GLASNIK MATEMATIČKI
Vol. 36(56)(2001), 155 – 167

ON THE MULTIDIMENSIONAL SAMPLING THEOREM

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Abstract. The well known sampling theorem is extended to the multidimensional weakly stationary (but not necessarily band-limited) processes. The mean square and almost sure convergence of the sampling expansion sum is derived for full spectrum multidimensional processes.

1. Introduction

The sampling theorem states that if $f(t)$ can be represented as

$$f(t) = \int_{-w}^{w} e^{it\lambda} dg(\lambda), \tag{1}$$

where $g(\lambda)$ is of bounded variation and continuous at the end points $\pm w$ of the interval $(-w, w)$, then (if $\operatorname{sinc}(x) : x^{-1}\sin(x)$):

$$f(t) = \sum_{n-\infty}^{\infty} f\left(\frac{n\pi}{\widetilde{w}}\right) \operatorname{sinc}(\widetilde{w}t - n\pi),$$

where $\widetilde{w} \geq w > 0$ ($\widetilde{w}$ is the sampling frequency) and the previous, so called sampling series converges uniformly on any bounded interval (or bounded region in the complex plane).

Stochastic versions of the sampling theorem have been derived for stationary processes among others by Balakrishnan [1], Belyaev [2], Lloyd [5], Gulyás [3], Pogány [6], Wong [8], Zakai [9], etc. In both cases ((1) and the stochastic

2000 *Mathematics Subject Classification.* 60G10.

Key words and phrases. sampling theorem, band-limited processes, multidimensional processes, L_2-convergence, a.s. convergence.

Editor's note: This article was accepted for publication in 1990, but its publication was put on hold because of the war situation in Croatia and UN sanctions against SR Yugoslavia. These circumstances have changed, and we are now publishing an updated version.

version) the function or the process has a suitably restricted frequency spectrum. So, the weakly stationary (WS) and mean square continuous process $\{X(t)|\, t\in \mathbf{R}\}$, $\mathbf{E}X(t)=0$ with the variance $\mathbf{D}X(t)=\sigma^2$ is said to be band-limited to frequency w if its correlation function $\mathcal{K}(t)=\mathbf{E}X(t)X^\star(0)$ has an integral like (1):

$$\mathcal{K}(t)=\int_{-w}^{w} e^{it\lambda}dF(\lambda), \tag{2}$$

where $F(\lambda)$ is the spectral distribution function of the process $X(t)$. Then it is also true that $X(t)$ has a spectral representation in the form

$$X(t)=\int_{-w}^{w} e^{it\lambda}dZ(\lambda), \tag{3}$$

where $Z(\lambda)$ (the spectral process of $X(t)$) is a process with orthogonal increments, and $\mathbf{E}|dZ(\lambda)|^2=dF(\lambda)$.

The stochastic sampling theorem is under foregoing conditions

$$X(t)=\sum_{n-\infty}^{\infty} X\left(\frac{n\pi}{\widetilde{w}}\right)\operatorname{sinc}(\widetilde{w}t-n\pi), \tag{4}$$

for arbitrary $\widetilde{w}\geq w>0$, where the equality in (4) is in mean square.

The sampling truncation error $\Delta_n=\mathbf{E}|X(t)-X_n(t)|^2$ satisfies the inequality

$$\Delta_n \leq \frac{64\sigma^2\widetilde{w}^2}{\pi^4 n^2}\cdot\frac{(\widetilde{w}|t|+\pi)^2}{(\widetilde{w}-w)^2}; \qquad \widetilde{w}>w. \tag{5}$$

Here $X_n(t)$ denotes the symmetric partial sum in (4) containing the $2n+1$ consecutive central addends, i.e.

$$X_n(t)=\sum_{|j|\leq n} X\left(\frac{j\pi}{\widetilde{w}}\right)\operatorname{sinc}(\widetilde{w}t-j\pi),$$

compare [2].

Remark 1. This article was accepted for publication more than ten years ago by the Editorial Board of Glasnik Matematički, but it was never published because of the war situation in Croatia (P. Peruničić was working at Mathematical Faculty, Belgrade, Yugoslavia). In the meantime, the formal reasons for nonpublishing disappeared. The original article contains the first three sections and the reference list. However, the original article needs certain comments about the new results concerning the matter exposed here. These comments are given in the new additional section entitled *Conclusions, further remarks*, written recently by the first author. The references concerning the multidimensional sampling extension problems appearing in the meantime are listed separately after the original *References* list, as *Supplementary references* item.

