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ON BOUNDED PERTURBATIONS OF OPERATORS OF KLEIN-GORDON TYPE

Peter Jonas
Technische Universität Berlin, Germany

Abstract. A class of nonnegative selfadjoint operators in a Krein space and bounded perturbations of them of a special form are considered. The perturbed operators, which include operators arising from the perturbed Klein-Gordon equation, are definitizable over a neighbourhood of infinity, but not necessarily definitizable. This paper can be regarded as a continuation of [9]; the compactness assumptions on the perturbation in [9] are dropped.

1. Introduction

The equation

$$\left\{-\left(-i\frac{\partial}{\partial t}-\overline{\mathcal{V}(x)}\right)\left(-i\frac{\partial}{\partial t}-\mathcal{V}(x)\right)-\sum_{j=1}^{n}\frac{\partial^2}{\partial x_j^2}+\mathcal{U}(x)+m^2\right\}u(x,t)=0,$$

where $m\geq 0$, $\mathcal{V},\mathcal{U}\in L^\infty(\mathbf{R}^n)$, $\mathcal{U}(x)=\overline{\mathcal{U}(x)}$, $u\in C^2(\mathbf{R}^n\times\mathbf{R})$, which is a perturbed Klein-Gordon ($m>0$) or wave ($m=0$) equation, can be written in the form

(1.1)
$$\frac{\partial}{\partial t}\begin{pmatrix} u(x,t)\\ v(x,t)\end{pmatrix}=i\begin{pmatrix} \mathcal{V}(x) & 1\\ -\sum_{j=1}^{n}\frac{\partial^2}{\partial x_j^2}+\mathcal{U}(x)+m^2 & \overline{\mathcal{V}(x)}\end{pmatrix}\begin{pmatrix} u(x,t)\\ v(x,t)\end{pmatrix}.$$

Here the derivatives are understood in the distribution sense.

We may consider an equation of the form (1.1) as a first order differential equation in $\mathcal{H}=L^2(\mathbf{R}^n)\times L^2(\mathbf{R}^n)$. Then if we provide $\mathcal{H}$ with the inner product $[\cdot,\cdot]$ defined by

$$\left[\binom{u_1}{v_1},\binom{u_2}{v_2}\right]:=(v_1,u_2)_{L^2}+(u_1,v_2)_{L^2},\qquad \binom{u_1}{v_1},\binom{u_2}{v_2}\in\mathcal{H},$$

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$(\mathcal{H}, [\cdot,\cdot])$ is a Krein space and the operator corresponding to the matrix in (1.1) with its natural domain is selfadjoint in this Krein space. In the unperturbed case (i.e. $\mathcal{V} = \mathcal{U} = 0$) this operator is nonnegative in $(\mathcal{H}, [\cdot,\cdot])$.

The first approach to the spectral theory of the Klein-Gordon equation employing the indefinite form $[\cdot,\cdot]$ as the basic inner product was a paper of K. Veselić ([18]). This work was stimulated by the preceding physical literature. Later this Krein space approach was elaborated by H. Langer and B. Najman in the unpublished paper [15], where some class of definitizable selfadjoint operators which contains operators arising from perturbed Klein-Gordon equations is studied. For an operator A of that class the quadratic form $[A\cdot,\cdot]$ has a finite number of negative squares and, hence, there is a finite set s of eigenvalues of A such that, roughly speaking, only on the spectral subspaces which correspond to $\mathbf{R}$-symmetric sets having common points with s, A is not nonnegative. Nonreal eigenvalues of A and points $\neq 0$ in which the spectral function of A has a singularity, belong to s. The paper [15] contains, among other things, upper estimates for the absolute values of these "exceptional eigenvalues" and for their number. In [15] this operator approach to the spectral theory of the Klein-Gordon equation is compared with another one (for references see [9, Introduction]).

In [9] besides some slight strengthening of the results of [15] a class of selfadjoint operators is considered which also contains operators associated to the perturbed wave equation ($m = 0$). In general, there is no longer a finite set of "exceptional points" as in [15] but the "exceptional set" "outside of which" the operator is nonnegative is contained in a circle the radius of which can be estimated in the same way as in [15]. The operators studied in [9] are locally definitizable in a neighbourhood of infinity (see Section 2.1).

The perturbations considered in [15] and [9] satisfy some compactness conditions. In [2] B. Ćurgus and B. Najman considered perturbations with compactness properties or of sufficiently small size such that the perturbed operator is definitizable.

In the present paper, which can be regarded as a continuation of [9], the compactness assumption of [9] on the perturbation is dropped and bounded perturbations of arbitrary size are considered. Therefore, also in the case $m > 0$, we have to deal with operators which are not definitizable. Again we prove that they are locally definitizable over a neighbourhood of infinity.

In Section 2 we recall some definitions and results of the spectral theory of selfadjoint operators in Krein spaces and the definition and some properties of an operator which will play the role of the unperturbed operator. In Section 3 we describe the spectrum and show definitizability properties of the perturbed operator. The results can be carried over to an associated operator whose spectral function is bounded in a neighbourhood of infinity. In Section 4 we consider a simple example of a Klein-Gordon or wave equation with a signum-type function $\mathcal{V}$ (see (1.1)).

It should be mentioned that in [19] K. Veselić studied a perturbation of an operator associated with the unperturbed Klein-Gordon equation which is not relatively bounded. It turned out that the perturbed operator has real spectrum and a regular spectral function, but it is not locally definitizable with respect to some natural Krein space inner product (which essentially coincides with that introduced above). For the class of bounded perturbations we are dealing with, it is not true in general (as simple examples show), that the perturbed operators are "globally" spectrally decomposable as the perturbed operator in [19].

