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BOUNDARY VALUE PROBLEMS IN KREĬN SPACES

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Dedicated to the memory of Branko Najman.

Abstract. Three abstract boundary value problems are considered in a Kreĭn setting

1. Introduction

The abstract boundary value problems in Kreĭn spaces which we study in this note are inspired by the following example (see [2]). For $\tau \geq 0$, find a function $u(x,t)$ defined for $x \in [a,b] = \Omega$, $a < 0 < b$ and $0 \leq t \leq 1$ such that

$$(\operatorname{sgn} x)|x|^{\tau} \frac{\partial u}{\partial t}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t) \ , \tag{1.1}$$

subject to the boundary conditions

$$\begin{aligned} u(a,t) = 0, \ \ u(b,t) = 0, \quad 0 \leq t \leq 1 \ , \\ u(x,0) = g_+(x), \ x > 0, \ \ u(x,1) = g_-(x), \ x < 0 \ , \end{aligned} \tag{1.2}$$

where g_+ and g_- are given functions defined on $(0,b]$ and $[a,0)$, respectively. If we define operators T and B in the Hilbert space $L^2(\Omega)$ of functions on the interval Ω by

$$(Tf)(x) := (\operatorname{sgn} x)|x|^{\tau} f(x), \quad \text{and} \quad (Bf)(x) := -\frac{d^2 f}{dx^2}(x), \quad x \in \Omega \ ,$$

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with the natural domain of B which is determined by the boundary conditions, then the problem (1.1)-(1.2) can be written in the form

$$T\frac{du}{dt} = -Bu \ , \tag{1.3}$$

$$P_+u(0) + P_-u(1) = g \ , \tag{1.4}$$

where $u : [0,1] \to L^2(\Omega)$, P_+ and P_- are spectral projections of T corresponding to $(0,+\infty)$ and $(-\infty,0)$, respectively, and g is an arbitrary function in $L^2(\Omega)$.

Problems of the form (1.3)-(1.4) in an abstract Hilbert space $(\mathcal{H}_0, \langle\,\cdot\,,\,\cdot\,\rangle_0)$ were considered in [2] under the conditions that T is an injective, selfadjoint, bounded operator and B is a bounded (with exception of two special cases), selfadjoint, nonnegative operator with closed range. Similar equations were studied under different assumptions on T and B by several authors, see the introduction and the references in [11]. In all of these papers B is either assumed to be bounded or, if unbounded B is allowed, it is assumed that 0 is its isolated eigenvalue.

In this note we give a Kreĭn space approach to the abstract problem (1.3)-(1.4). The assumptions that B has closed range and that it is bounded are replaced by Kreĭn space regularity conditions on the operator $T^{-1}B$. A result from an earlier version of this note appeared in [8, Section 4.2, Remark 2]

Let T be an injective, selfadjoint, bounded operator in $(\mathcal{H}_0, \langle\,\cdot\,,\,\cdot\,\rangle_0)$. A convenient setting to study the problem (1.3)-(1.4) is the Hilbert space $(\mathcal{H}, \langle\,\cdot\,,\,\cdot\,\rangle)$ obtained by completing the original Hilbert space $(\mathcal{H}_0, \langle\,\cdot\,,\,\cdot\,\rangle_0)$ with respect to the norm generated by the inner product $\langle |T|\,\cdot\,,\,\cdot\rangle_0$. The inner product $\langle\,\cdot\,,\,\cdot\,\rangle$ is the continuous extension of $\langle |T|\,\cdot\,,\,\cdot\rangle_0$ onto $\mathcal{H}$. The operators $P_\pm$ and $\operatorname{sgn}(T)$ are densely defined bounded operators in $\mathcal{H}$. Therefore we can consider these operators to be defined on the entire space $\mathcal{H}$. Denote by $[\,\cdot\,,\,\cdot\,]$ the continuous extension of the indefinite inner product $\langle T\,\cdot\,,\,\cdot\rangle_0$ onto $\mathcal{H}$. Then $(\mathcal{H}, [\,\cdot\,,\,\cdot\,])$ is a Kreĭn space. The subspaces $\mathcal{K}_\pm = P_\pm\mathcal{H}$ are mutually orthogonal with respect to $[\,\cdot\,,\,\cdot\,]$, the spaces $(\mathcal{K}_\pm, \pm[\,\cdot\,,\,\cdot\,])$ are Hilbert spaces, and the sum $\mathcal{H} = \mathcal{K}_+ \dot{+} \mathcal{K}_-$ is direct. Such a decomposition of a Kreĭn space is called a fundamental decomposition, $P_\pm$ are corresponding fundamental projections and the operator $\operatorname{sgn}(T) = P_+ - P_-$ is called a fundamental symmetry. The fundamental symmetry establishes the following connection between $[\,\cdot\,,\,\cdot\,]$ and $\langle\,\cdot\,,\,\cdot\,\rangle$: $[x,y] = \langle \operatorname{sgn}(T)x, y\rangle$, $x, y \in \mathcal{H}$. If B in (1.3) is an injective, selfadjoint, positive operator in $(\mathcal{H}_0, \langle\,\cdot\,,\,\cdot\,\rangle_0)$ such that the range of T is contained in the range of B, then the operator $B^{-1}T$ is bounded on $(\mathcal{H}_0, \langle\,\cdot\,,\,\cdot\,\rangle_0)$ and positive in $(\mathcal{H}, [\,\cdot\,,\,\cdot\,])$. It follows from [1, Theorem 3.3.15] that the densely defined operator $B^{-1}T$ is bounded in $(\mathcal{H}, [\,\cdot\,,\,\cdot\,])$. Therefore, the operator $B^{-1}T$, and its densely defined inverse $T^{-1}B$, are positive essentially selfadjoint operators in the Kreĭn space $(\mathcal{H}, [\,\cdot\,,\,\cdot\,])$. Denote by A the positive selfadjoint closure of $T^{-1}B$ in $\mathcal{H}$. With this operator we can rewrite

(1.3) as

$$\frac{du}{dt} = -Au$$

and consider this equation in the Kreĭn space $(\mathcal{H}, [\,\cdot\,,\,\cdot\,])$.