2. Multidimensional band - limited case

Let $\{\mathbf{X}(t) = (X_1(t), \cdots, X_q(t)) | t \in \mathbf{R}\}$ be a WS q-dimensional stochastic process with band-limited coordinates (qBLP) to the frequencies $w_j, j \in Q := \{1, \cdots, q\}$, and let $\mathbf{E}X_j(t) = 0,\ \mathbf{D}X_j(t) = \sigma_j^2$. The end points of the intervals $(-w_j, w_j)$ are required to be continuity points of $F_{jj}(\lambda)$, where

$$\mathcal{K}_{jj}(t) = \int_{-w_j}^{w_j} e^{it\lambda} dF_{jj}(\lambda), \tag{6}$$

and $\mathcal{K}_{jj}(t), j \in Q$ are diagonal elements of the correlation matrix

$$\mathbf{K}(t) = (\mathbf{E}X_j(t)X_k^\star(0))_{q\times q} = (\mathcal{K}_{jk}(t))_{q\times q}$$

of the qBLP $\mathbf{X}(t)$. The cross-correlation functions $\mathcal{K}_{jk}(t)$ have also integral representations and it is not difficult to show that

$$\mathcal{K}_{jk}(t) = \int_{-w}^{w} e^{it\lambda} dF_{jk}(\lambda), \tag{7}$$

where $w = \min\{w_j, w_k\};\ j,k \in Q$. For its simplicity the proof of the relation (7) is omitted.

Let us introduce

$$X_{ab}(t) = \sum_{|j|\le b} X_a\left(\frac{j\pi}{\widetilde{w}_a}\right) \mathrm{sinc}(\widetilde{w}_a t - j\pi), \tag{8}$$

where $\widetilde{w}_a > w_a;\ a \in Q$, b positive integer. The following result gives us the evaluation of the sampling truncation error for the cross-correlations in $\mathbf{K}(t)$.

Theorem 1. *Let*

$$\Delta_{mn}(j,k) = \mathbf{E}\left(X_j(t) - X_{jm}(t)\right)\left(X_k(t) - X_{kn}(t)\right)^\star. \tag{9}$$

Then it follows that

$$|\Delta_{mn}(j,k)| \le \frac{64\sigma_j\sigma_k\widetilde{w}_j\widetilde{w}_k(\widetilde{w}_j|t|+\pi)(\widetilde{w}_k|t|+\pi)}{mn\pi^4(\widetilde{w}_j - w)(\widetilde{w}_k - w)}, \qquad j,k \in Q. \tag{10}$$

Proof. It is clear form (7) and (8) that

$$\Delta_{mn}(j,k) =$$
$$\int_{-w}^{w} \sum_{|p|>m}\sum_{|r|>n} \exp\left\{i\lambda\pi\left(\frac{p}{\widetilde{w}_j} - \frac{r}{\widetilde{w}_k}\right)\right\} \mathrm{sinc}(\widetilde{w}_j - p\pi)\mathrm{sinc}(\widetilde{w}_k - r\pi) dF_{jk}(\lambda),$$

where $w = \min\{w_j, w_k\};\ j,k \in Q$. Denote

$$y_m^{(j)}(\lambda) = \sum_{|p|>m} \exp\left\{i\lambda\frac{p\pi}{\widetilde{w}_j}\right\} \mathrm{sinc}(\widetilde{w}_j - p\pi). \tag{11}$$

As

$$\left|y_m^{(j)}(\lambda)\right| \le \frac{8\widetilde{w}_j(\widetilde{w}_j|t|+\pi)}{m\pi^2(\widetilde{w}_j - \lambda)},$$

(consult e.g. [2],[4],[7]), we obtain that

$$\begin{aligned}|\Delta_{mn}(j,k)| &\leq \frac{64}{mn\pi^2}\left(\frac{\widetilde{w}_j|t|}{\pi}+1\right)\left(\frac{\widetilde{w}_j|t|}{\pi}+1\right)\left|\int_{-w}^{w}\frac{dF_{jk}(\lambda)}{\left(1-\frac{\lambda}{\widetilde{w}_j}\right)\left(1-\frac{\lambda}{\widetilde{w}_k}\right)}\right| \\ &\leq \frac{64}{mn\pi^2}\left(\frac{\widetilde{w}_j|t|}{\pi}+1\right)\left(\frac{\widetilde{w}_j|t|}{\pi}+1\right)\frac{|\mathcal{K}_{jk}(0)|}{\left(1-\frac{w}{\widetilde{w}_j}\right)\left(1-\frac{w}{\widetilde{w}_k}\right)}.\end{aligned}$$

From the Schwarz inequality it follows: $|\mathcal{K}_{jk}(t)|^2 \leq \sigma_j^2\sigma_k^2$ and we get the assertion of the Theorem. ∎

We can clearly show that (10) is a simple generalization of the Belyaev - formula (5). We shall apply this evaluation in the mean square convergence investigations.

Finally, the multidimensional sampling theorem for WS, qBLP $\mathbf{X}(t)$ based on the Theorem 1 is given as the

THEOREM 2. *Denote* $\mathbf{X}_\mathbf{N}(t) = \big(X_{1n_1}(t), \cdots, X_{qn_q}(t)\big)$. *Then we have*

$$\text{l.i.m.}_{n\to\infty}\mathbf{X}_\mathbf{N}(t) = \mathbf{X}(t), \tag{12}$$

uniformly on all compact t-sets from $\mathbf{R}$. *Here is* $n = \min_Q(n_j)$.