2. Notation and Preliminaries

2.1. *Locally definitizable selfadjoint operators in Krein spaces.* In this section we recall the notation and some definitions and results from [6], [9], [11] and [14]. Let $(\mathcal{H}, [\cdot,\cdot])$ be a Krein space and let A be a selfadjoint operator in $\mathcal{H}$ with nonempty resolvent set $\rho(A)$. The operator A is called *nonnegative* if $[Ax, x] \geq 0$ for every $x \in \mathcal{D}(A)$. The operator A is called *definitizable* if there exists a polynomial p, $p \neq 0$, such that $[p(A)x, x] \geq 0$ for every $x \in \mathcal{D}(p(A))$.

An open subset Δ of $\overline{\mathbf{R}}$ ($\overline{\mathbf{R}}$ is the closure of $\mathbf{R}$ in the complex sphere $\overline{\mathbf{C}}$) is said to be of *positive type (negative type)* with respect to A if the following conditions (i), (ii), (iii) are fulfilled:

(i) No point of Δ is an accumulation point of $\sigma(A) \setminus \mathbf{R}$.

(ii) For every closed subset δ of Δ there exist a positive integer m and $M > 0$ such that
$$\|(A - z)^{-1}\| \leq M(1 + |z|)^{2m-2} |\operatorname{Im} z|^{-m}$$
for all z in some neighbourhood of δ (in $\overline{\mathbf{C}}$) with $z \neq \infty$ and $\operatorname{Im} z \neq 0$.

(iii) For every nonnegative (resp. nonpositive) $f \in C^\infty(\overline{\mathbf{R}})$ [1] with $\operatorname{supp} f \subset \Delta$ the operator $f(A)$ (defined, in view of (ii), by extension of the Riesz-Dunford-Taylor functional calculus, see [6, Proposition 1.3]) is nonnegative.

Condition (ii) means that the resolvent of the Cayley transform of A grows not faster than of finite order near to the arc of the unit circle corresponding to Δ. In this definition (ii) and (iii) can be replaced by a condition on the resolvent of A. We give this condition under the additional assumption that $\infty \notin \Delta$: If (i) holds and $\Delta \subset \mathbf{R}$, then (ii) and (iii) are true if and only if the following holds ([11], cf. [6, Remark 2.5]):

(iii′) For every $x \in \mathcal{H}$ and for almost every $t \in \Delta$ we have
$$\liminf_{\epsilon \downarrow 0} -i\,[\{(A - t - i\epsilon)^{-1} - (A - t + i\epsilon)^{-1}\}x, x] \geq 0$$

[1] Here $\overline{\mathbf{R}}$ is regarded as real-analytic manifold in the usual way.

(resp. ≤ 0 with $\liminf$ replaced by $\limsup$); and for every $x \in \mathcal{H}$, every compact subinterval $\delta \subset \Delta$, and sufficiently small $\epsilon_0 > 0$ there exists an $M > 0$ such that

$$-i\,[\{(A-t-i\epsilon)^{-1} - (A-t+i\epsilon)^{-1}\}x, x] \geq -M \qquad (\text{resp.} \leq M)$$

for every $t \in \delta$ and $\epsilon \in (0, \epsilon_0]$.

In [14] H. Langer, A. Markus and V. Matsaev introduced sign types for spectral points of A in the following way: A real boundary point of $\sigma(A)$ is said to be of *positive type* (*negative type*), if for every sequence $(x_n) \subset \mathcal{D}(A)$ with $\|x_n\| = 1$, $n = 1, 2, \ldots$, and $\lim_{n\to\infty} \|(A-\lambda)x_n\| = 0$ it holds $\liminf_{n\to\infty} [x_n, x_n] > 0$ (resp. $\limsup_{n\to\infty} [x_n, x_n] < 0$). In [14] it was assumed that A is bounded.

An open subset Δ of $\mathbf{R}$ which satisfies Condition (i) above is of positive (negative) type with respect to A if and only if all points of $\Delta \cap \sigma(A)$ are of positive type (resp. negative type) (see [11]). In Theorem 3.3 below we need the "if"-part of this statement and we will prove it in part 4 of the proof of Theorem 3.3.

We say that an open set $\Delta \subset \mathbf{R}$ is of *definite type* if it is of positive or of negative type.

The operator A is called *definitizable over* the open set $\Delta \subset \overline{\mathbf{R}}$ if the above-mentioned conditions (i) and (ii) are fulfilled and for every $t \in \Delta$ all sufficiently small one-sided open neighbourhoods of t are of definite type. A selfadjoint operator A with $\sigma(A)\backslash\mathbf{R}$ consisting of no more than a finite number of poles of the resolvent is definitizable over $\overline{\mathbf{R}}$ if and only if it is definitizable.

Let A be definitizable (definitizable over Δ). A point t (resp. $t \in \Delta$) is called a *critical point of* A (resp. *of* A *in* Δ) if there is no open neighbourhood of t of definite type. The most important consequence of local definitizability is the existence of a local spectral function. If A is definitizable over Δ, its spectral function $E(\cdot, A)$ is defined for all connected sets $\delta \subset \Delta$ whose endpoints belong to Δ and are not critical points. For such an δ, $E(\delta, A)$ is a selfadjoint projection in $\mathcal{H}$ and $A|E(\delta, A)\mathcal{H}$ is definitizable. For the definition of the spectral function and a construction of it with the help of the extension of the functional calculus of A (observe condition (iii) above) we refer to [6, Section 2.2] (for a unitary operator, e.g. the Cayley transform of A, see [5]). For the operators occurring in the present paper the additional condition of discreteness of the nonreal spectrum in [6] plays no role and could be omitted. For another construction of the spectral function (for bounded operators) we refer to [14]. An open subset $\Delta_0 \subset \Delta$ is of positive type (negative type) with respect to A if and only if for every connected subset δ of Δ_0, $E(\delta, A)$ is defined and a nonnegative (resp. nonpositive) projection.

