \in \mathrm{Im}(\mu)$. Then μ is a fuzzy g-ideal of G if and only if the level subset

$$\mu_t := \{x \in G | \mu(x) \geq t\}$$

is a g-ideal of G, which is called a *level g-ideal* of μ.

PROOF. Assume that μ is a fuzzy g-ideal of G. Clearly $0 \in \mu_t$. Let $a \in \mu_t$ and $b \in G$. Then $\mu(a) \geq t$ and so $\mu(ab) \geq \mu(a) \geq t$, which implies that $ab \in \mu_t$. Now let $a_1, a_2 \in \mu_t$ and $b \in G$. Then

$$\mu(b(ba_1 \cdot a_2)) \geq \min\{\mu(a_1), \mu(a_2)\} \geq t$$

and thus $b(ba_1 \cdot a_2) \in \mu_t$. Hence μ_t is a g-ideal of G.

Conversely suppose that μ_t is a g-ideal of G. If there exists $a_0 \in G$ such that $\mu(0) < \mu(a_0)$, then $\mu(0) < \frac{1}{2}(\mu(0) + \mu(a_0)) < \mu(a_0)$ and hence $a_0 \in \mu_s$ where $s = \frac{1}{2}(\mu(0) + \mu(a_0))$. Since $0 \in \mu_s$, we have $\mu(0) \geq s$, a contradiction. Assume that $\mu(a_0b_0) < \mu(a_0)$ for some $a_0, b_0 \in G$. Taking $u = \frac{1}{2}(\mu(a_0b_0) + \mu(a_0))$, then $\mu(a_0b_0) < u < \mu(a_0)$ and thus $a_0 \in \mu_u$ and $a_0b_0 \notin \mu_u$. This is a contradiction. Finally suppose that there exist $a_1, a_2, b \in G$ such that

$$\mu(b(ba_1 \cdot a_2)) < \min\{\mu(a_1), \mu(a_2)\}.$$

If we take $v = \frac{1}{2}(\mu(b(ba_1 \cdot a_2)) + \min\{\mu(a_1), \mu(a_2)\})$, then $\mu(b(ba_1 \cdot a_2)) < v < \min\{\mu(a_1), \mu(a_2)\}$ and so $a_1, a_2 \in \mu_v$ and $b(ba_1 \cdot a_2) \notin \mu_v$, a contradiction. This completes the proof. ∎

LEMMA 4.5. Let μ be a fuzzy g-ideal of a BCC-algebra G and let $x \in G$. Then $\mu(x) = t$ if and only if $x \in \mu_t$ and $x \notin \mu_s$ for all $s > t$.

PROOF. Straightforward. ∎

THEOREM 4.6. For a fuzzy g-ideal μ of a BCC-algebra G, the following are equivalent:
(i) μ is fuzzy characteristic.
(ii) Each level g-ideal of μ is characteristic.

PROOF. Assume that μ is a fuzzy characteristic and let $t \in \mathrm{Im}(\mu)$, $f \in \mathrm{Aut}(G)$ and $x \in \mu_t$. Then $\mu^f(x) = \mu(x) \geq t$, i.e., $\mu(f(x)) \geq t$, and so $f(x) \in \mu_t$, i.e., $f(\mu_t) \subset \mu_t$. Now let $x \in \mu_t$ and let $y \in G$ be such that $f(y) = x$. Then $\mu(y) = \mu^f(y) = \mu(f(y)) = \mu(x) \geq t$, whence $y \in \mu_t$, so that $x = f(y) \in f(\mu_t)$. Consequently $\mu_t \subset f(\mu_t)$. Hence $f(\mu_t) = \mu_t$ and μ_t is characteristic.