PROOF. We have to prove that the matrix

$$\Delta = \mathbf{E}(\mathbf{X}(t)-\mathbf{X}_\mathbf{N}(t))'(\mathbf{X}(t)-\mathbf{X}_\mathbf{N}(t))^\star$$

vanishes it n tends to infinity ($'$ denote the transponate and * the complex conjugate). Indeed, it is clear that $\Delta = \big(\Delta_{n_jn_k}(j,k)\big)_{q\times q}$. Now, from (10) it follows for $n = \min_Q(n_j)$ that $|\Delta_{n_jn_k}(j,k)| \to 0$ as $n\to\infty$. But this means exactly the assertion of the Theorem. ∎

So, the multidimensional sampling expansion procedure is possible. Namely, we apply the scalar sampling expansion result to the multidimesional BLP coordinatewise. The mean square limit of the multidimensional sampling expansion partial sum vector $\mathbf{X}_\mathbf{N}(t)$ is the initial qBLP $\mathbf{X}(t)$, see the relation (12). The upper bound of the mean square truncation error of the sampling expansion is the matrix

$$\underline{\Delta} = \big(|\Delta_{n_jn_k}(j,k)|\big)_{q\times q}\,.$$

But it contains also the cross-errors which upper bounds are given by (10).

The idea of the following considerations (scalar case) was given in the paper [2]. We shall consider the almost sure convergence of the multidimensional truncated sampling expansion sequence $\mathbf{X}_\mathbf{N}(t)$ if the minimal sampling size n tends to the infinity.

THEOREM 3.

$$\mathbf{P}\{\lim_{n\to\infty}\mathbf{X}_\mathbf{N}(t) = \mathbf{X}(t)\} = 1, \tag{13}$$

uniformly in t when it is belonging to certain compact subset of $\mathbf{R}$.

PROOF. $\mathbf{X}(t)$ is defined on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with the measurable space $(\mathbf{R}_q, \mathcal{B}_q)$. All metrics on the finite-dimensional Euclidean spaces are equivalent to each other. Then can use the max - metric defining a norm:

$$\|\mathbf{X}(t) - \mathbf{X}_{\mathbf{N}}(t)\| = \max_Q |X_j(t) - X_{jn_j}(t)|.$$

Then we have, for every $\varepsilon > 0$

$$\begin{aligned}
\mathbf{P}\{\|\mathbf{X}(t) - \mathbf{X}_{\mathbf{N}}(t)\| \geq \varepsilon\} &= \mathbf{P}\left(\bigcup_Q \{|X_j(t) - X_{jn_j}(t)| \geq \varepsilon\}\right) \\
&\leq \sum_{j=1}^{q} \mathbf{P}\{|X_j(t) - X_{jn_j}(t)| \geq \varepsilon\} \\
&\leq \varepsilon^{-2} \sum_{j=1}^{q} \mathbf{E}|X_j(t) - X_{jn_j}(t)|^2 \\
&\leq \varepsilon^{-2} \sum_{j=1}^{q} \Delta_{n_j n_j}(j,j) \\
&\leq \frac{C}{\varepsilon^2} \sum_{j=1}^{q} \frac{1}{n_j^2} \leq \frac{Cq}{\varepsilon^2 n^2},
\end{aligned}$$

where

$$C = \frac{64}{\pi^4} \max_Q \frac{\sigma_j^2 \widetilde{w}_j^2 (\widetilde{w}_j |t| + \pi)^2}{(\widetilde{w}_j - w)^2},$$

and $n = \min_Q(n_j)$.

So, by the Borel - Cantelli Lemma

$$\mathbf{P}\{\|\mathbf{X}(t) - \mathbf{X}_{\mathbf{N}}(t)\| \geq \varepsilon \quad \text{infinitely often}\} = 0,$$

and it follows that $\mathbf{P}\{\lim_{n\to\infty} \mathbf{X}_{\mathbf{N}}(t) = \mathbf{X}(t)\} = 1$. □

3. Multidimensional full spectrum case

Consider a special class of the WS processes having a full spectrum, i.e. there does not exist an interval of positive Lebesgue measure on which the spectral measure of the process is identically equal to zero (sometimes these processes are called non band - limited). In other words

$$\int_{\mathbf{R}} e^{it\lambda} dZ(\lambda)$$

is the spectral representation of such process $X(t)$.

Let $\mathcal{L}$ be a linear transformation (filter) with the spectral characteristic

$$h_w(\lambda) = \mathbf{1}_{(-w,w)}(\lambda),$$

i.e. the indicator function of the interval $(-w, w)$. If we apply this filter to arbitrary full spectrum process $X(t)$, it follows that

$$(\mathcal{L}X)(t) = \mathcal{L}\int_{\mathbf{R}} e^{it\lambda} dZ(\lambda) = \int_{\mathbf{R}} e^{it\lambda} \mathbf{1}_{(-w,w)}(\lambda) dZ(\lambda) = \int_{-w}^{w} e^{it\lambda} dZ(\lambda). \tag{14}$$

(In the signal processing terminology speaking the process $(\mathcal{L}X)(t)$ is actually the response of the so-called *ideal low-pass filter* $\mathcal{L}$ applied to $X(t)$). Obviously $(\mathcal{L}X)(t)$ is a WS, 1BLP. The identity filter $\mathcal{I}$ has the characteristic $h_\infty(\lambda) \equiv 1$ on the whole real axis, and $(\mathcal{I}X)(t) \equiv X(t)$ for all $t \in \mathbf{R}$.