The space $\mathcal{H}$ for (1.1)-(1.2) is the weighted Hilbert space $L^2(\Omega, |x|^\tau)$ and the indefinite and definite inner products, respectively, are defined by

$$[f,g] := \int_\Omega f(x)\overline{g(x)}x^\tau\,dx \quad \text{and} \quad (f,g) := \int_\Omega f(x)\overline{g(x)}|x|^\tau\,dx\ . \tag{1.5}$$

The corresponding fundamental symmetry is $(Jf)(x) := (\operatorname{sgn} x)f(x),\ x \in \Omega$. The operator A is defined by

$$(Af)(x) := -(\operatorname{sgn} x)|x|^{-\tau}\frac{d^2 f}{dx^2}(x),\ x \in \Omega\ ,$$

with the natural domain which is determined by the boundary conditions and the smoothness of f.

In this note we consider an abstract *Kreĭn space* $\mathcal{H}$, that is, a vector space with an indefinite inner product $[\,\cdot\,,\,\cdot\,] : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ for which there exist subspaces $\mathcal{K}_+$ and $\mathcal{K}_-$ of $\mathcal{H}$ which are mutually orthogonal with respect to $[\,\cdot\,,\,\cdot\,]$ and such that $(\mathcal{K}_\pm, \pm[\,\cdot\,,\,\cdot\,])$ are Hilbert spaces. We write

$$\mathcal{H} = \mathcal{K}_+ [\dot{+}] \mathcal{K}_-\ , \tag{1.6}$$

where $[\dot{+}]$ denotes the direct sum orthogonal with respect to the inner product $[\,\cdot\,,\,\cdot\,]$. The decomposition (1.6) is called a *fundamental decomposition* of a Kreĭn space $\mathcal{H}$. Let P_+ and P_- be the *fundamental projections* onto $\mathcal{K}_+$ and $\mathcal{K}_-$, respectively, induced by (1.6). Then $J_\mathcal{K} := P_+ - P_-$ is the corresponding *fundamental symmetry*, the inner product $\langle x, y\rangle := [J_\mathcal{K} x, y],\ x, y \in \mathcal{H}$, is positive definite and $(\mathcal{H}, \langle\,\cdot\,,\,\cdot\,\rangle)$ is a Hilbert space. A fundamental decomposition *reduces* an operator A in $\mathcal{H}$ if A commutes with the corresponding fundamental symmetry. Different fundamental decompositions generate different positive definite inner products, but the corresponding norms are equivalent. For this and other facts about Kreĭn spaces and their operators see [1, 13]. For the applications of the Kreĭn space operator theory to differential operators see [8], and the references quoted therein.

Let A be a closed, densely defined operator in $\mathcal{H}$. We will consider three abstract boundary value problems in the Kreĭn space $(\mathcal{H}, [\,\cdot\,,\,\cdot\,])$.

PROBLEM 1.1. *For a given $g \in \mathcal{H}$, find a differentiable function $u : [0, +\infty) \to \mathcal{H}$ satisfying*

$$u'(t) = -Au(t)\ , \tag{1.7}$$

$$P_+ u(0) = P_+ g, \quad \textit{and} \quad u(t), \quad t \geq 0, \quad \textit{is bounded}\ . \tag{1.8}$$

Problem 1.2. *For a given* $g \in \mathcal{H}$*, find a differentiable function* $u : [0, +\infty) \to \mathcal{H}$ *satisfying* (1.7) *and the boundary conditions*

$$P_+ u(0) = P_+ g, \quad \text{and} \quad \lim_{t \to +\infty} u(t) = 0 \ . \tag{1.9}$$

Problem 1.3. *For a given* $g \in \mathcal{H}$*, find a differentiable function* $u : [0, 1] \to \mathcal{H}$ *satisfying*

$$u'(t) = -Au(t) \ , \tag{1.10}$$

$$P_+ u(0) + P_- u(1) = g \ . \tag{1.11}$$

A useful tool for these problems is the theory of strongly continuous semigroups in a Hilbert space $(\mathcal{H}, \langle \cdot\,, \cdot \rangle)$. For basic definitions and properties of strongly continuous semigroups see for example [12]. For results specific to semigroups in Hilbert spaces see [17]. By $\mathbb{R}_+$ we denote the set of all nonnegative real numbers and by $\mathcal{B}(\mathcal{H})$ we denote the space of all bounded linear operators on $\mathcal{H}$. Recall that a semigroup $T : \mathbb{R}_+ \to \mathcal{B}(\mathcal{H})$ is *uniformly bounded* if $\|T(t)\| \leq M$ for some real M and all $t \in \mathbb{R}_+$. If we can choose $M = 1$, then semigroup $T : \mathbb{R}_+ \to \mathcal{B}(\mathcal{H})$, is called a *contraction semigroup*. The connection between a strongly continuous semigroup $T : \mathbb{R}_+ \to \mathcal{B}(\mathcal{H})$, and the equation (1.7) on $\mathbb{R}_+$ is given as: All the solutions of (1.7) defined on $\mathbb{R}_+$ are given by $u(t) := T(t)f, \ t \in \mathbb{R}_+, \ f \in \mathcal{D}(A)$, if and only if the operator $-A$ is the generator of the strongly continuous semigroup T. An operator $-A$ is the generator of a strongly continuous contraction semigroup if and only if $\mathrm{Re}\langle Ax, x \rangle \geq 0$ for every $x \in \mathcal{D}(A)$, and each negative number is in the resolvent set of A. An operator A with the preceding property is said to be *maximal accretive operator in the Hilbert space* $(\mathcal{H}, \langle \cdot\,, \cdot \rangle)$. Let $(\mathcal{H}, [\,\cdot\,, \cdot\,])$ be a Kreĭn space with a fundamental symmetry $J_{\mathcal{K}}$. An operator A is *maximal accretive in the Kreĭn space* $(\mathcal{H}, [\,\cdot\,, \cdot\,])$ if the operator $J_{\mathcal{K}} A$ is maximal accretive in the Hilbert space $(\mathcal{H}, [J_{\mathcal{K}} \cdot\,, \cdot\,])$. It follows from [1, Chapter II,§2] that this definition does not depend on the choice of a fundamental symmetry $J_{\mathcal{K}}$. Note that A is maximal accretive if and only if iA, where $i = \sqrt{-1}$, is maximal dissipative, and all the results about maximal dissipative operators carry over to maximal accretive operators.

