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SOME NEW 2-(17,4,3) AND 2-(52,13,4) DESIGNS

Dean Crnković and Sanja Rukavina
University of Rijeka, Croatia

Abstract. We have constructed three hundred and seventy-seven non-isomorphic 2-(17,4,3) designs as derived and six hundred and forty-nine non-isomorphic 2-(52,13,4) designs as residual designs of symmetric (69,17,4) designs admitting an action of Z_6.

1. Introduction and preliminaries

A $2-(v,k,\lambda)$ design is a finite incidence structure $(\mathcal{P},\mathcal{B},I)$, where $\mathcal{P}$ and $\mathcal{B}$ are disjoint sets and $I \subseteq \mathcal{P}\times\mathcal{B}$, with the following properties:

1. $|\mathcal{P}| = v$,
2. every element of $\mathcal{B}$ is incident with exactly k elements of $\mathcal{P}$,
3. every pair of elements of $\mathcal{P}$ is incident with exactly λ elements of $\mathcal{B}$.

Elements of $\mathcal{B}$ are called blocks, and elements of $\mathcal{P}$ are called points. For $x \in \mathcal{B}$ and $P \in \mathcal{P}$, denote $\langle x\rangle = \{Q \in \mathcal{P} | (Q,x) \in I\}$, $\langle P\rangle = \{y \in \mathcal{B} | (P,y) \in I\}$. For two designs $\mathcal{D}_1 = (\mathcal{P}_1,\mathcal{B}_1,I_1)$ and $\mathcal{D}_2 = (\mathcal{P}_2,\mathcal{B}_2,I_2)$ an isomorphism from $\mathcal{D}_1$ onto $\mathcal{D}_2$ is a bijection which maps points onto points and blocks onto blocks preserving the incidence. If there exists an isomorphism from $\mathcal{D}_1$ onto $\mathcal{D}_2$, we shall say that $\mathcal{D}_1$ and $\mathcal{D}_2$ are isomorphic and write $\mathcal{D}_1 \cong \mathcal{D}_2$. An isomorphism from $\mathcal{D}$ onto $\mathcal{D}$ is an automorphism of $\mathcal{D}$. The set of all automorphism of the design $\mathcal{D}$ is a group called the full automorphism group of $\mathcal{D}$, and it is denoted by $Aut\mathcal{D}$. Each subgroup of the $Aut\mathcal{D}$ is an automorphism group of $\mathcal{D}$. A symmetric (v,k,λ) design is a $2-(v,k,\lambda)$ design with $|\mathcal{P}| = |\mathcal{B}| = v$.

First known symmetric (69,17,4) design was constructed by Shrikhande and Singhi (see [7]). Designs with these parameters belong to the family of symmetric designs satisfying conditions $v = m^3+m+1$, $k = m^2+1$

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and $\lambda = m$, where both $m-1$ and m^2-m+1 are prime powers (see [8]). Recently Z. Božikov (see [1]) constructed two symmetric designs with parameters (69,17,4) having Frobenius group F_{39} as an automorphism group, out of which one was new and the other was previously constructed by Shrikhande and Singhi. In august 1998., S. Topalova reported that there are four symmetric (69,17,4) designs with automorphisms of order 13 (see [2]). Finally, S. Rukavina constructed all symmetric (69,17,4) designs admitting an action of cyclic group of order six (see [6]), out of which fifty-seven designs were new and two designs admitting an action of Frobenius group F_{39} were previously known.

2. Derived and residual designs

Let $\mathcal{D}$ be a symmetric (v,k,λ) design. The derived design $\mathcal{D}_x$ with respect to a block x is the incidence structure whose points are the points of x and whose blocks are the sets $\langle y\rangle \cap \langle x\rangle$, where y is any block other than x. $\mathcal{D}_x$ is a $2-(k,\lambda,\lambda-1)$ design. The residual design $\mathcal{D}^x$ with respect to a block x is the incidence structure whose points are those points of $\mathcal{D}$ which are not incident with x and whose blocks are the sets $\langle y\rangle - \langle x\rangle$, where y ranges over all blocks other than x. It is a $2-(v-k,k-\lambda,\lambda)$ design.