It is clear that if $X(t)$ is defined on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$, then $(\mathcal{L}X)(t)$ is defined on the same probability space.

Let $\{w_n\}_1^\infty$ be a positive divergent monotonically increasing real sequence. Consider the sequence of filters $\{\mathcal{L}_n\}_1^\infty$. The belonging sequence of spectral characteristics is $\{h_n(\lambda) = \mathbf{1}_{(-w_n.w_n)}(\lambda)\}_1^\infty$, such that $h_n(\lambda)$ characterizes $\mathcal{L}_n$. Then we have

$$(\mathcal{L}_n X)(t) = \int_{-w_n}^{w_n} e^{\lambda} dZ(\lambda). \tag{15}$$

Therefore it is easy to show that

$$\lim_{n\to\infty} \mathbf{E}|X(t) - (\mathcal{L}_n X)(t)|^2 = 0. \tag{16}$$

The stochastic sampling theorem is now applicable to the sequence of WS, 1BLP $\{(L_n X)(t)\}_1^\infty$ elementwise. This procedure gives us the full spectrum stochastic scalar sampling theorem in the sense of the "in medio" convergence. Namely, let us take

$$X_{n\|m}(t) := (\mathcal{L}_n X(t))_m = \sum_{|k|\le m} (\mathcal{L}_n X)\left(\frac{k\pi}{\widetilde{w}_n}\right) \operatorname{sinc}(\widetilde{w}_n t - k\pi), \tag{17}$$

where $\widetilde{w}_n > w_n$ for any positive integer n. Thus

$$\lim_{m,n\to\infty} \mathbf{E}|X(t) - X_{n\|m}(t)|^2 = 0, \tag{18}$$

if

$$\lim_{m,n\to\infty} \frac{\widetilde{w}_n}{m} = 0 \qquad \lim_{n\to\infty} \frac{w_n}{\widetilde{w}_n} = \alpha < 1,$$

where it is

$$\alpha = \begin{cases} \lim_{n\to\infty} \frac{w_n}{\widetilde{w}_n} & \text{if } \{w_n/\widetilde{w}_n\} \text{ increases,} \\ \max_n \frac{w_n}{\widetilde{w}_n} & \text{if } \{w_n/\widetilde{w}_n\} \text{ decreases.} \end{cases}$$

This result is exposed in detail in [6]. In this way the sampling theorem to the class of full spectrum processes is extended.

The correlation function of the full spectrum WS process $X(t)$ has an integral representation

$$\mathcal{K}(t) = \int_{\mathbf{R}} e^{it\lambda} dF(\lambda). \tag{19}$$

We have already supposed that $\pm w_n$ are the continuity points of $F(\lambda)$. If $\mathbf{D}X(t) = \sigma^2$, the n^{th} step sampling truncation error $\mathbf{E}|X(t) - X_{n\|m}(t)|^2$ will be bounded above with

$$\sigma^2 - F(w_n) + F(-w_n) + \left(\frac{8\sigma \widetilde{w}_n(\widetilde{w}_1|t| + \pi)}{m\pi^2 \widetilde{w}_1(1-\alpha)} \right)^2, \tag{20}$$

compare also [6].

Suppose that $\mathbf{X}(t) = (X_1(t), \cdots, X_q(t))$ is a full spectrum WS process. Let $\mathbf{L} = (\mathcal{L}_1, \cdots, \mathcal{L}_q)$ be a q-dimensional filter with coordinates like (14). The filter $\mathcal{L}_j$ has the spectral characteristics $h^{(j)}(\lambda) = \mathbf{1}_{(-w^{(j)}, w^{(j)})}(\lambda),\ j \in Q$. The process $(\mathbf{L}\,\mathbf{X})(t) = ((\mathcal{L}_1 X_1)(t), \cdots, (\mathcal{L}_q X_q)(t))$ is qBLP. According to the scalar (1-dimensional) case $\mathbf{I} = (\mathcal{I}, \cdots, \mathcal{I})_{1\times q}$ is the q-dimensional identity filter with the spectral characteristics $(h_\infty(\lambda), \cdots, h_\infty(\lambda))_{1\times q}$ and $(\mathbf{I}\,\mathbf{X})(t) \equiv X(t)$.

The spectral representation of the coordinates of the q-dimensional WS process $\mathbf{X}(t)$ is $X_j(t) = \int_{\mathbf{R}} \exp(it\lambda) dZ_j(\lambda)$, and the elements/entries of the correlation matrix $\mathbf{K}(t) = (\mathcal{K}_{jk}(t))_{q\times q}$ can be represented as $\mathcal{K}_{jk}(t) = \int_{\mathbf{R}} \exp(it\lambda) dF_{jk}(\lambda)$.