2. Preliminaries

Let $\mathcal{H} = \mathcal{K}_+ [\dot{+}] \mathcal{K}_-$ and $\mathcal{H} = \mathcal{L}_+ [\dot{+}] \mathcal{L}_-$ be two fundamental decompositions of $\mathcal{H}$. Denote by $J_{\mathcal{K}}$ and $J_{\mathcal{L}}$ the corresponding fundamental symmetries and by

$$P_\pm = \frac{1}{2}(I \pm J_{\mathcal{K}}) \quad \text{and} \quad Q_\pm = \frac{1}{2}(I \pm J_{\mathcal{L}})$$

the corresponding projections onto $\mathcal{K}_\pm$ and $\mathcal{L}_\pm$, respectively. In the rest of the note our notation will follow the pattern established in the previous sentence.

Let $\langle f|g\rangle_{\mathcal{K}} := [J_{\mathcal{K}} f|g]$ and $\langle f|g\rangle_{\mathcal{L}} := [J_{\mathcal{L}} f|g]$, be the corresponding Hilbert space inner products. Since

$$\langle J_{\mathcal{K}} J_{\mathcal{L}} f|g\rangle_{\mathcal{K}} = [J_{\mathcal{L}} f|g] = \langle f|g\rangle_{\mathcal{L}}$$

the operator $J_{\mathcal{K}} J_{\mathcal{L}}$ is a uniformly positive bounded operator in the Hilbert space $(\mathcal{H}, \langle \cdot\,, \cdot\rangle_{\mathcal{K}})$. Analogously, the operator $J_{\mathcal{L}} J_{\mathcal{K}}$ is a uniformly positive operator in the Hilbert space $(\mathcal{H}, \langle \cdot\,, \cdot\rangle_{\mathcal{L}})$. Since these two operators are inverses of each other, each one is uniformly positive in each Hilbert space $(\mathcal{H}, \langle \cdot\,, \cdot\rangle_{\mathcal{K}})$ and $(\mathcal{H}, \langle \cdot\,, \cdot\rangle_{\mathcal{L}})$. Put

$$V := P_+ Q_+ + P_- Q_- = \frac{1}{2}(I + J_{\mathcal{K}} J_{\mathcal{L}}) \ . \tag{2.12}$$

Lemma 2.1. *The operator V is a uniformly positive bounded operator in both Hilbert spaces $(\mathcal{H}, \langle \cdot\,, \cdot\rangle_{\mathcal{K}})$ and $(\mathcal{H}, \langle \cdot\,, \cdot\rangle_{\mathcal{L}})$. The operator $I - V^{-1}$ is a selfadjoint strict contraction in the Hilbert space $(\mathcal{H}, \langle \cdot\,, \cdot\rangle_{\mathcal{L}})$, i.e.,*

$$\|I - V^{-1}\|_{\mathcal{L}} < 1.$$

Proof. The first claim follows from (2.12) and the previously proved properties of $J_{\mathcal{K}} J_{\mathcal{L}}$. It follows that the spectrum of V is contained in the interval $(1/2, \|V\|]$. Consequently, the spectrum of the operator $I - V^{-1}$ is contained in $(-1, 1 - \|V\|^{-1}]$. The second statement follows from this and the fact that $I - V^{-1}$ is a selfadjoint operator in $(\mathcal{H}, \langle \cdot\,, \cdot\rangle_{\mathcal{L}})$. ◻

It follows from (2.12) that $2V - I = J_{\mathcal{K}} J_{\mathcal{L}}$. Therefore, $2V - I$ is bounded and boundedly invertible. Next we generalize this fact.

Lemma 2.2. *Let V be as defined in (2.12) and let K be a contraction in the Hilbert space $(\mathcal{H}, \langle \cdot\,, \cdot\rangle_{\mathcal{L}})$, i.e., $\|K\| \le 1$. Then $V - (V - I)K$ is a bounded and boundedly invertible operator.*

Proof. Since V and K are bounded operators, the operator $V - (V - I)K$ is bounded. By Lemma 2.1 the operator $I - V^{-1}$ is a strict contraction. As K is a contraction, the operator $(I - V^{-1})K$ is a strict contraction. Therefore $I - (I - V^{-1})K$ has a bounded inverse. Since V also has a bounded inverse, the equality

$$V - (V - I)K = V(I - (I - V^{-1})K)$$

yields that $V - (V - I)K$ has a bounded inverse. The lemma is proved. ◻

The following three statements collect few simple properties of bounded semigroups in Banach spaces.

Lemma 2.3. *Let A be an operator in a Banach space $(\mathcal{H}, \|\cdot\|)$ which is the generator of a strongly continuous semigroup $T : \mathbb{R}_+ \to \mathcal{B}(\mathcal{H})$ and let $u : \mathbb{R}_+ \to \mathcal{H}$ be a solution of the initial value problem*

$$u'(t) = -Au(t), \quad t \in \mathbb{R}_+,\ u(0) = f, \tag{2.13}$$

with $f \in \mathcal{D}(A)$. Then $T(t)u(t) = f$ for all $t \in \mathbb{R}_+$.