Conversely suppose that each level g-ideal of μ is characteristic and let $x \in G$, $f \in \mathrm{Aut}(G)$ and $\mu(x) = t$. Then, by virtue of Lemma 4.5, $x \in \mu_t$ and $x \notin \mu_s$ for all $s > t$. It follows from hypothesis that $f(x) \in f(\mu_t) = \mu_t$, so that $\mu^f(x) = \mu(f(x)) \geq t$. Let $s = \mu^f(x)$ and assume that $s > t$. Then $f(x) \in \mu_s = f(\mu_s)$, which implies from the injectivity of f that $x \in \mu_s$, a contradiction. Hence $\mu^f(x) = \mu(f(x)) = t = \mu(x)$ showing that μ is fuzzy characteristic. ∎

5. Cartesian product of fuzzy g-ideals

Definition 5.1. ([1]). *A* fuzzy relation *on any set S is a fuzzy set*

$$\mu : S \times S \to [0,1].$$

Definition 5.2. ([1]). *If μ is a fuzzy relation on a set S and ν is a fuzzy set in S, then μ is a* fuzzy relation *on ν if*

$$\mu(x,y) \le \min\{\nu(x),\nu(y)\}, \ \ \forall x,y \in S.$$

Definition 5.3. ([1]). *Let μ and ν be fuzzy sets in a set S. The* Cartesian product *of μ and ν is defined by*

$$(\mu \times \nu)(x,y) = \min\{\mu(x),\nu(y)\}, \ \ \forall x,y \in S.$$

Lemma 5.4. ([1]). Let μ and ν be fuzzy sets in a set S. Then
(i) $\mu \times \nu$ is a fuzzy relation on S,
(ii) $(\mu \times \nu)_t = \mu_t \times \nu_t$ for all $t \in [0,1]$.

Definition 5.5. ([1]). *If ν is a fuzzy set in a set S, the* strongest fuzzy relation *on S that is a fuzzy relation on ν is μ_ν, given by*

$$\mu_\nu(x,y) = \min\{\nu(x),\nu(y)\}, \ \ \forall x,y \in S.$$

Lemma 5.6. ([1]). For a given fuzzy set ν in a set S, let μ_ν be the strongest fuzzy relation on S. Then for $t \in [0,1]$, we have that $(\mu_\nu)_t = \nu_t \times \nu_t$.

Proposition 5.7. For a given fuzzy set ν in a BCC-algebra G, let μ_ν be the strongest fuzzy relation on G. If μ_ν is a fuzzy g-ideal of $G \times G$, then $\nu(a) \le \nu(0)$ for all $a \in G$.

Proof. From the fact that μ_ν is a fuzzy g-ideal of $G \times G$, it follows from (FK1) that $\mu_\nu(a,a) \le \mu_\nu(0,0)$ for all $a \in G$, where $(0,0)$ is the zero element of $G \times G$. But this means that $\min\{\nu(0),\nu(0)\} \ge \min\{\nu(a),\nu(a)\}$, which implies that $\nu(0) \ge \nu(a)$. □

The following proposition is an immediate consequence of Lemma 5.6, and we omit the proof.

Proposition 5.8. If ν is a fuzzy g-ideal of a BCC-algebra G, then the level g-ideals of μ_ν are given by $(\mu_\nu)_t = \nu_t \times \nu_t$ for all $t \in [0,1]$.

Theorem 5.9. Let μ and ν be fuzzy g-ideals of a BCC-algebra G. Then $\mu \times \nu$ is a fuzzy g-ideal of $G \times G$.

Proof. Note first that for every $(x,y) \in G \times G$,

$$(\mu \times \nu)(0,0) = \min\{\mu(0),\nu(0)\} \ge \min\{\mu(x),\nu(y)\} = (\mu \times \nu)(x,y).$$

Let $(a_1,a_2),(b_1,b_2) \in G \times G$. Then

$$\begin{aligned}&(\mu \times \nu)((a_1,a_2) * (b_1,b_2)) = (\mu \times \nu)(a_1b_1, a_2b_2)\\ &= \min\{\mu(a_1b_1),\nu(a_2b_2)\} \ge \min\{\mu(a_1),\nu(a_2)\} = (\mu \times \nu)(a_1,a_2).\end{aligned}$$