Let $\left\{ w_n^{(j)} \right\}_1^\infty,\ j \in Q$ be a positive divergent monotonically increasing real sequence, such that the spectral distribution function $F_{jj}(\lambda)$ is continuous at the points $\pm w_n^{(j)},\ j \in Q$. The q-dimensional sequence of filters $\mathbf{L_N} = (\mathcal{L}_{1n_1}, \cdots, \mathcal{L}_{qn_q})$ gives us the qBLP

$$(\mathbf{L_N}\mathbf{X})(t) = ((\mathcal{L}_{1n_1} X_1)(t), \cdots, (\mathcal{L}_{qn_q} X_q)(t))$$

with following coordinates

$$(\mathcal{L}_{jn_j} X_j)(t) = \int_{-w_n^{(j)}}^{w_n^{(j)}} e^{it\lambda} dZ_j(\lambda). \tag{21}$$

Naturally the spectral characteristics $h_n^{(j)}(\lambda) = \mathbf{1}_{(-w_n^{(j)}, w_n^{(j)})}(\lambda)$ belongs to the filter $\mathcal{L}_{jn_j},\ j \in Q$.

For fixed $j \in Q$ we have

$$\lim_{n\to\infty} \mathbf{E}|X_j(t) - (\mathcal{L}_{jn_j} X_j)(t)|^2 = 0. \tag{22}$$

We can write (having in mind the notation in (17), (11) and (9) respectively):

1. $$\begin{aligned} X_{j\|n\|m}(t) =& [(\mathcal{L}_{jn_j} X_j)(t)]_m \\ =& \sum_{|p|\le m} (\mathcal{L}_{jn_j} X_j)\left(\frac{p\pi}{\widetilde{w}_n^{(j)}} \right) \operatorname{sinc}(\widetilde{w}_n^{(j)} t - p\pi); \end{aligned}$$
2. $y_{m,n}^{(j)}(\lambda) = \sum_{|p|>m} \exp\left\{ i\lambda \frac{p\pi}{\widetilde{w}_n^{(j)}} \right\} \operatorname{sinc}(\widetilde{w}_n^{(j)} - p\pi)$;
3. $\Delta_{nm}^{rs}(j,k) = \mathbf{E}(X_j(t) - X_{j\|n\|r}(t))(X_k(t) - X_{k\|m\|s}(t))^\star$,

where $\widetilde{w}_n^{(j)} > w_n^{(j)}$ for all positive integer n.

Theorem 4. *Let* $\widetilde{w}_n^{(j)} > \widetilde{w}_m^{(k)}$ *for fixed* $j,k \in Q$ *and for fixed* $n,m \geq 1$, *and assume*

$$1 > \alpha_p = \begin{cases} \lim_{r\to\infty} \frac{w_r^{(p)}}{\widetilde{w}_r^{(p)}} & \text{if } \left\{ w_r^{(p)}/\widetilde{w}_r^{(p)} \right\} \text{ increases} \\ \max_r \frac{w_r^{(p)}}{\widetilde{w}_r^{(p)}} & \text{if } \left\{ w_r^{(p)}/\widetilde{w}_r^{(p)} \right\} \text{ decreases,} \end{cases}$$

for all $p \in Q$. *Then*

$$|\Delta_{nm}^{rs}(j,k)| \leq \tag{23}$$

$$\leq \sqrt{\left[\sigma_j^2 - F_{jj}\left(\widetilde{w}_n^{(j)}\right) + F_{jj}\left(-\widetilde{w}_n^{(j)}\right)\right]\left[\sigma_k^2 - F_{kk}\left(\widetilde{w}_m^{(k)}\right) + F_{kk}\left(-\widetilde{w}_m^{(k)}\right)\right]}$$

$$+\sqrt{\sigma_k^2 - F_{kk}\left(\widetilde{w}_m^{(k)}\right) + F_{kk}\left(-\widetilde{w}_m^{(k)}\right)}\,\frac{8\sigma_j \widetilde{w}_n^{(j)}(\widetilde{w}_1^{(j)}|t|+\pi)}{r\pi^2 \widetilde{w}_1^{(j)}(1-\alpha_j)}(1-\delta_{jk})$$

$$+\frac{64\sigma_j\sigma_k \widetilde{w}_n^{(j)}(\widetilde{w}_1^{(j)}|t|+\pi)\widetilde{w}_m^{(k)}(\widetilde{w}_1^{(k)}|t|+\pi)}{rs\pi^4 \widetilde{w}_1^{(j)}\widetilde{w}_1^{(k)}(1-\alpha_j)(1-\alpha_k)},$$

where δ_{jk} *is the Kronecker symbol.*

Proof. It is not hard to see that

(24)