Theorem 1. *Let α be an isomorphism of symmetric designs $\mathcal{D}=(\mathcal{P},\mathcal{B},I)$ and $\mathcal{D}'=(\mathcal{P}',\mathcal{B}',I')$. Further, let $x\in\mathcal{B}$ and $x\alpha = x' \in \mathcal{B}'$. Then $\alpha_1 = \alpha|_{\mathcal{D}_x}$ is an isomorphism from $\mathcal{D}_x$ onto $\mathcal{D}'_{x'}$ and $\alpha_2 = \alpha|_{\mathcal{D}^x}$ is an isomorphism from $\mathcal{D}^x$ onto $\mathcal{D}'^{x'}$.*

Proof. Let $x_1 \in \mathcal{B}$. $(\langle x_1\rangle \cap \langle x\rangle)\alpha_1 = \langle x_1\rangle\alpha_1 \cap \langle x\rangle\alpha_1 = \langle x_1\alpha_1\rangle \cap \langle x'\rangle$ and $(\langle x_1\rangle - \langle x\rangle)\alpha_2 = \langle x_1\rangle\alpha_2 - \langle x\rangle\alpha_2 = \langle x_1\alpha_2\rangle - \langle x'\rangle$, which means that α_1 maps blocks from $\mathcal{D}_x$ onto blocks from $\mathcal{D}'_{x'}$ and α_2 maps blocks from $\mathcal{D}^x$ onto blocks from $\mathcal{D}'^{x'}$. □

Corollary 1. *Let $\mathcal{D}=(\mathcal{P},\mathcal{B},I)$ be a symmetric design, $x,x'\in\mathcal{B}$ and $G \le Aut\mathcal{D}$. If $x' \in xG$, then $\mathcal{D}_x \cong \mathcal{D}_{x'}$ and $\mathcal{D}^x \cong \mathcal{D}^{x'}$.*

Proof. There exists $\alpha \in G$ such that $x' = x\alpha$. $\alpha|_{\mathcal{D}_x}$ is an isomorphism from $\mathcal{D}_x$ onto $\mathcal{D}_{x'}$ and $\alpha|_{\mathcal{D}^x}$ is an isomorphism from $\mathcal{D}^x$ onto $\mathcal{D}^{x'}$. □

Corollary 2. *Let $\mathcal{D}=(\mathcal{P},\mathcal{B},I)$ be a symmetric design, $x\in\mathcal{B}$ and $\alpha \in Aut\mathcal{D}$. If $x\alpha = x$ then $\alpha|_{\mathcal{D}_x} \in Aut\mathcal{D}_x$ and $\alpha|_{\mathcal{D}^x} \in Aut\mathcal{D}^x$.*

3. Some new 2-(17,4,3) designs

Excluding one block and all points that do not belong to that block from symmetric (69,17,4) design we obtain 2-(17,4,3) design.

In [6] fifty-nine mutually non-isomorphic symmetric (69,17,4) designs are presented. In that article, representatives of block orbits for constructed designs and corresponding automorphisms are given. From corollary 1 follows that construction of derived designs corresponding to block orbit representatives led us, after eliminating isomorphic copies, to all mutually non-isomorphic 2-(17,4,3) designs that can be obtained from given symmetric (69,17,4) design. For all designs in [6], except designs 54 and 58, representatives of block orbits of full automorphism groups of those designs are given. Designs 54 and 58 are presented in [6] by representatives of subgroups of full automorphism groups. Therefore, except in those two cases, from each (69,17,4) design we shall construct derived designs with respect to the representatives of block orbits given in [6]. On designs 54 and 58 full automorphism groups of those designs act in three block orbits. For construction of derived designs we shall consider one block from each of those three orbits. Precisely, from design 54 we shall exclude the following blocks:

0 1 2 3 4 21 22 23 24 25 26 27 28 29 30 31 32
1 5 6 7 8 21 22 23 36 37 38 51 52 53 60 61 62
0 5 9 10 11 21 22 23 33 34 35 39 40 41 45 46 47

From design 58 we shall exclude the following blocks:
1 5 6 7 8 21 22 23 36 37 38 51 52 53 57 58 59
0 1 2 3 4 21 22 23 24 25 26 27 28 29 30 31 32
0 5 9 10 11 21 22 23 33 34 35 39 40 41 45 46 47

Described process gave us six hundred and forty-nine 2-(17,4,3) designs. Using computer program by V. Krčadinac (see [3], [5]) we obtain that between them there are three hundred and seventy-seven mutually non-isomorphic designs.