A critical point t of A in Δ is called *regular* if there exists a neighbourhood $\mathcal{U}_t$ (in $\overline{\mathbf{R}}$) of t such that $\sup \|E(\delta, A)\| < \infty$, where the supremum is taken over all intervals $\delta \subset \mathcal{U}_t$ such that $E(\delta, A)$ is defined. A non-regular critical point of A is called *singular*.

2.2. *The unperturbed operator.* Let $(\mathcal{G}_0, (\cdot,\cdot)_0)$ be a Hilbert space and let H_0 be a nonnegative selfadjoint operator in $\mathcal{G}_0$ such that $0 \in \sigma(H_0)$. We set $H_m := H_0 + m^2$, $m \geq 0$. Define a scalar product $(\cdot,\cdot)_\alpha$ on $\mathcal{D}(H_1^\alpha)$, $\alpha \in \mathbf{R}$, by $(x,y)_\alpha := (H_1^\alpha x, H_1^\alpha y)_0$, $x, y \in \mathcal{D}(H_1^\alpha)$. By $\mathcal{G}_\alpha$, $\alpha \in \mathbf{R}$, we denote the completion of $\mathcal{D}(H_1^\alpha)$ with respect to the norm $\|\cdot\|_\alpha$, $\|x\|_\alpha := (x,x)_\alpha^{\frac{1}{2}}$, $x \in \mathcal{D}(H_1^\alpha)$. The form $(\cdot,\cdot)_0$ can be extended by continuity to $\mathcal{G}_\alpha \times \mathcal{G}_{-\alpha}$ for every $\alpha \in \mathbf{R}$. In the following, extensions of forms by continuity will be denoted in the same way as the forms theirselves. Every continuous linear functional on $\mathcal{G}_\alpha$ is of the form $(\cdot, y)_0$, $y \in \mathcal{G}_{-\alpha}$.

We provide the linear space $\mathcal{H} = \mathcal{G}_0 \times \mathcal{G}_0$ with the Krein space inner product $[\cdot,\cdot]$ defined by

$$\left[\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right] = (u_2, v_1)_0 + (u_1, v_2)_0, \qquad u_1, u_2, v_1, v_2 \in \mathcal{G}_0. \tag{2.2}$$

Set

$$J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

with respect to the decomposition $\mathcal{H} = \mathcal{G}_0 \times \mathcal{G}_0$. J is a fundamental symmetry of $(\mathcal{H}, [\cdot,\cdot])$ and we have

$$\left(\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right) := \left[J\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right] = (u_1, v_1)_0 + (u_2, v_2)_0, \quad u_1, u_2, v_1, v_2 \in \mathcal{G}_0.$$

Set $\|x\| := (x,x)^{\frac{1}{2}}$, $x \in \mathcal{H}$.

In what follows the operator A in $\mathcal{H}$ defined by

$$A := \begin{pmatrix} 0 & 1 \\ H_m & 0 \end{pmatrix}, \qquad \mathcal{D}(A) := \mathcal{G}_1 \times \mathcal{G}_0, \tag{2.3}$$

$m \geq 0$, will play the role of the unperturbed operator in our abstract setting. The operator matrix in (1.1) with $\mathcal{V}, \mathcal{U} = 0$ is a special case of A.

Since JA is selfadjoint in the Hilbert space $(\mathcal{H}, (\cdot,\cdot))$, A is selfadjoint in the Krein space $(\mathcal{H}, [\cdot,\cdot])$. Moreover, A is a nonnegative operator: we have $\mathbf{C} \setminus \mathbf{R} \subset \rho(A)$ (see [9, (2.1)]) and $[Ax, x] \geq 0$ for every $x \in \mathcal{D}(A)$. There are explicit formulas for the resolvent and the spectral function of A ([15], [9]). If H_0 is unbounded, then ∞ is a singular critical point of A. The point 0 is a singular critical point of A if and only if $m = 0$ and 0 is an accumulation point of $\sigma(H_0)$.

2.3. *The "regularized" operator A_r.* If

$$\mathcal{H}_{\frac{1}{2}} := \mathcal{G}_{\frac{1}{2}} \times \mathcal{G}_0, \qquad \mathcal{H}_{-\frac{1}{2}} := \mathcal{G}_0 \times \mathcal{G}_{-\frac{1}{2}},$$

we have $\mathcal{H}_{\frac{1}{2}} \subset \mathcal{H} \subset \mathcal{H}_{-\frac{1}{2}}$ and the form $[\cdot,\cdot]$ restricted to $\mathcal{H}_{\frac{1}{2}} \times \mathcal{H}$ ($\mathcal{H} \times \mathcal{H}_{\frac{1}{2}}$) can be extended by continuity to $\mathcal{H}_{\frac{1}{2}} \times \mathcal{H}_{-\frac{1}{2}}$ (resp. $\mathcal{H}_{-\frac{1}{2}} \times \mathcal{H}_{\frac{1}{2}}$). Every continuous linear functional on $\mathcal{H}_{\frac{1}{2}}$ has the form $[\cdot, y]$ for some $y \in \mathcal{H}_{-\frac{1}{2}}$. A can be extended by continuity to an operator $\widetilde{A} \in \mathcal{L}(\mathcal{H}_{\frac{1}{2}}, \mathcal{H}_{-\frac{1}{2}})$.