For any $(b_1, b_2), (x_1, x_2), (y_1, y_2) \in G \times G$, we have

$$\begin{aligned}
&(\mu \times \nu)((b_1, b_2) * (((b_1, b_2) * (x_1, x_2)) * (y_1, y_2))) \\
&= (\mu \times \nu)((b_1, b_2) * ((b_1x_1, b_2x_2) * (y_1, y_2))) \\
&= (\mu \times \nu)((b_1, b_2) * (b_1x_1 \cdot y_1, b_2x_2 \cdot y_2)) \\
&= (\mu \times \nu)(b_1(b_1x_1 \cdot y_1), b_2(b_2x_2 \cdot y_2)) \\
&= \min\{\mu(b_1(b_1x_1 \cdot y_1)), \nu(b_2(b_2x_2 \cdot y_2))\} \\
&\geq \min\{\min\{\mu(x_1), \mu(y_1)\}, \min\{\nu(x_2), \nu(y_2)\}\} \\
&= \min\{\min\{\mu(x_1), \nu(x_2)\}, \min\{\mu(y_1), \nu(y_2)\}\} \\
&= \min\{(\mu \times \nu)(x_1, x_2), (\mu \times \nu)(y_1, y_2)\}.
\end{aligned}$$

Hence $\mu \times \nu$ is a fuzzy g-ideal of $G \times G$. □

THEOREM 5.10. Let μ and ν be fuzzy sets in a BCC-algebra G such that $\mu \times \nu$ is a fuzzy g-ideal of $G \times G$. Then
(i) either $\mu(x) \leq \mu(0)$ or $\nu(x) \leq \nu(0)$ for all $x \in G$.
(ii) if $\mu(x) \leq \mu(0)$ for all $x \in G$, then either $\mu(x) \leq \nu(0)$ or $\nu(x) \leq \nu(0)$.
(iii) if $\nu(x) \leq \nu(0)$ for all $x \in G$, then either $\mu(x) \leq \mu(0)$ or $\nu(x) \leq \mu(0)$.
(iv) either μ or ν is a fuzzy g-ideal of G.

PROOF. (i) Suppose that $\mu(x) > \mu(0)$ and $\nu(y) > \nu(0)$ for some $x, y \in G$. Then $(\mu\times\nu)(x, y) = \min\{\mu(x), \nu(y)\} > \min\{\mu(0), \nu(0)\} = (\mu\times\nu)(0, 0)$, which is a contradiction and we obtain (i).

(ii) Assume that there exist $x, y \in G$ such that $\mu(x) > \nu(0)$ and $\nu(y) > \nu(0)$. Then $(\mu \times \nu)(0, 0) = \min\{\mu(0), \nu(0)\} = \nu(0)$ and hence

$$(\mu \times \nu)(x, y) = \min\{\mu(x), \nu(y)\} > \nu(0) = (\mu \times \nu)(0, 0).$$

This is a contradiction. Hence (ii) holds.

(iii) is by similar method to part (ii).

(iv) Since, by (i), either $\mu(x) \leq \mu(0)$ or $\nu(x) \leq \nu(0)$ for all $x \in G$, without loss of generality we may assume that $\nu(x) \leq \nu(0)$ for all $x \in G$. It follows from (iii) that either $\mu(x) \leq \mu(0)$ or $\nu(x) \leq \mu(0)$. If $\nu(x) \leq \mu(0)$ for any $x \in G$, then

$$\begin{aligned}
\nu(x) \quad &= \min\{\mu(0), \nu(x)\} = (\mu \times \nu)(0, x) \\
&\leq (\mu \times \nu)((0, x) * (y_1, y_2)) = (\mu \times \nu)(0y_1, xy_2) \\
&= (\mu \times \nu)(0, xy_2) = \nu(xy_2)
\end{aligned}$$

for all $x, y_1, y_2 \in G$, which proves that ν satisfies the condition (FI1). Now