$$\begin{aligned} \Delta_{nm}^{rs}(j,k) &= \int_{\mathbf{R}} \left[e^{it\lambda}\left(1 - h_n^{(j)}(\lambda)\right) + y_{n,r}^{(j)}(\lambda)h_n^{(j)}(\lambda)\right] \\ &\quad \times \left[e^{-it\lambda}\left(1 - h_m^{(k)}(\lambda)\right) + y_{m,s}^{(k)}(-\lambda)h_m^{(k)}(\lambda)\right] dF_{jk}(\lambda) \\ &= \int_{\mathbf{R}} \left(1 - h_n^{(j)}(\lambda)\right)\left(1 - h_m^{(k)}(\lambda)\right) dF_{jk}(\lambda) && (:= \mathcal{J}_1) \\ &\quad + \int_{\mathbf{R}} e^{it\lambda}\left(1 - h_n^{(j)}(\lambda)\right) y_{m,s}^{(k)}(-\lambda)h_m^{(k)}(\lambda)dF_{jk}(\lambda) && (:= \mathcal{J}_2) \\ &\quad + \int_{\mathbf{R}} e^{-it\lambda}\left(1 - h_m^{(k)}(\lambda)\right) y_{n,r}^{(j)}(\lambda)h_n^{(j)}(\lambda)dF_{jk}(\lambda) && (:= \mathcal{J}_3) \\ &\quad + \int_{\mathbf{R}} h_n^{(j)}(\lambda)h_m^{(k)}(\lambda)y_{n,r}^{(j)}(\lambda)y_{m,s}^{(k)}(-\lambda)dF_{jk}(\lambda). && (:= \mathcal{J}_4) \end{aligned}$$

From the first condition of the theorem it follows that $\left(1 - h_n^{(j)}(\lambda)\right)h_m^{(k)}(\lambda)$ vanishes, therefore $\mathcal{J}_2 \equiv 0$. If $\widetilde{w}_n^{(j)} < \widetilde{w}_m^{(k)}$ then vanishes $\mathcal{J}_3$.

From the Schwarz inequality we have $|dF_{jk}(\lambda)|^2 \leq dF_{jj}(\lambda)dF_{kk}(\lambda)$ and we can easily evaluate $\mathcal{J}_1, \mathcal{J}_3$ and $\mathcal{J}_4$:

$$\begin{aligned}
(25) \qquad |\mathcal{J}_1|^2 &\leq \int_{\mathbf{R}} \left(1 - h_n^{(j)}(\lambda)\right) dF_{jj}(\lambda) \int_{\mathbf{R}} \left(1 - h_m^{(k)}(\lambda)\right) dF_{kk}(\lambda) \\
&= \left[\sigma_j^2 - F_{jj}\left(\widetilde{w}_n^{(j)}\right) + F_{jj}\left(-\widetilde{w}_n^{(j)}\right)\right] \\
&\quad \times \left[\sigma_k^2 - F_{kk}\left(\widetilde{w}_m^{(k)}\right) + F_{kk}\left(-\widetilde{w}_m^{(k)}\right)\right]
\end{aligned}$$

$$(26) \qquad |\mathcal{J}_3|^2 \leq \int_{\mathbf{R}} \left(1 - h_m^{(k)}(\lambda)\right) dF_{kk}(\lambda) \int_{\mathbf{R}} h_n^{(j)}(\lambda)|y_{n,r}^{(j)}(\lambda)|^2 dF_{jj}(\lambda)$$

The first right-side integral in (26) is equal to the second term in the squared bracketts in (25). The second right-side integral in (26) has an upper bound like (5).Namely,

$$|\mathcal{J}_3| \leq \frac{8\sigma_j \widetilde{w}_n^{(j)}(\widetilde{w}_1^{(j)}|t| + \pi)}{r\pi^2 \widetilde{w}_1^{(j)}(1 - w_n^{(j)}/\widetilde{w}_n^{(j)})} \sqrt{\sigma_k^2 - F_{kk}\left(\widetilde{w}_m^{(k)}\right) + F_{kk}\left(-\widetilde{w}_m^{(k)}\right)}\,.$$

Since $\alpha_j > w_n^{(j)}/\widetilde{w}_n^{(j)}$ it follows that

$$(27) \qquad |\mathcal{J}_3| \leq \frac{8\sigma_j \widetilde{w}_n^{(j)}(\widetilde{w}_1^{(j)}|t| + \pi)}{r\pi^2 \widetilde{w}_1^{(j)}(1 - \alpha_j)} \sqrt{\sigma_k^2 - F_{kk}\left(\widetilde{w}_m^{(k)}\right) + F_{kk}\left(-\widetilde{w}_m^{(k)}\right)}\,.$$

Similarly asin the foregoing considerations we get

$$(28) \qquad |\mathcal{J}_4| \leq \frac{64\sigma_j\sigma_k \widetilde{w}_n^{(j)}(\widetilde{w}_1^{(j)}|t| + \pi)\widetilde{w}_m^{(k)}(\widetilde{w}_1^{(k)}|t| + \pi)}{rs\pi^4 \widetilde{w}_1^{(j)}\widetilde{w}_1^{(k)}(1 - \alpha_j)(1 - \alpha_k)}.$$