Let

$$\mathcal{H}_r = \mathcal{G}_{\frac{1}{4}} \times \mathcal{G}_{-\frac{1}{4}}$$

and define a Krein space inner product on $\mathcal{H}_r$ by (2.2) with u_1, $v_1 \in \mathcal{G}_{\frac{1}{4}}$, u_2, $v_2 \in \mathcal{G}_{-\frac{1}{4}}$. We have $\mathcal{H}_{\frac{1}{2}} \subset \mathcal{H}_r \subset \mathcal{H}_{-\frac{1}{2}}$. In the case of the Klein-Gordon equation the space $\mathcal{H}_r$ regarded as Hilbert space was called, in [19], the "number norm" Hilbert space because of its physical meaning. The operator A_r in $\mathcal{H}_r$ introduced in [7, Section 2.4] (in that paper denoted by A') which is defined by

$$\mathcal{D}(A_r) := \{x \in \mathcal{H}_{\frac{1}{2}} : \widetilde{A}x \in \mathcal{H}_r\}, \quad A_r x := \widetilde{A}x,\ x \in \mathcal{D}(A_r), \tag{2.4}$$

is a nonnegative operator in $\mathcal{H}_r$ and ∞ is not a singular critical point of A_r (see [7, Lemma 2.1]). By this definition, $\mathcal{D}(A_r) = \mathcal{G}_{\frac{3}{4}} \times \mathcal{G}_{\frac{1}{4}}$. We have $\sigma(A) = \sigma(A_r)$. Moreover, the spectral subspaces with respect to A and A_r corresponding to an arbitrary bounded interval (which are contained in $\mathcal{D}(A)$ or $\mathcal{D}(A_r)$, respectively) coincide and A and A_r coincide on these subspaces.

Triples $\mathcal{H}_{\frac{1}{2}} \subset \mathcal{H} \subset \mathcal{H}_{-\frac{1}{2}}$, $\mathcal{H}_{\frac{1}{2}} \subset \mathcal{H}_r \subset \mathcal{H}_{-\frac{1}{2}}$ and "regularized" operators A_r can be assigned to more general selfadjoint operators A in $\mathcal{H}$; we refer to [12], [7] and [10]. In more general cases $\mathcal{H}_{\frac{1}{2}}$ is defined as the middle space in the interpolation scale between $\mathcal{D}(A)$ with the graph norm and $\mathcal{H}$, $\mathcal{H}_{-\frac{1}{2}}$ is defined with the help of $[\cdot,\cdot]$ and $\mathcal{H}_r$ is again the middle space in the interpolation scale between $\mathcal{H}_{\frac{1}{2}}$ and $\mathcal{H}_{-\frac{1}{2}}$.

3. A class of bounded perturbations of A

3.1. We consider operators in $\mathcal{H} = \mathcal{G}_0 \times \mathcal{G}_0$ of the form

$$B := \begin{pmatrix} V_0 & 1 \\ H_m + U_0 & V_0^* \end{pmatrix},$$

where U_0, $V_0 \in \mathcal{L}(\mathcal{G}_0)$, $U_0 = U_0^*$. Since B arises from A (see (2.3)) by the symmetric bounded perturbation

$$Z_0 = \begin{pmatrix} V_0 & 0 \\ U_0 & V_0^* \end{pmatrix},$$

B is selfadjoint in $\mathcal{H}$. Our aim is to describe the spectrum of B and its sign properties. Theorem 3.2 below will show that $\rho(B)$ is not empty. For describing the nonreal part of $\sigma(B)$ we shall need the numbers (cf. [9, Section 3.2])

$$\begin{array}{rcl} \widetilde{\gamma}_0 & := & \sup\{((V_0^*V_0 - U_0 - H_m)u, u)_0 (u,u)_0^{-1} : u \in \mathcal{G}_1\}, \\ \gamma_0 & := & (\max\{\widetilde{\gamma}_0, 0\})^{\frac{1}{2}}, \\ \gamma_l & := & \inf\{\mathrm{Re}\,(V_0 u, u)_0\,(u,u)_0^{-1} : u \in \mathcal{G}_0\}, \\ \gamma_r & := & \sup\{\mathrm{Re}\,(V_0 u, u)_0\,(u,u)_0^{-1} : u \in \mathcal{G}_0\} \end{array}$$

and the following

LEMMA 3.1. *If $L(\lambda)$ is the operator polynomial*

$$L(\lambda) = \lambda^2 - \lambda(V_0 + V_0^*) - H_m - U_0 + V_0^* V_0, \tag{3.5}$$

in the Hilbert space $\mathcal{G}_0$ defined on $\mathcal{G}_1$, then the following holds:

(i) $\lambda \in \sigma(B)$ *if and only if* $0 \in \sigma(L(\lambda))$.

(ii) $\lambda \in \sigma_p(B)$ *if and only if* $0 \in \sigma_p(L(\lambda))$.

PROOF. If $0 \in \rho(L(\lambda))$ for some $\lambda \in \mathbf{C}$, then it can easily be verified that

$$\begin{pmatrix} L(\lambda)^{-1}(V_0^* - \lambda) & -L(\lambda)^{-1} \\ 1 - (V_0 - \lambda)L(\lambda)^{-1}(V_0^* - \lambda) & (V_0 - \lambda)L(\lambda)^{-1} \end{pmatrix} \tag{3.6}$$

is the inverse of $B - \lambda$, that is $\lambda \in \rho(B)$.

If $\lambda \in \rho(B)$, then the system

$$\begin{array}{ccccc} (V_0 - \lambda)u_1 & + & u_2 & = & 0 \\ (H_m + U_0)u_1 & + & (V_0^* - \lambda)u_2 & = & f_2 \end{array}$$

has a unique solution $\binom{u_1}{u_2} \in \mathcal{D}(B)$ for every $f_2 \in \mathcal{G}_0$. This implies that for every $f_2 \in \mathcal{G}_0$, $-L(\lambda)u_1 = f_2$ has a unique solution $u_1 \in \mathcal{D}(H_0)$, that is $0 \in \rho(L(\lambda))$, which proves (i). An easy computation shows that (ii) holds. ∎

For the case when the nonreal spectrum of B consists of isolated eigenvalues the first part of the following theorem was proved in [15] (see also [17], [9, Proposition 3.3]).