$$\begin{aligned}
&\min\{\nu(a_1), \nu(a_2)\} \\
&= \min\{\min\{\mu(0), \nu(a_1)\}, \min\{\mu(0), \nu(a_2)\}\} \\
&= \min\{(\mu \times \nu)(0, a_1), (\mu \times \nu)(0, a_2)\} \\
&\le (\mu \times \nu)((b_1, b_2) * (((b_1, b_2) * (0, a_1)) * (0, a_2))) \\
&= (\mu \times \nu)((b_1, b_2) * (b_1 0 \cdot 0, b_2 a_1 \cdot a_2)) \\
&= (\mu \times \nu)((b_1, b_2) * (b_1, b_2 a_1 \cdot a_2)) \\
&= (\mu \times \nu)(b_1 b_1, b_2(b_2 a_1 \cdot a_2)) \\
&= (\mu \times \nu)(0, b_2(b_2 a_1 \cdot a_2)) \\
&= \min\{\mu(0), \nu(b_2(b_2 a_1 \cdot a_2))\} \\
&= \nu(b_2(b_2 a_1 \cdot a_2))
\end{aligned}$$

for all $a_i, b_j \in G$, $i = 1, 2$; $j = 1, 2$. Hence ν is a fuzzy g-ideal of G. Now we consider the case $\mu(x) \le \mu(0)$ for all $x \in G$. Suppose that $\nu(y) > \mu(0)$ for some $y \in G$. Then $\nu(0) \ge \nu(y) > \mu(0)$. Since $\mu(0) \ge \mu(x)$ for all $x \in G$, it follows that $\nu(0) > \mu(x)$ for any $x \in G$. Hence $(\mu \times \nu)(x, 0) = \min\{\mu(x), \nu(0)\} = \mu(x)$ for all $x \in G$. Thus

$$\begin{aligned}
&\mu(x) = (\mu \times \nu)(x, 0) \le (\mu \times \nu)((x, 0) * (y_1, y_2)) \\
&= (\mu \times \nu)(x y_1, 0 y_2) = (\mu \times \nu)(x y_1, 0) = \mu(x y_1)
\end{aligned}$$

for all $x, y_1, y_2 \in G$. Moreover

$$\begin{aligned}
&\min\{\mu(a_1), \mu(a_2)\} \\
&= \min\{(\mu \times \nu)(a_1, 0), (\mu \times \nu)(a_2, 0)\} \\
&\le (\mu \times \nu)((b_1, b_2) * (((b_1, b_2) * (a_1, 0)) * (a_2, 0))) \\
&= (\mu \times \nu)((b_1, b_2) * (b_1 a_1 \cdot a_2, b_2 0 \cdot 0)) \\
&= (\mu \times \nu)(b_1(b_1 a_1 \cdot a_2), b_2 b_2) \\
&= (\mu \times \nu)(b_1(b_1 a_1 \cdot a_2), 0) \\
&= \mu(b_1(b_1 a_1 \cdot a_2))
\end{aligned}$$

for all $a_i, b_j \in G$, $i = 1, 2$; $j = 1, 2$, which proves that μ is a fuzzy g-ideal of G. This completes the proof. □

Now we give an example to show that if $\mu \times \nu$ is a fuzzy g-ideal of $G \times G$, then μ and ν both need not be fuzzy g-ideals of G.

Example 5.11. Let G be a BCC-algebra with $|G| \ge 2$ and let $s, t \in [0, 1)$ be such that $s \le t$. Define fuzzy sets μ and ν in G by $\mu(x) = s$ and

$$\nu(x) = \begin{cases} t & \text{if } x = 0, \\ 1 & \text{otherwise,} \end{cases}$$

for all $x \in G$, respectively. Then $(\mu \times \nu)(x, y) = \min\{\mu(x), \nu(y)\} = s$ for all $(x, y) \in G \times G$, that is, $\mu \times \nu$ is a constant function and so $\mu \times \nu$ is a fuzzy g-ideal of $G \times G$. Now μ is a fuzzy g-ideal of G, but ν is not a fuzzy g-ideal of G since for $x \ne 0$ we have $\nu(0) = t < 1 = \nu(x)$.