From the triangle inequality we conclude $|\Delta_{nm}^{rs}(j,k)| \leq |\mathcal{J}_1| + |\mathcal{J}_3| + |\mathcal{J}_4|$. Now, from (25),(27) and (28) clearly follows the evaluation (23). For $j = k$ the integrals $\mathcal{J}_2, \mathcal{J}_3$ vanish (see the relation (24)), so the upper bound (23) reduces to the coordinatewise boundary as (20) of the process $(\mathbf{L_N X})(t)$. The proof is complete. ∎

THEOREM 5. *Let m_j be the j^{th} coordinate sample size of the process*

$$(\mathbf{L_{N,M} X})(t) = (X_{1\|n_1\|m_1}(t), \cdots, X_{q\|n_q\|m_q}(t)).$$

When the following two conditions are satified

1. $\lim_{n_j\to\infty} \frac{w_{n_j}^{(j)}}{\widetilde{w}_{n_j}^{(j)}} = \alpha_j < 1,$
2. $\lim_{n_j,m_j\to\infty} \frac{\widetilde{w}_{n_j}^{(j)}}{m_j} = 0,\ j \in Q$

then we have

$$(29) \qquad \lim_{n,m\to\infty} \mathbf{E}|\mathbf{X}(t) - (\mathbf{L_{N,M} X})(t)|^2 = 0,$$

where $n = \min_Q(n_j),\ m = \min_Q(m_j)$.

Proof. Since

$$\mathbf{E}(\mathbf{X}(t)-(\mathbf{L}_{\mathbf{N},\mathbf{M}}\mathbf{X})(t))'(\mathbf{X}(t)-(\mathbf{L}_{\mathbf{N},\mathbf{M}}\mathbf{X})(t))^\star = \left(\Delta_{n_j,n_k}^{m_j,m_k}(j,k)\right)_{q\times q}, \tag{30}$$

it is sufficient to prove that the matrix (30) tends to zero matrix of order $q\times q$ if n,m tends to infinity.

Suppose that the WS full spectrum process $X(t)$ has $\kappa\geq 0$ mean square derivatives. In other words there exists $\mathcal{K}^{(2\kappa)}(0)$ finite. Naturally, it is

$$\begin{aligned}\mathcal{K}^{(2\kappa)}(0) = \mathbf{E}|X_j^{(\kappa)}(t)|^2 &= \int_{\mathbf{R}}\lambda^{2\kappa}dF_{jj}(\lambda)\\ &= \int_{|\lambda|\geq x}\lambda^{2\kappa}dF_{jj}(\lambda)+\int_{|\lambda|<x}\lambda^{2\kappa}dF_{jj}(\lambda),\end{aligned}$$

for some real positive x. It is clear that the function

$$B\to\int_B\lambda^{2\kappa}dF_{jj}(\lambda)$$

is a finite measure on the Borel σ-field $\mathcal{B}_1$, for all Borel sets $B\in\mathcal{B}_1$. Such a function is continuous with respect to the decreasing sequences. So,

$$\lim_{x\to\infty}\int_{|\lambda|\geq x}\lambda^{2\kappa}dF_{jj}(\lambda)=0.$$

But, on the other hand

$$\int_{|\lambda|\geq x}\lambda^{2\kappa}dF_{jj}(\lambda)\geq x^{2\kappa}\int_{|\lambda|\geq x}dF_{jj}(\lambda)=x^{2\kappa}(\sigma_j^2-F_{jj}(x)+F_{jj}(-x)).$$

Taking $x=w_n^{(j)}$ it is true that

$$\sigma_j^2-F_{jj}\left(w_n^{(j)}\right)+F_{jj}\left(-w_n^{(j)}\right)=o\left(\left(w_n^{(j)}\right)^{-2\kappa}\right). \tag{31}$$

From the conditions 1,2. using (31) it is straightforward that the upper bound of $\left|\Delta_{n_j,n_k}^{m_j,m_k}(j,k)\right|$ vanishes with the n,m growing. This finishes the proof of the theorem. ∎

The convergence "in the probability" of $(\mathbf{L}_{\mathbf{N},\mathbf{M}}\mathbf{X})(t)$ to $\mathbf{X}(t)$ is a simple consequence of the mean square convergence result which is proved in the Theorem 5. It is well-known that there exists a subsequence of $(\mathbf{L}_{\mathbf{N},\mathbf{M}}\mathbf{X})(t)$, e.g. $(\mathbf{L}_{\mathbf{N}',\mathbf{M}'}\mathbf{X})(t)$ with the property:

$$\mathbf{P}\left\{\lim_{n',m'\to\infty}(\mathbf{L}_{\mathbf{N}',\mathbf{M}'}\mathbf{X})(t)=\mathbf{X}(t)\right\}=1.$$

But we cannot be satisfied with this obvious result, we will prove that $(\mathbf{L}_{\mathbf{N},\mathbf{M}}\mathbf{X})(t)$ converges almost surely to $\mathbf{X}(t)$ as well.