THEOREM 3.2. *The spectrum of B is contained in $\mathbf{R} \cup S$, where*

$$S := \{\lambda : |\lambda| \le \gamma_0\} \cap \{\lambda : \gamma_l \le \mathrm{Re}\lambda \le \gamma_r\}.$$

Moreover, there exists a neighbourhood $\mathcal{U}$ of $\overline{\mathbf{R}} \setminus S$ in $\overline{\mathbf{C}}$ and a constant c such that

$$\|(B - \lambda)^{-1}\| \le c(1 + |\lambda|)^2\, |\mathrm{Im}\,\lambda|^{-2}, \qquad \lambda \in \mathcal{U} \setminus \overline{\mathbf{R}}.$$

PROOF. 1. We consider the numerical root range $R(L)$ of L (see (3.5))

$$R(L) := \{\lambda \in \mathbf{C} : \text{there exists}\, f \in \mathcal{G}_1 \,\text{with}\, \|f\| = 1 \,\text{and}\, (L(\lambda)f, f) = 0\}.$$

The relation $(L(\lambda)f, f) = 0$ holds if and only if

$$(\lambda - \mathrm{Re}\,(V_0 f, f))^2 = (\mathrm{Re}\,(V_0 f, f))^2 + ((H_m + U_0 - V_0^* V_0)f, f).$$

This equation has a nonreal solution λ if and only if

$$((-H_m - U_0 + V_0^* V_0)f, f) - (\mathrm{Re}\,(V_0 f, f))^2 > 0.$$

In this case

$$\lambda = \mathrm{Re}\,(V_0 f, f) \pm i\sqrt{((-H_m - U_0 + V_0^* V_0)f, f) - (\mathrm{Re}\,(V_0 f, f))^2}.$$

This implies

$$|\lambda|^2 = ((-H_m - U_0 + V_0^* V_0)f, f).$$

Hence $R(L) \subset \mathbf{R} \cup S$.

Now we apply [16, Theorem 26.7]. The proof of that theorem remains true if one coefficient of the operator polynomial is unbounded. We obtain that $\lambda \notin \mathbf{R} \cup S$ implies $0 \in \rho(L(\lambda))$ and

$$\|L(\lambda)^{-1}\| \leq (\mathrm{dist}\,\{\lambda, \mathbf{R} \cup S\})^{-2}.$$

In view of Lemma 3.1, this proves the first assertion of the theorem.

It is easy to see that there is a neighbourhood $\mathcal{U}$ of $\overline{\mathbf{R}} \setminus S$ in $\overline{\mathbf{C}}$ such that, for some constant c',

$$\|L(\lambda)^{-1}\| \leq c' |\mathrm{Im}\,\lambda|^{-2}, \qquad \lambda \in \mathcal{U}.$$

Then the second assertion of the theorem follows from the matrix representation (3.6) for $(B - \lambda)^{-1}$, and Theorem 3.2 is proved. □

THEOREM 3.3. *The interval* $(\min\{\gamma_r, \gamma_0\}, \infty)$ *is of positive type with respect to* B *and the interval* $(-\infty, \max\{\gamma_l, -\gamma_0\})$ *is of negative type with respect to* B*.* B *is definitizable over*

$$(\min\{\gamma_r, \gamma_0\}, \infty) \cup \{\infty\} \cup (-\infty, \max\{\gamma_l, -\gamma_0\}). \tag{3.7}$$

PROOF. 1. Let $\lambda \in \mathbf{R}$ be a real boundary point of the spectrum of B. Consider a sequence of elements $x_n = \binom{u_n}{v_n} \in \mathcal{D}(B)$ such that $\|x_n\| = 1$ and $\lim_{n\to\infty} \|(B - \lambda)x_n\| = 0$. Then, for $n \to \infty$,

$$V_0 u_n - \lambda u_n + v_n \;\to\; 0, \tag{3.8}$$

$$(H_m + U_0)u_n + V_0^* v_n - \lambda v_n \;\to\; 0. \tag{3.9}$$

In view of $[x_n, x_n] = 2\,\mathrm{Re}\,(v_n, u_n)$ relation (3.8) gives

$$\mathrm{Re}\,(V_0 u_n, u_n) - \lambda(u_n, u_n) + \frac{1}{2}[x_n, x_n] \to 0 \tag{3.10}$$

for $n \to \infty$. From (3.8) and (3.9) it follows that

$$\{H_m + U_0 - (V_0^* - \lambda)(V_0 - \lambda)\}u_n \to 0$$

and

$$((H_m + U_0 - V_0^* V_0)u_n, u_n) + 2\lambda\,\mathrm{Re}\,(V_0 u_n, u_n) - \lambda^2(u_n, u_n) \to 0.$$

Making use of (3.10) we obtain

$$\lambda^2(u_n, u_n) + ((H_m + U_0 - V_0^* V_0)u_n, u_n) - \lambda[x_n, x_n] \to 0. \tag{3.11}$$

2. Let $\lambda \in (\gamma_r, \infty) \cap \sigma(B)$. Then λ is a boundary point of the spectrum of B. Let $x_n = \binom{u_n}{v_n}$ be as in the first part of the proof. Then by (3.10)

$$\liminf_{n\to\infty} [x_n, x_n] = \\ = 2 \liminf_{n\to\infty} \{\lambda(u_n, u_n) - \mathrm{Re}(V_0 u_n, u_n)\} \geq 2(\lambda - \gamma_r) \liminf_{n\to\infty} (u_n, u_n).$$