Proof. It follows from the Uniform Boundedness Theorem and the Triangle Inequality that if a function $F : [0,\gamma) \to \mathcal{B}(\mathcal{H})$ has the strong limit $F_0 := s - \lim_{t\downarrow 0} F(t)$ and a function $\phi : [0,\gamma) \to \mathcal{H}$ has the limit $\phi_0 := \lim_{t\downarrow 0} \phi(t)$, then $\lim_{t\downarrow 0} F(t)\phi(t) = F_0\phi_0$. A consequence of this fact is that the function $g(t) := T(t)u(t),\ t \in \mathbb{R}_+$ is differentiable. Indeed, let $t \in \mathbb{R}_+$ and $h \in \mathbb{R}$ be such that $t+h \in \mathbb{R}_+$. Then

$$\begin{aligned}\frac{g(t+h)-g(t)}{h} &= \frac{1}{h}\Big(T(t+h)u(t+h) - T(t)u(t)\Big)\\ &= T(t)\frac{1}{h}\Big(T(h)u(t+h) - T(h)u(t) + T(h)u(t) - u(t)\Big)\\ &= T(t)T(h)\frac{u(t+h)-u(t)}{h} + T(t)\frac{T(h)u(t)-u(t)}{h}\ .\end{aligned}$$

Since

$$\lim_{h\to 0} T(h)\frac{u(t+h)-u(t)}{h} = u'(t) \quad \text{and} \quad \lim_{h\to 0} \frac{T(h)u(t)-u(t)}{h} = Au(t)\ ,$$

it follows that

$$\begin{aligned}\lim_{h\to 0} \frac{g(t+h)-g(t)}{h} &= T(t)u'(t) + T(t)Au(t)\\ &= -T(t)Au(t) + T(t)Au(t) = 0\ .\end{aligned} \tag{2.14}$$

Thus the function $g : \mathbb{R}_+ \to \mathcal{H}$ is constant, that is, $g(t) = g(0) = f,\ t \in \mathbb{R}_+$. This completes the proof of the lemma. ∎

In the next two corollaries we assume that A satisfies the assumptions of Lemma 2.3.

Corollary 2.4. *Assume that at least one of the operators $T(t),\ t \in \mathbb{R}_+$, is a strict contraction. Then all the solutions of* (2.13) *with* $f \neq 0$ *are unbounded.*

Proof. Let $s > 0$ be such that $\|T(s)\| = r < 1$. Then

$$0 < \|f\| = \|g(ns)\| \leq \|T(ns)\|\|u(ns)\| \leq r^n\|u(ns)\|, \quad n \in \mathbb{N}\ . \tag{2.15}$$

Since $r < 1$, the inequality (2.15) implies that the function $t \mapsto u(t),\ t \in \mathbb{R}_+$, is unbounded. ∎

Corollary 2.5. *Assume that all the operators $T(t),\ t \in \mathbb{R}_+$, are injective. Then the initial value problem* (2.13) *has a unique solution.*

Proof. Lemma 2.3 implies that $f = T(t)u(t),\ t \in \mathbb{R}_+$. Injectivity of $T(t),\ t \in \mathbb{R}_+$, yields $u(t) = T(t)^{-1}f,\ t \in \mathbb{R}_+$. ∎

Remark 2.6. Let $-A$ be a normal accretive operator in a Hilbert space $(\mathcal{H}, \langle\,\cdot\,,\,\cdot\,\rangle)$. Then $-A$ is maximal accretive and the operator A is the generator of a strongly continuous contraction semigroup $T : \mathbb{R}_+ \to \mathcal{B}(\mathcal{H})$. It follows from the Spectral Theorem for normal operators that this semigroup satisfies the assumption of Corollary 2.5.

PROPOSITION 2.7. *Let A be a maximal accretive operator in a Hilbert space $(\mathcal{H}, \langle\,\cdot\,,\,\cdot\,\rangle)$ such that the intersection of the spectrum of A and the imaginary axes is countable and contains no eigenvalues. Then:*

(a) *All the solutions of the initial value problem*

$$u'(t) = -Au(t), \quad t \in \mathbb{R}_+, \quad u(0) = f, \quad f \in \mathcal{D}(A)\ , \tag{2.16}$$

have the property

$$\lim_{t\to+\infty} u(t) = 0\ .$$

(b) *All the solutions of the initial value problem*

$$u'(t) = Au(t), \quad t \in \mathbb{R}_+, \quad u(0) = f \neq 0, \quad f \in \mathcal{D}(A)\ , \tag{2.17}$$

are unbounded.

PROOF. This proof uses the terminology of and results from [17]. Let $U = (A-I)(A+I)^{-1}$ be the Cayley transform of A. Then U is a contraction defined on $\mathcal{H}$. The function $z \mapsto (z-1)/(z+1)$, $z \in \mathbb{C} \setminus \{-1\}$ maps the spectrum of A onto $\sigma(U) \setminus \{1\}$. The number 1 is not an eigenvalue of U. Thus, the intersection of the spectrum of U and the unit circle in $\mathbb{C}$ contains no eigenvalues and it is a countable set. By [17, Theorem I.3.2] there exists an orthogonal decomposition $\mathcal{H} = \mathcal{H}_0 \langle\dot{+}\rangle \mathcal{H}_1$ which reduces U, such that $U|\mathcal{H}_0$ is unitary on $\mathcal{H}_0$, and $U|\mathcal{H}_1$ is completely non-unitary on $\mathcal{H}_1$. The above description of the intersection of the spectrum of U and the unit circle in $\mathbb{C}$ implies that the spectrum of the unitary operator $U|\mathcal{H}_0$ is at most countable and contains no eigenvalues. Since each countable closed set in $\mathbb{C}$ has an isolated point, this is possible only if $\mathcal{H}_0 = \{0\}$. Thus, U is a completely non-unitary contraction. Let $T : \mathbb{R}_+ \to \mathcal{B}(\mathcal{H})$ be a strongly continuous semigroup whose generator is $-A$. The cogenerator of T is U. Since the intersection of the spectrum of U and the unit circle is countable, [17, Proposition II.6.7 and Proposition III.9.1] imply that

$$\lim_{t\to+\infty} T(t)f = 0 \ \text{ and } \ \lim_{t\to+\infty} T(t)^* f = 0 \ \text{ for all } \ f \in \mathcal{H}\ . \tag{2.18}$$