THEOREM 6. *There exists the convenient choice of the sequence* $\left\{w_n^{(j)}\right\}_1^\infty$, $j \in Q$ *such that*

$$\mathbf{P}\left\{\lim_{n,m\to\infty}(\mathbf{L}_{\mathbf{N},\mathbf{M}}\mathbf{X})(t) = \mathbf{X}(t)\right\} = 1.$$

PROOF. Let $X(t)$ be a WS scalar full spectrum process and we denote with $X_{n\|m}(t)$ the samling expansion of the 1BLP $(\mathcal{L}_n X)(t)$. Let us consider a.s. convergence $X_{n\|m}(t)$ to $X(t)$ as $n, m \to \infty$. As

$$\mathbf{P}\left\{\lim_{m\to\infty} X_{n\|m}(t) = X(t)\right\} = 1,$$

(band-limited case), we are interested in a.s. convergence $(\mathcal{L}_n X)(t)$ to $X(t)$ as n tends to infinity. But we have

$$\mathbf{E}|X(t) - (\mathcal{L}_n X)(t)|^2 = \sigma^2 - F(w_n) + F(-w_n).$$

In the full spectrum case, a.s. convergence depends on the choice of the sequence $\{w_n\}_1^\infty$. As $F(\cdot)$ is a finite measure on $\mathbf{R}$, the convenient choice of the sequence $\{w_n\}_1^\infty$ is possible such that

$$\sum_{n1}^{\infty}\int_{\mathbf{R}}(1 - h_n(\lambda))dF(\lambda) < \infty.$$

Finally using the Borel-Cantelli Lemma and the Chebyshev inequality we conclude that

$$\mathbf{P}\left\{\lim_{n\to\infty}(\mathcal{L}_n X)(t) = \mathbf{X}(t)\right\} = 1.$$

Now, it is not hard to show that the coordinatewise a.s. convergence in $(\mathbf{L}_{\mathbf{N},\mathbf{M}}\mathbf{X})(t)$ to the initial coordinate processes in $\mathbf{X}(t)$ is equivalent to the assertion of Theorem 6. □

4. CONCLUSIONS, FURTHER REMARKS

The main achievment of the paper is the fact that if the coordinate processes (not necessarily with full spectrum!) of a q-dimensional vectorial WS process admit a sampling extension with some coordinatwise sampling frequencies $\widetilde{w}_j$, $j \in Q$, then the process itself admits a sampling extension too. Following Lloyd's ideas Pourahmadi proved this result in different way then we did here, but under some periodicity condition on the range of the spectral measure of the initial q-dimensional process assuming the equal sampling frequencies $\widetilde{w}_j\pi/h$, $j \in Q$, compare [19].

We use the oversampling technique proposed strongly mathematically by Belyaev for the band-limited WS processes, [2]. The implementation of the oversampling with the growing sampling frequency $\widetilde{w}_j$ which is closely connected to the ideal low-pass filter $\mathcal{L}$ acting on the full spectrum process $X(t)$ is introduced for the approximative sampling reconstruction procedure in [6]. The results in this aritcle generalize in the mean square and almost sure sense the results of [6]. For the signal processing specialists it is very important to estimate the optimal sampling frequency under already known error level appearing by the finite sampling approximation procedure. This problem is solved in the papers [16], [17] where some robust aliasing and truncation error upper bound inequalities are obtained upon the optimal equal - coordinatewise sampling frequencies having on mind the convergence results of the Theorems 4,5,6. in the recent article.

At this point the first author has to give thanks to Professor A.Ya.Olenko, Mathematical Faculty, Kiev University for sending him many new articles and results about the sampling method introduced by the first author in [6], applied by Olenko himself, Professor M.I.Yadrenko and his former Ph.D. student S.I.Khalikulov, who extend the mentioned method to full spectrum homogeneous time isotropic random fields on cylinder $\mathbf{R} \times \mathbf{S}_2$ and on the sphere $\mathbf{R} \times \mathbf{S}_n$ in the papers [10], [12], [13], [14], [15]. The Thesis [11] contains some additional results on the subject. Generalizations of the results of [6] and the recent paper to the full spectrum homogeneous random fields are close to the achievments of the article [18] by Pogány & Peruničić. For all kind of these random signals the sampling extension theorem is proved in the mean square manner in approximating the full spectrum signal (process or field) with the sequence of band-limited ones. All these results are the consequences of the Belyaev - type truncation error upper bounds like (5), which allow the convergence rate evaluation such that cannot exceed $\mathcal{O}(N^{-2D})$, where the truncated sampling reconstruction sum contains $2N + 1$ addends for the recontructed D-variate signal, consult [10], [13], [15],[18]. Unfortunately the above listed references are not well-known worldwide, and most of them are not translated to English from Russian and Ukrainian.

It has to be mentioned endly that the sampling approximation of multidimensional variants of all above listed stochatic signals and additionally for all nonstationary (harmonizable processes, Piranashvili α-processes etc.) await a serious investigation in the future.

Finally, this article is in the same time devoted to the memory of my late friend, coauthor and excellent mathematician Predrag M. Peruničić.

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Received: 17.04.1989.
Revised: 23.01.1990.
Revised: 08.05.1990.
Revised: 18.08.2000.