We have $\liminf_{n\to\infty} (u_n, u_n) > 0$. Indeed, by (3.8) $\liminf_{n\to\infty} (u_n, u_n) = 0$ would contradict $\|x_n\| = 1$. Hence

$$\liminf_{n\to\infty} [x_n, x_n] > 0.$$

Similarly, $\lambda \in (-\infty, \gamma_l) \cap \sigma(B)$ implies

$$\limsup_{n\to\infty} [x_n, x_n] < 0.$$

3. Now let $\lambda \in (\gamma_0, \infty) \cap \sigma(B)$ and let $x_n = \binom{u_n}{v_n}$ be as above. Then by (3.11)

$$\liminf_{n\to\infty} [x_n, x_n] = \\ = \lambda^{-1} \liminf_{n\to\infty} \{\lambda^2 (u_n, u_n) + ((H_m + U_0 - V_0^* V_0) u_n, u_n)\} \geq \\ \geq \lambda^{-1}(\lambda^2 - \gamma_0^2) \liminf_{n\to\infty} (u_n, u_n).$$

As above it follows that

$$\liminf_{n\to\infty} [x_n, x_n] > 0.$$

Similarly, $\lambda \in (-\infty, -\gamma_0) \cap \sigma(B)$ implies

$$\limsup_{n\to\infty} [x_n, x_n] < 0.$$

4. The sign properties of B mentioned in the theorem follow now with the help of [11] from what was proved in parts 2 and 3 of this proof. For completeness we shall prove this here.

Let t be an arbitrary point of $(\min\{\gamma_r, \gamma_0\}, \infty)$, and $x \in \mathcal{H}$. We claim that

$$\liminf_{\epsilon \downarrow 0} \; -i\,[\{(B - t - i\epsilon)^{-1} - (B - t + i\epsilon)^{-1}\} x, x] \geq 0. \tag{3.12}$$

Suppose that this is not true. Then there exist an $\beta < 0$ and a sequence (ϵ_n), $\epsilon_n \downarrow 0$, such that

$$-i\,[\{(B - t - i\epsilon_n)^{-1} - (B - t + i\epsilon_n)^{-1}\} x, x] = \\ = 2\epsilon_n\, [(B - t - i\epsilon_n)^{-1} x, (B - t - i\epsilon_n)^{-1} x] \leq \beta. \tag{3.13}$$

This implies $b_n := \|(B - t - i\epsilon_n)^{-1} x\| \to \infty$ for $n \to \infty$ and, hence, $t \in \sigma(B)$. If $x_n := b_n^{-1}(B - t - i\epsilon_n)^{-1} x$, then $\|x_n\| = 1$ and

$$\|(B - t) x_n\| = b_n^{-1} \|x + i\epsilon_n (B - t - i\epsilon_n)^{-1} x\| \to 0.$$

Hence by parts 2 and 3 of this proof

$$0 < b_n^{-2}[(B-t-i\epsilon_n)^{-1}x, (B-t-i\epsilon_n)^{-1}x]$$

for sufficiently large n. This contradicts (3.13). Hence (3.12) holds.

Let δ be a compact subinterval of $(\min\{\gamma_r, \gamma_0\}, \infty)$, and $x \in \mathcal{H}$. We claim that for a sufficiently small $\epsilon_0 > 0$ there exists an $M > 0$ such that

$$-i\,[\{(B-t-i\epsilon)^{-1} - (B-t+i\epsilon)^{-1}\}x, x] \geq -M \tag{3.14}$$

for all $t \in \delta$ and $\epsilon \in (0, \epsilon_0]$.

Suppose that this is not true. Then there exist a sequence $(t_n) \subset \delta$, $\lim_{n\to\infty} t_n =: t \in \delta$ and a sequence (ϵ_n), $\epsilon_n \downarrow 0$, such that

$$\epsilon_n\,[(B-t_n-i\epsilon_n)^{-1}x, (B-t_n-i\epsilon_n)^{-1}x] \to -\infty. \tag{3.15}$$

This implies $b_n' := \|(B-t_n-i\epsilon_n)^{-1}x\| \to \infty$ for $n \to \infty$ and, hence, $t \in \sigma(B)$. If $x_n' := b_n'^{-1}(B-t_n-i\epsilon_n)^{-1}x$, then $\|x_n'\| = 1$ and $\|(B-t)x_n'\| \to 0$. Therefore, by what was proved above

$$0 < b_n'^{-2}\,[(B-t_n-i\epsilon_n)^{-1}x, (B-t_n-i\epsilon_n)^{-1}x]$$

for sufficiently large n, which contradicts (3.15). A similar reasoning applies for $(\min\{\gamma_r, \gamma_0\}, \infty)$ replaced by $(-\infty, \max\{\gamma_l, -\gamma_0\})$. In this case we obtain relations similar to (3.12) and (3.14) with opposite signs.

5. By Theorem 3.2 the connected set (3.7) fulfils condition (ii) in Section 2.1 with $m = 2$. Then part 4 of the proof together with Condition (iii$'$) in Section 2.1 imply the sign properties of B stated in Theorem 3.3. ◻

Theorem 3.3 implies the existence of a local spectral function for B with the sign properties mentioned in Section 2.1. Then for every closed connected subset of

$$(\min\{\gamma_r, \gamma_0\}, \infty) \cup \{\infty\} \cup (-\infty, \max\{\gamma_l, -\gamma_0\})$$

with finite endpoints we may decompose $\mathcal{H}$ into the corresponding spectral subspace of B and its orthogonal complement. In the following theorem we put down the properties of such a decomposition.