Here * denotes the adjoint in $(\mathcal{H}, \langle\,\cdot\,,\,\cdot\,\rangle)$. This proves (a).

Let $u : \mathbb{R}_+ \to \mathcal{D}(A)$ be a solution of (2.17). By Lemma 2.3 we have $T(t)u(t) = u(0) = f$ for all $t \in \mathbb{R}_+$. Assuming that $\|u(t)\| \leq M$ for all $t \in \mathbb{R}_+$, we get

$$\begin{aligned} 0 \leq |\langle f, f\rangle| &= \lim_{t\to+\infty} |\langle T(t)u(t), f\rangle| = \lim_{t\to+\infty} |\langle u(t), T(t)^* f\rangle| \\ &\leq M \lim_{t\to+\infty} \|T(t)^* f\| = 0\ . \end{aligned}$$

This proves (b). ∎

Remark 2.8. If A is an arbitrary maximal accretive operator in a Hilbert space $(\mathcal{H}, \langle\,\cdot\,,\,\cdot\,\rangle)$, it follows from accretivity that for each solution of the initial value problem

$$u'(t) = Au(t), \quad t \in \mathbb{R}_+, \quad u(0) = f, \quad f \in \mathcal{D}(A)\ ,$$

the derivative of the function $t \mapsto \langle u(t), u(t)\rangle$, $t \in \mathbb{R}_+$, is nonnegative and therefore this function is nondecreasing, but it could be bounded.

3. Solutions of boundary value problems

In this section we use the notation introduced in Section 2 and we give solutions to the problems stated in Section 1. If B is the generator of a continuous semigroup in a Hilbert space $(\mathcal{H}, \langle\,\cdot\,,\,\cdot\,\rangle)$, then the operator values in $\mathcal{B}(\mathcal{H})$ of this unique semigroup are denoted by e^{tB}, $t \in \mathbb{R}_+$.

Theorem 3.1. *Let A be a maximal accretive operator in a Kreĭn space $(\mathcal{H}, [\,\cdot\,,\,\cdot\,])$. Assume that there exists a fundamental symmetry $J_{\mathcal{L}} = Q_+ - Q_-$ in $(\mathcal{H}, [\,\cdot\,,\,\cdot\,])$ which commutes with A. Let $\mathcal{H} = \mathcal{L}_+[\dot{+}]\mathcal{L}_-$ be the corresponding fundamental decomposition. Assume that the intersection of the spectrum of $A|\mathcal{L}_-$ and the imaginary axes is countable and that it contains no eigenvalues. Then* Problem 1.1 *has a solution for each $g \in \mathcal{H}$, such that $P_+g \in V\mathcal{D}(A)$. All such solutions are given by*

$$u(t) = e^{-tAQ_+}Q_+V^{-1}P_+g + v_-(t), \quad t \in \mathbb{R}_+\ , \tag{3.19}$$

where $v_- : \mathbb{R}_+ \to \mathcal{L}_-$ is a bounded solution of the initial value problem

$$v'(t) = -\big(A|\mathcal{L}_-\big)v(t), \quad t \in \mathbb{R}_+, \quad v(0) = 0\ . \tag{3.20}$$

The solution (3.19) *is unique if the problem* (3.20) *has a unique solution in $(\mathcal{L}_-, -[\,\cdot\,,\,\cdot\,])$.*

Proof. Since A is maximal accretive in $(\mathcal{H}, [\,\cdot\,,\,\cdot\,])$ and since A commutes with $J_{\mathcal{L}}$, the operator $A|\mathcal{L}_+$ is a maximal accretive in the Hilbert space $(\mathcal{L}_+, [\,\cdot\,,\,\cdot\,])$. Therefore the function in (3.19) is a solution of Problem 1.1.

Let $u : \mathbb{R}_+ \to \mathcal{H}$ be a solution of Problem 1.1. Then the function Q_+u is a solution of the initial value problem

$$v'(t) = -\big(A|\mathcal{L}_+\big)v(t), \quad t \in \mathbb{R}_+, \quad v(0) = Q_+u(0)\ .$$

As the operator $A|\mathcal{L}_+$ is maximal accretive in the Hilbert space $(\mathcal{L}_+, [\,\cdot\,,\,\cdot\,])$, $-A|\mathcal{L}_+$ is the generator of a contraction semigroup. Therefore, $Q_+u(t) = e^{-tAQ_+}Q_+u(0)$, $t \in \mathbb{R}_+$. Similarly, the function Q_-u is a bounded solution of the initial value problem

$$v'(t) = -\big(A|\mathcal{L}_-\big)v(t), \quad t \in \mathbb{R}_+, \quad v(0) = Q_-u(0)\ .$$

The operator $-A|\mathcal{L}_-$ is maximal accretive in the Hilbert space $(\mathcal{L}_-, -[\,\cdot\,,\,\cdot\,])$, and it satisfies the assumptions of Proposition 2.7. Since the function Q_-u is bounded, it follows that $Q_-u(0) = 0$. Consequently the general bounded solution of (1.7) is given by $u(t) = e^{-tAQ_+}Q_+f + v_-(t)$, $t \in \mathbb{R}_+$, where $Q_+f \in \mathcal{D}(A)$ and v_- is a bounded solution of (3.20). In order to match the first condition in (1.8) we need $P_+Q_+f = P_+g$, or, equivalently, $VQ_+f = P_+g$. By Lemma 2.1 V is invertible. Therefore each solution of Problem 1.1 is given by (3.19). The theorem is proved. □

Remark 3.2. Sufficient conditions for the uniqueness of the solution of (3.20) are given in Corollary 2.5 and Remark 2.6. Other sufficient conditions can, for example, be found in [18, Chapter 3].