THEOREM 5.12. Let ν be a fuzzy set in a BCC-algebra G and let μ_ν be the strongest fuzzy relation on G. Then ν is a fuzzy g-ideal of G if and only if μ_ν is a fuzzy g-ideal of $G \times G$.

PROOF. Assume that ν is a fuzzy g-ideal of G. Clearly $\mu_\nu(0,0) \geq \mu_\nu(x,y)$ for any $(x,y) \in G \times G$. Now

$$\begin{aligned} \mu_\nu(a_1,a_2) &= \min\{\nu(a_1),\nu(a_2)\} \leq \min\{\nu(a_1b_1),\nu(a_2b_2)\} \\ &= \mu_\nu(a_1b_1,a_2b_2) = \mu_\nu((a_1,a_2)*(b_1,b_2)) \end{aligned}$$

for all $(a_1,a_2),(b_1,b_2) \in G \times G$, and

$$\begin{aligned} &\min\{\mu_\nu(a_1,a_2),\mu_\nu(b_1,b_2)\} \\ &= \min\{\min\{\nu(a_1),\nu(a_2)\},\min\{\nu(b_1),\nu(b_2)\}\} \\ &= \min\{\min\{\nu(a_1),\nu(b_1)\},\min\{\nu(a_2),\nu(b_2)\}\} \\ &\leq \min\{\nu(x(xa_1 \cdot b_1)),\nu(y(ya_2 \cdot b_2))\} \\ &= \mu_\nu(x(xa_1 \cdot b_1),y(ya_2 \cdot b_2)) \\ &= \mu_\nu((x,y)*(((x,y)*(a_1,a_2))*(b_1,b_2))) \end{aligned}$$

for all $(x,y),(a_1,a_2),(b_1,b_2) \in G \times G$. Hence μ_ν is a fuzzy g-ideal of $G \times G$.

Conversely suppose that μ_ν is a fuzzy g-ideal of $G \times G$. Then

$$\min\{\nu(0),\nu(0)\} = \mu_\nu(0,0) \geq \mu_\nu(x,y) = \min\{\nu(x),\nu(y)\}$$

for all $(x,y) \in G \times G$. It follows that $\nu(x) \leq \nu(0)$ for all $x \in G$. Now we have

$$\begin{aligned} \nu(a) &= \min\{\nu(a),\nu(0)\} = \mu_\nu(a,0) \leq \mu_\nu((a,0)*(b_1,b_2)) \\ &= \mu_\nu(ab_1,0b_2) = \mu_\nu(ab_1,0) = \min\{\nu(ab_1),\nu(0)\} = \nu(ab_1) \end{aligned}$$

for all $a,b_1 \in G$, and

$$\begin{aligned} &\min\{\min\{\nu(a_1),\nu(a_2)\},\min\{\nu(b_1),\nu(b_2)\}\} \\ &= \min\{\mu_\nu(a_1,a_2),\mu_\nu(b_1,b_2)\} \\ &\leq \mu_\nu((x,y)*(((x,y)*(a_1,a_2))*(b_1,b_2))) \\ &= \mu_\nu(x(xa_1 \cdot b_1),y(ya_2 \cdot b_2)) \\ &= \min\{\nu(x(xa_1 \cdot b_1)),\nu(y(ya_2 \cdot b_2))\} \end{aligned}$$

for all $(x,y),(a_1,a_2),(b_1,b_2) \in G \times G$. Taking $a_2 = b_2 = 0$ (resp. $a_1 = b_1 = 0$) and using (2) and (4), then

$$\min\{\nu(a_1),\nu(b_1)\} \leq \nu(x(xa_1 \cdot b_1))$$

$$(\text{resp. } \min\{\nu(a_2),\nu(b_2)\} \leq \nu(x(xa_2 \cdot b_2))).$$

Hence ν is a fuzzy g-ideal of G. ∎

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