For each of three hundred and seventy-seven mutually non-isomorphic 2-(17,4,3) design we present (or describe) block that should be excluded from symmetric (69,17,4) design. Ordinal numbers of symmetric (69,17,4) designs correspond to those in [6].

design 1:
excluding fixed block from (69,17,4) design 1
designs 2 - 54:
excluding block 0 5 9 10 11 21 22 23 33 34 35 39 40 41 45 46 47
from (69,17,4) designs 1 -53
designs 55 - 107:
excluding block 1 5 6 7 8 21 22 23 36 37 38 51 52 53 60 61 62
from (69,17,4) designs 1 - 53
designs 108:
excluding block 0 5 9 10 11 21 22 23 33 34 35 39 40 41 45 46 47
from (69,17,4) design 1
designs 109:

excluding block 1 5 6 7 8 21 22 23 36 37 38 51 52 53 60 61 62
from (69,17,4) design 1
designs 110 - 321:
excluding representatives of block orbits of length 12 from (69,17,4) designs 1-53
design 322:
excluding block 1 5 6 7 8 21 22 23 36 37 38 51 52 53 60 61 62
from (69,17,4) design 54
design 323:
excluding block 0 5 9 10 11 21 22 23 33 34 35 39 40 41 45 46 47
from (69,17,4) design 54
design 324:
excluding fixed block from (69,17,4) design 55
designs 325 - 336:
excluding representatives of block orbits of length 2 from (69,17,4) designs 55 - 57
design 337:
excluding block 1 2 9 27 30 33 36 39 42 45 48 51 52 54 55 57 60
from (69,17,4) design 55
design 338:
excluding block 3 4 12 21 24 39 42 46 47 49 50 53 56 57 60 63 66
from (69,17,4) design 55
designs 339 - 341:
excluding block 18 21 24 27 30 33 36 40 41 43 44 58 59 61 62 63 66
from (69,17,4) designs 55 - 57
designs 342 - 365:
excluding representatives of block orbits of length 6 from (69,17,4) designs 55 - 57
designs 366 - 368:
excluding previously presented blocks from (69,17,4) design 58
designs 369 - 377:
excluding all nine block representatives from (69,17,4) design 59

Distribution of orders of full automorphism groups of obtained 2-(17,4,3) designs is: (240,1), (4,2), (122,3), (3,4), (1,6), (2,12), (1,16), (1,32), (1,39), (1,48) and (1,78), where the first number is the number of designs and the second number is the order of an automorphism group.

Full automorphism groups of designs 110 - 321, 342 - 365 and 374 -377 are trivial and full automorphism groups of designs 337, 338, 372 and 373 are isomorphic the group Z_2. Furthermore, full automorphism groups of designs 2 - 107, 323, 325 - 336, 367, 370 and 371 are isomorphic to the group Z_3 and full automorphism groups of designs 339 - 341 are isomorphic to the group Z_4. Full automorphism group of design 324 is isomorphic to the group Z_6.

Full automorphism groups of designs 366 and 369 are isomorphic to the group Z_{12}. Full automorphism group of design 108 is non-abelian group of order 16. Generators a, b, c and d of that group satisfy conditions: $a^2 = b^2 = c^2 = d^2 = 1$, $a^b = a^c = a^d = a$, $b^c = ab$, $b^d = c$ and $c^d = b$. Full automorphism group of design 109 is non-abelian group of order 32. Generators a, b, c, d and e of that group satisfy conditions: $a^2 = b^2 = c^2 = d^2 = e^2 = 1$, $a^b = a^c = a^d = a^e = a$, $b^c = b^d = b^e = b$, $c^d = c^e = c$ and $d^e = bd$. Full automorphism group of design 322 is isomorphic to the group F_{39}. Full automorphism group of design 1 is non-abelian group of order 48. Generators a, b, c, d and e of that group satisfy conditions: $a^2 = b^2 = c^2 = d^3 = e^2 = 1$, $a^b = a^c = a^d = a^e = a$, $b^c = b^d = b$, $b^e = ab$, $c^d = c^e = c$ and $d^e = d^2$. Full automorphism group of design 368 is isomorphic to the group $Z_2 \times F_{39}$.

4. Some new 2-(52,13,4) designs

Excluding one block and all points that belong to that block from a symmetric (69,17,4) design we obtain a 2-(52,13,4) design.

From fifty-nine symmetric (69,17,4) designs presented in [6] we have constructed six hundred forty-nine residual 2-(52,13,4) designs excluding block orbit representatives from all designs as we did in previous section for derived designs. Corollary 1 imply that such a construction of residual designs led us, after eliminating isomorphic copies, to all mutually non-isomorphic 2-(52,13,4) designs that can be obtained from given symmetric (69,17,4) designs. Using computer program by V. Krčadinac (see [3], [5]) we obtain that all of six hundred and forty-nine constructed 2-(52,13,4) designs are mutually non-isomorphic.