Theorem 3.3. *Let A be a maximal accretive operator in a Kreĭn space $(\mathcal{H}, [\,\cdot\,,\,\cdot\,])$. Assume that there exists a fundamental symmetry $J_{\mathcal{L}} = Q_+ - Q_-$ in $(\mathcal{H}, [\,\cdot\,,\,\cdot\,])$ which commutes with A. Assume that the intersection of the spectrum of A and the imaginary axes is countable and that it contains no eigenvalues. Then* Problem 1.2 *has a unique solution for each $g \in \mathcal{H}$, such that $P_+g \in V\mathcal{D}(A)$. This solution is given by*

$$u(t) = e^{-tAQ_+}Q_+V^{-1}P_+g, \quad t \in \mathbb{R}_+ \ . \tag{3.21}$$

Proof. As before, the operator $A|\mathcal{L}_+$ is maximal accretive in the Hilbert space $(\mathcal{L}_+, [\,\cdot\,,\,\cdot\,])$. Since the intersection of the spectrum of $A|\mathcal{L}_+$ with the imaginary axes is countable, Proposition 2.7(a) implies that

$$\lim_{t\to+\infty} e^{-tAQ_+}Q_+V^{-1}P_+g = 0 \ . \tag{3.22}$$

Hence, the function in (3.21) is a solution of Problem 1.2. Let $u : \mathbb{R}_+ \to \mathcal{H}$ be an arbitrary solution of Problem 1.2. By Theorem 3.1, u is given by (3.19). The assumption that $\lim_{t\to+\infty} u(t) = 0$ and (3.22) imply that $\lim_{t\to+\infty} v_-(t) = 0$. On the other hand, the function v_- is a solution of (3.20) with the operator $-A|\mathcal{L}_-$ being maximal accretive in the Hilbert space $(\mathcal{L}_-, -[\,\cdot\,,\,\cdot\,])$. By Remark 2.8 the function $t \mapsto -[v_-(t), v_-(t)]$, $t \in \mathbb{R}_+$, is nondecreasing and has value 0 at $t = 0$. Hence, $v_-(t) = 0$ for all $t \in \mathbb{R}_+$. Thus, (3.21) gives all solutions of Problem 1.2. □

Theorem 3.4. *Let A be a maximal accretive operator in a Kreĭn space $(\mathcal{H}, [\,\cdot\,,\,\cdot\,])$. Assume that there exists a fundamental symmetry $J_{\mathcal{L}} = Q_+ - Q_-$ in $(\mathcal{H}, [\,\cdot\,,\,\cdot\,])$ which commutes with A. Let*

$$W = V - (V - I)(e^{-AQ_+}Q_+ + e^{AQ_-}Q_-) \ . \tag{3.23}$$

Then Problem 1.3 *has a unique solution for each $g \in W\mathcal{D}(A)$. This solution is given by*

$$u(t) = \left(e^{-tAQ_+}Q_+ + e^{(1-t)AQ_-}Q_-\right)W^{-1}g \ , \quad 0 \le t \le 1 \ . \tag{3.24}$$

Proof. Let $Q_\pm\mathcal{H} = \mathcal{L}_\pm$. The operator $A|\mathcal{L}_+$ is maximal accretive in the Hilbert space $(\mathcal{L}_+, [\,\cdot\,,\,\cdot\,])$ and the operator $-A|\mathcal{L}_-$ is maximal accretive in the Hilbert space $(\mathcal{L}_-, -[\,\cdot\,,\,\cdot\,])$. It follows from the basic properties of the semigroups generated by the operators $-A|\mathcal{L}_+$ and $A|\mathcal{L}_-$ in the corresponding Hilbert spaces that (3.24) is a solution of Problem 1.3.

Let $u : [0,1] \to \mathcal{H}$ be an arbitrary solution of Problem 1.3. Then the function Q_+u is also a solution of

$$v'(t) = -\big(A|\mathcal{L}_+\big)v(t), \quad 0 \leq t \leq 1 \ .$$

Since the operator $-A|\mathcal{L}_+$ is the generator of a contraction semigroup in $(\mathcal{L}_+, [\,\cdot\,,\,\cdot\,])$ we have

$$Q_+u(t) = e^{-tAQ_+}Q_+u(0), \ 0 \leq t \leq 1 \ .$$

Similarly, the function Q_-u is a solution of

$$v'(t) = -\big(A|\mathcal{L}_-\big)v(t), \quad 0 \leq t \leq 1 \ . \tag{3.25}$$

and the operator $A|\mathcal{L}_-$ is the generator of a contraction semigroup in the Hilbert space $(\mathcal{L}_-, -[\,\cdot\,,\,\cdot\,])$. A change of variable in (3.25) yields

$$Q_-u(t) = e^{(1-t)AQ_-}Q_-u(1), \quad 0 \leq t \leq 1 \ .$$

Consequently, with $f = Q_+u(0) + Q_-u(1) \in \mathcal{D}(A)$,

$$u(t) = \Big(e^{-tAQ_+}Q_+ + e^{(1-t)AQ_-}Q_-\Big)\, f, \quad 0 \leq t \leq 1 \ .$$