Theorem 3.4. *For every open interval* $\Delta = (\alpha, \beta)$ *containing* 0, $\min\{\gamma_r, \gamma_0\}$ *and* $\max\{\gamma_l, -\gamma_0\}$ *there exists a selfadjoint projection* E_∞ *commuting with every bounded operator which commutes with the resolvent of* B *such that the diagonal representation of* B,

$$B = \begin{pmatrix} B_\infty & 0 \\ 0 & B_{(\infty)} \end{pmatrix},$$

with respect to the decomposition

$$\mathcal{H} = E_\infty\mathcal{H} + (1-E_\infty)\mathcal{H}$$

has the following properties:

(i) *B_∞ is a uniformly positive operator in the Krein space $(E_\infty\mathcal{H},[\cdot,\cdot])$, that is, there exists some $\mu>0$ such that $[B_\infty x,x]\geq\mu\|x\|^2$ for all $x\in\mathcal{D}(B_\infty)$. $B_{(\infty)}$ is bounded.*

(ii) $\sigma(B_\infty)\subset\mathbf{R}\setminus\Delta$, $\quad\sigma(B_{(\infty)})\subset\bar{\Delta}\cup S$ *(see* Theorem 3.2*).*

(iii) α, $\beta\notin\sigma_p(B_{(\infty)})$.

(iv) *If $E(\cdot)$ and $E(\cdot;B_\infty)$ are the spectral functions of B and B_∞, respectively, then*

$$E(\delta)=E(\delta;B_\infty)E_\infty$$

for all connected subsets δ of $\overline{\mathbf{R}}\setminus\Delta$ such that ∞ is not an endpoint of δ.

(v) *If H_0 is unbounded, then ∞ is a singular critical point of B_∞ and, hence, of B.*

Proof. Let $E(\cdot)$ be the local spectral function of B. We set $E_\infty:=E(\overline{\mathbf{R}}\setminus\Delta)$. Then the fact that $B_{(\infty)}$ is bounded, and the assertions (ii) and (iii) are consequences of properties of the local spectral function of B (see [6, Theorem 2.6]). For every connected subset Δ' of $\overline{\mathbf{R}}$ such that ∞ is not an endpoint of Δ', we have

$$E(\Delta';B_\infty)=E(\Delta'\cap(\overline{\mathbf{R}}\setminus\Delta))|E_\infty\mathcal{H}. \tag{3.16}$$

This implies (iv).

By (3.16) and Theorem 3.3 $E(\Delta';B_\infty)$ is nonnegative if $\Delta'\subset(0,\infty)$, and nonpositive if $\Delta'\subset(-\infty,0)$. Then it follows from [3, § 3.2, Satz 5 and its proof] that there exists a positive integer m such that $[B_\infty^{-m}x,x]\geq 0$ for all $x\in E_\infty\mathcal{H}$. From this fact and the injectivity of B_∞^{-1} we infer that

$$\operatorname{closp}\{E((-n,n);B_\infty)E_\infty\mathcal{H}:\,n\in\mathbf{N}\}=E_\infty\mathcal{H} \tag{3.17}$$

([13], see also [4, § 1.1]). The relation (3.17) implies $[B_\infty^{-1}x,x]\geq 0$ for every $x\in E_\infty\mathcal{H}$ and, consequently, the first part of assertion (i) holds.

To verify (v) we assume that H_0 is unbounded. Then ∞ is a singular critical point of A. Since the nonnegative operators A and $\begin{pmatrix} B_\infty & 0\\ 0 & 0\end{pmatrix}$ differ by a bounded operator, we find by [1, Corollary 3.3] that ∞ is a singular critical point of B_∞ and, hence, of B. This proves Theorem 3.4. ∎

3.2. Now we consider the operator A_r in $\mathcal{H}_r=\mathcal{G}_{\frac14}\times\mathcal{G}_{-\frac14}$ as unperturbed operator (see Section 2.3). The perturbation Z_0 of A introduced at the beginning of Section 3.1 can, in general, not be considered as a bounded operator in $\mathcal{H}_r$. We are going to define a perturbation of A_r which corresponds to the perturbation Z_0 of A in a natural way.

Let V denote the restriction of V_0 to an operator of $\mathcal{G}_{\frac12}$ into $\mathcal{G}_0$. Then $V^*\in\mathcal{L}(\mathcal{G}_0,\mathcal{G}_{-\frac12})$. Here " $*$ " denotes the adjoint with respect to the $(\cdot,\cdot)_0$-duality in the scale $\mathcal{G}_\alpha$, $\alpha\in\mathbf{R}$, see Section 2.2. The restriction of U_0 to $\mathcal{G}_{\frac12}$

regarded as an operator of $\mathcal{G}_{\frac{1}{2}}$ into $\mathcal{G}_{-\frac{1}{2}}$ will be denoted by U. Then $U = U^*$. We consider the operator

$$Z = \begin{pmatrix} V & 0 \\ U & V^* \end{pmatrix} \in \mathcal{L}(\mathcal{H}_{\frac{1}{2}}, \mathcal{H}_{-\frac{1}{2}}).$$