Distribution of orders of full automorphism groups of obtained 2-(52,13,4) designs is: (240,1), (14,2), (122,3), (212,4), (3,6), (56,12), and (2,39), where the first number is the number of designs and the second number is the order of an automorphism group. Full automorphism groups are presented below. Trivial full automorphism groups:

- designs obtained from (69,17,4) designs 1 - 53 excluding block orbit representatives of block orbits of length twelfth,
- designs obtained from (69,17,4) designs 55 - 57 excluding block orbit representatives of block orbits of length six,
- designs obtained from (69,17,4) design 59 excluding block orbit representatives of block orbits of length twelfth.

Full automorphism groups isomorphic to the group Z_2:

- designs obtained from (69,17,4) designs 55 - 57 excluding block orbit representatives of block orbits of length three,
- designs obtained from (69,17,4) design 59 excluding block orbit representatives of block orbits of length six.

Full automorphism groups isomorphic to the group Z_3:

- designs obtained from (69,17,4) designs 1 - 53 excluding block orbit representatives of block orbits of length four,
- designs obtained from (69,17,4) designs 55 - 57 excluding block orbit representatives of block orbits of length two,
- designs obtained from (69,17,4) design 59 excluding block orbit representatives of block orbits of length four,
- design obtained from (69,17,4) design 54 excluding block 1 5 6 7 8 21 22 23 36 37 38 51 52 53 60 61 62,
- design obtained from (69,17,4) design 58 excluding block 1 5 6 7 8 21 22 23 36 37 38 51 52 53 57 58 59.

Full automorphism groups isomorphic to the group Z_4:

- designs obtained from (69,17,4) designs 1 - 53 excluding block orbit representatives of block orbits of length four.

Full automorphism groups isomorphic to the group Z_6:

- designs obtained from (69,17,4) designs 55 - 57 excluding fixed blocks.

Full automorphism groups isomorphic to the group Z_{12}:

- designs obtained from (69,17,4) designs 1 - 53 and 55 - 57 excluding fixed blocks,
- designs obtained from (69,17,4) designs 54 and 58 excluding block 0 5 9 10 11 21 22 23 33 34 35 39 40 41 45 46 47.

Full automorphism groups isomorphic to the group F_{39}:

- designs obtained from (69,17,4) designs 54 and 58 excluding block 0 1 2 3 4 21 22 23 24 25 26 27 28 29 30 31 32 .

REMARK 4.1. We constructed up to isomorphism all 2-(52,13,4) designs that can be obtained as residual designs of symmetric (69,17,4) designs with an automorphism of order six. However, an explicit construction of non-embeddable 2-(52,13,4) design, given by Tran van Trung (see [9]) proves the existence of 2-(52,13,4) design that cannot be obtained as residual design of any (69,17,4) design..

References

[1] Z. Božikov, Symmetric designs with parameters (69,17,4) and F_{39} as a group of automorphisms, J. Comb. Des. 6, No. 4, (1998), 231-233.

[2] J. Dinitz, New Results in Combinatorial Designs, Part 1, el. document, August 2000, http://www.emba.uvm.edu/~dinitz/newresults.part1.html

[3] V. Krčadinac, Steiner 2-designs $S\left(k, 2k^2 - 2k + 1\right)$, MSc thesis, University of Zagreb (1999).

[4] E. Lander, Symmetric Designs: An Algebraic Approach, Cambridge University Press (1983).

[5] B.D. McKay, Nauty Users Guide (version 1.5) Technical Report TR-CS-90-02, Department of Computer Science, Australian National University, 1990.

[6] S. Rukavina, Enumeration of symmetric (69,17,4) designs admitting Z_6 as an automorphism group, Glasnik matematički, 34 (54) (1999), 129-145.

[7] S. S. Shrikhande and N. M. Singhi, Construction of geometroids, Utilitas Math. 8, (1975), 187-192.

[8] Tran van Trung, Symmetric Designs, in The CRC Handbook of Combinatorial Designs, (ed. C. J. Colbourn and J. H. Dinitz), CRC press (1996), 75-87.

[9] Tran van Trung, Nonembeddable quasi-residual designs, Contemp. Math. 111, (1990), 237-278.

Department of mathematics
Faculty of philosophy in Rijeka
Omladinska 14, 51000 Rijeka, Croatia
E-mail: `deanc@mapef.pefri.hr`

Department of mathematics
Faculty of philosophy in Rijeka
Omladinska 14, 51000 Rijeka, Croatia
E-mail: `sanjar@mapef.pefri.hr`

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