Note that

$$u(0) = \big(Q_+ + e^{AQ_-}Q_-\big)\, f \quad \text{and} \quad u(1) = \big(e^{-AQ_+}Q_+ + Q_-\big)\, f \ .$$

In order to match the condition (1.11) we need $P_+\left(Q_+ + e^{AQ_-}Q_-\right) f = P_+g$ and $P_-\left(e^{-AQ_+}Q_+ + Q_-\right) f = P_-g$. Therefore

$$\Big(V + P_+Q_-e^{AQ_-}Q_- + P_-Q_+e^{-AQ_+}Q_+\Big)f = g$$

Noting that $P_+Q_- + P_-Q_+ = I - V$ we get

$$\Big(V - (V-I)\left(e^{AQ_-}Q_- + e^{-AQ_+}Q_+\right)\Big)f = g \ . \tag{3.26}$$

It follows from the theorem's assumptions that $e^{AQ_-}Q_- + e^{-AQ_+}Q_+$ is a contraction in the Hilbert space $(\mathcal{H}, \langle\,\cdot\,,\,\cdot\,\rangle_{\mathcal{L}})$. Thus, Lemma 2.2 implies that the operator

$$W = V - (V-I)\left(e^{AQ_-}Q_- + e^{-AQ_+}Q_+\right)$$

has a bounded inverse. Therefore Problem 1.3 has a unique solution for all $g \in W\mathcal{D}(A)$ and that solution is given by (3.24). ∎

Note that W defined by (3.23) has the property $W(\mathcal{L}_\pm) = \mathcal{K}_\pm$.

4. Remarks and examples

In this section we give classes of differential operators satisfying the assumptions of the theorems in Section 3. All differential operators in these examples have nonempty resolvent sets and are nonnegative in their Kreĭn spaces.

REMARK 4.1. The spectrum of a nonnegative operator A with the nonempty resolvent set in a Kreĭn space $(\mathcal{H}, [\,\cdot\,,\,\cdot\,])$ is real. Such an operator has a projector valued spectral function analogous to the spectral function for selfadjoint operators in a Hilbert space; the only exception being that this spectral function might be unbounded in a neighborhoods of 0 and ∞. If the spectral function is unbounded in neighborhood of a point (0 or ∞), that point is said to be a *singular critical point* of A. If neither 0 nor ∞ is a singular critical point of A and if $\ker(A^2) = \ker(A)$, then the spectral function can be used for the construction of a fundamental symmetry which commutes with A. Thus, such an operator A satisfies the assumptions of Theorem 3.4. If, in addition, $\ker(A) \subset \mathcal{L}_+$, then A satisfies the assumptions of Theorem 3.1, and if $\ker(A) = \{0\}$, A satisfies the assumptions of Theorem 3.3. For more details about the spectral theory of nonnegative operators in Kreĭn spaces see [13, 1].

REMARK 4.2. If $A = zS$, where $z \in \mathbb{C}$, $\operatorname{Re} z \geq 0$, and S is a positive operator in a Kreĭn space with nonempty resolvent set and such that 0 and ∞ are not singular critical points of S, then A satisfies the assumptions of Theorem 3.4.

REMARK 4.3. The following perturbation result is proved in [11] using a perturbation theorem for bisemigroups: Let A be a uniformly positive operator in a Krein space $(\mathcal{H}, [\,\cdot\,,\,\cdot\,])$ such that ∞ is not a singular critical point of A. Let S be an accretive operator in $(\mathcal{H}, [\,\cdot\,,\,\cdot\,])$ such that $A^{-1}S$ is a trace class operator in $\mathcal{H}$. Let A_1 be the closure of the operator $A + S$. Assume that $\ker(A_1) = \ker(A_1 + A_1^{+}) = \{0\}$. Then A_1 is a maximal accretive operator in $(\mathcal{H}, [\,\cdot\,,\,\cdot\,])$ and there exists a fundamental symmetry in $\mathcal{H}$ which commutes with A_1.

Next we give several examples of specific operators T and B in (1.3)-(1.4) for which the corresponding operator A satisfies the assumptions of theorems in Section 3.

EXAMPLE 4.4. Let $\Omega \subseteq \mathbb{R}$ be a bounded or unbounded interval. Let $w : \Omega \to \mathbb{R}$ be a locally integrable function which changes sign on Ω. Assume that w has only finitely many turning points at which it satisfies the "turning point condition" of Beals, see [3, Definition 3.1], and [9, Chapter 3]. For more general conditions on w near its turning points see [16]. Let $\mathcal{H}$ be a weighted Hilbert space $L^2(\Omega, |w|)$ and let $[f, g] = \int_\Omega f\,\overline{g}\,w$. Let B be a uniformly positive selfadjoint differential operator in $\mathcal{H}_0 = L^2(\Omega)$ associated with a quasi-differential expression of even order

$$\ell(f) := (-1)^m (p_0 f^{(m)})^{(m)} + (-1)^{m-1} (p_1 f^{(m-1)})^{(m-1)} + \ldots + p_m f \tag{4.27}$$

defined on Ω. Assume that the coefficients $p_0, p_1, \ldots, p_m$ are real measurable functions and we assume that the functions $\dfrac{1}{p_0}, p_1, \ldots, p_m$ are locally integrable over Ω, see [14, Chapter V]. Note that the uniform positivity of B imposes additional restrictions on the coefficients. Let T be the operator of multiplication by the function w. Then A is a uniformly positive operator in the Kreĭn space $(\mathcal{H}, [\,\cdot\,,\,\cdot\,])$ and ∞ is not a singular critical point of A, see [3] and [9] for the case $m = 1$.

EXAMPLE 4.5. Let $\Omega \subseteq \mathbb{R}^n$ and $w : \Omega \to \mathbb{R}$ be a measurable function which changes sign in Ω and such that $0 < c \le |w| \le C$ for some numbers c and C. Assume that the sets

$$\Omega^+ = \{x \in \mathbb{R}^n : w(x) > 0\} \quad \text{and} \quad \Omega^- = \{x \in \mathbb{R}^n : w(x) < 0\}$$

are unions of finitely many domains with sufficiently smooth boundaries. Let T be the bounded and boundedly invertible operator of multiplication by the function w. Let $\mathcal{H}_0 = \mathcal{H} = L^2(\mathbb{R}^n)$, let $[f,g] = \int f\,\overline{g}\,w$, and $B = -\Delta + 1$. The operator A in this case is $A = \frac{1}{w}(-\Delta + 1)$, with domain $\mathcal{D}(A) = H^2(\mathbb{R}^n)$. The operator A is a uniformly positive operator in the Kreĭn space $(L^2(\Omega), [\,\cdot\,,\,\cdot\,])$ and ∞ is not its singular critical point, see [4].

The operator $-\Delta + 1$ may be replaced by a symmetric elliptic operator L of order $2m$ defined on a different domain $\Omega \subset \mathbb{R}^n$. In this case the Dirichlet form of L needs to be defined on a closed subspace of $H^m(\Omega)$ specified by boundary conditions in the usual way and this Dirichlet form must be uniformly positive on its domain. For more details see [4, 15].

EXAMPLE 4.6. Let $\Omega = \mathbb{R}$. Let $w(x) = (\operatorname{sgn} x)|x|^\tau, x \in \mathbb{R}$, where $\tau > -1$. Let $\mathcal{H}$ be a weighted Hilbert space $L^2(\Omega, |x|^\tau)$, and let

$$B = -\frac{d^2}{dx^2}\ .$$

Then B is a positive differential operator in $\mathcal{H}_0 = L^2(\mathbb{R})$ and the range of B is dense in $L^2(\mathbb{R})$. Let T be the operator of multiplication by the function w. Let

$$(Af)(x) := -(\operatorname{sgn} x)|x|^{-\tau} \frac{d^2 f}{dx^2}(x), x \in \mathbb{R}\ .$$

It was proved in [10, Theorem 2.7] (see also [5]), that 0 and ∞ are not singular critical points of A. For more general operators B see [6].

EXAMPLE 4.7. Let $\Omega = \mathbb{R}^n$ and $w(x) = \operatorname{sgn}(x_n), x \in \mathbb{R}^n$, where x_n is the n-th component of $x \in \mathbb{R}^n$. Let $\mathcal{H}_0 = \mathcal{H} = L^2(\mathbb{R}^n)$ and let $[f,g] = \int f\,\overline{g}\,w$. Let $B = -\Delta$. The operator A in this case is $A = -(\operatorname{sgn} x_n)\Delta$, with domain $\mathcal{D}(A) = H^2(\mathbb{R}^n)$. The operator A is a positive selfadjoint operator in the

Kreĭn space $(L^2(\mathbb{R}^n), [\,\cdot\,,\,\cdot\,])$ and by [7, Theorem 4.6(b)] points 0 and ∞ are not singular critical points of A. In fact [7, Theorem 4.6(b)] relates to more general positive symmetric partial differential operators with constant coefficients.

Example 4.8. Let $\Omega = [-1, 1]$ and $w(x) = \operatorname{sgn}(x), x \in \Omega$. Let $\mathcal{H}_0 = \mathcal{H} = L^2(\Omega)$ and let $[f, g] = \int_\Omega f\,\overline{g}\,w$. Denote by $AC(\Omega)$ the set of all absolutely continuous functions on Ω. Let $(Af)(x) := -(\operatorname{sgn} x)f'(x), x \in \Omega$, with the domain

$$\mathcal{D}(A) = \{f \in \mathcal{H} : f \in AC(\Omega),\ f(-1) = f(1)\}\ .$$

The operator A is antiselfadjoint in the Kreĭn space $(L^2(\Omega), [\,\cdot\,,\,\cdot\,])$. Consider Problem 1.3 for this operator. Let $G : [0, 2] \to \mathbb{C}$ be an absolutely continuous function. The general solution of the equation (1.10) in this case is given by

$$u(t) = \begin{cases} G(t-x), & -1 \le x \le 0, \\ G(t+x), & 0 \le x \le 1\ . \end{cases} \tag{4.28}$$

Clearly $u(t) \in \mathcal{D}(A)$ for all $0 \le t \le 1$. Therefore, Problem 1.3 has a solution if and only if the function

$$x \mapsto \begin{cases} g(x), & -1 \le x < 0, \\ g(1-x), & 0 \le x \le 1\ . \end{cases} \tag{4.29}$$

is absolutely continuous on Ω. If the condition (4.29) is satisfied, the solution of Problem 1.3 is given by (4.28) with

$$G(\xi) := \begin{cases} g(\xi), & 0 \le \xi \le 1, \\ g(1-\xi), & 1 \le \xi \le 2\ . \end{cases}$$

It is interesting to note that this solution is not of the form (3.24). Namely, the solution (3.24) has the property that if we choose $g \in W\mathcal{L}_+ \subset \mathcal{D}(A)$ then the solution given by (3.24) stays in the uniformly positive subspace $\mathcal{L}_+$ for all $0 \le t \le 1$. This is not the case for the solution here since for arbitrary G and $0 \le t \le 1$ the solution (4.28) is an even function of x and therefore it is a neutral vector in the Krein space $(L^2(\Omega), [\,\cdot\,,\,\cdot\,])$. Since the operator A satisfies all the assumptions of Theorem 3.4 except the condition about the existence of a commuting fundamental symmetry, we conclude that there exists no fundamental symmetry in $(L^2(\Omega), [\,\cdot\,,\,\cdot\,])$ which commutes with A.

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