The set of all $\lambda \in \mathbf{C}$ for which $\widetilde{A} + Z - \lambda E$, where $\widetilde{A}$ is the extension of A (see Section 2.3) and E is the natural imbedding of $\mathcal{H}_{\frac{1}{2}}$ in $\mathcal{H}_{-\frac{1}{2}}$, is an isomorphism of $\mathcal{H}_{\frac{1}{2}}$ onto $\mathcal{H}_{-\frac{1}{2}}$ coincides with $\rho(B) = \rho(A + Z_0)$ ([12, Section 1.1], [7, Section 2.1]). Then $\rho(B) \neq \emptyset$ implies that the operator $A_r \uplus Z$ defined by

$$\mathcal{D}(A_r \uplus Z) = \{x \in \mathcal{H}_{\frac{1}{2}} : (\widetilde{A} + Z)x \in \mathcal{H}_r\}, \quad A_r \uplus Z = \widetilde{A} + Z|\mathcal{D}(A_r \uplus Z)$$

is closed and densely defined. Moreover, $\rho(B) \subset \rho(A_r \uplus Z)$. If $\lambda \in \rho(B) \setminus \mathbf{R}$, then by

$$[(\widetilde{A} + Z - \lambda E)x, y] = [x, (\widetilde{A} + Z - \bar{\lambda} E)y]$$

for all $x, y \in \mathcal{H}_{\frac{1}{2}}$ and, hence, for all $x, y \in \mathcal{D}(A_r \uplus Z)$. That is,

$$((A_r \uplus Z - \lambda)^{-1})^+ = (A_r \uplus Z - \bar{\lambda})^{-1}$$

and $A_r \uplus Z$ is selfadjoint in $\mathcal{H}_r$. The operator $A_r \uplus Z$ will be regarded as the operator in $\mathcal{H}_r$ which corresponds to the perturbed operator $A + Z_0$ in $\mathcal{H}$. Both operators $A + Z_0$ and $A_r \uplus Z$ are restrictions of $\widetilde{A} + Z$.

On the other hand, we may consider the "regularization" B_r corresponding to B. As $\mathcal{D}(B) = \mathcal{D}(A)$, the operator B_r is an operator in $\mathcal{H}_r$. Since the extension by continuity $\widetilde{B} \in \mathcal{L}(\mathcal{H}_{\frac{1}{2}}, \mathcal{H}_{-\frac{1}{2}})$ of B coincides with $\widetilde{A} + Z$ the operators B_r and $A_r \uplus Z$ coincide (see [7, (2.6)]). Then, as a consequence of [7, Lemma 2.1] we obtain the following.

Theorem 3.5. Theorem 3.4 *remains true if $\mathcal{H}$ and B are replaced by $\mathcal{H}_r$ and $A_r \uplus Z$, respectively. Moreover, if ∞ is a critical point of $A_r \uplus Z$ it is a regular critical point. In particular, $i\,(A_r \uplus Z)$ generates a C_0-group of unitary operators in $\mathcal{H}_r$.*

4. An example

Let $\mathcal{G}_0 = L^2(\mathbf{R})$, let H_0 be the operator $-\frac{d^2}{dx^2}$ with its usual domain, $m \geq 0$. We set $U_0 = 0$, and assume that V_0 is the operator $\mathcal{V}_{a,b}\cdot$ of multiplication with the function $\mathcal{V}_{a,b}$ defined by

$$\mathcal{V}_{a,b}(x) = \begin{cases} a & \text{if} \quad x \in (0, \infty) \\ b & \text{if} \quad x \in (-\infty, 0), \end{cases}$$

where a and b are complex numbers. We consider the operator $B = A + Z_0$, which can be written as

$$B = \begin{pmatrix} \mathcal{V}_{a,b}\cdot & 1 \\ -\frac{d^2}{dx^2} + m^2 & \overline{\mathcal{V}_{a,b}}\cdot \end{pmatrix}.$$

Then the spectrum of B has the following properties:

(i) $\sigma_p(B) = \emptyset$.

(ii) $\sigma(B)$ is real if and only if $|\operatorname{Im} a| \leq m$ and $|\operatorname{Im} b| \leq m$. In this case $\sigma(B)$ is the union of the following four intervals:

$$\left(-\infty, \operatorname{Re} a - \sqrt{m^2 - (\operatorname{Im} a)^2}\right], \qquad \left(-\infty, \operatorname{Re} b - \sqrt{m^2 - (\operatorname{Im} b)^2}\right],$$
$$\left[\operatorname{Re} a + \sqrt{m^2 - (\operatorname{Im} a)^2}, \infty\right), \qquad \left[\operatorname{Re} b + \sqrt{m^2 - (\operatorname{Im} b)^2}, \infty\right).$$

(iii) Assume that $\sigma(B) \subset \mathbf{R}$ and $\operatorname{Re} a < \operatorname{Re} b$. Then the interval

$$\left(\operatorname{Re} a - \sqrt{m^2 - (\operatorname{Im} a)^2}, \infty\right) \cap \left(\operatorname{Re} b - \sqrt{m^2 - (\operatorname{Im} b)^2}, \infty\right)$$

is of positive type with respect to B, the interval

$$\left(-\infty, \operatorname{Re} a + \sqrt{m^2 - (\operatorname{Im} a)^2}\right) \cap \left(-\infty, \operatorname{Re} b + \sqrt{m^2 - (\operatorname{Im} b)^2}\right)$$

is of negative type with respect to B, and no open subinterval of

$$\left(\operatorname{Re} a + \sqrt{m^2 - (\operatorname{Im} a)^2}, \infty\right) \cap \left(-\infty, \operatorname{Re} b - \sqrt{m^2 - (\operatorname{Im} b)^2}\right)$$

is of definite type.

(iv) Assume that $|\operatorname{Im} a| > m$ and $|\operatorname{Im} b| > m$ holds. Then

$$\begin{aligned}\sigma(B) = \mathbf{R} \cup \{\operatorname{Re} a + it\sqrt{(\operatorname{Im} a)^2 - m^2} : t \in [-1, 1]\} \cup \\ \cup \{\operatorname{Re} b + it\sqrt{(\operatorname{Im} b)^2 - m^2} : t \in [-1, 1]\}.\end{aligned}$$

If, in addition, $\operatorname{Re} a < \operatorname{Re} b$, then the intervals $(\operatorname{Re} b, \infty)$ and $(-\infty, \operatorname{Re} a)$ are of positive and negative type, respectively, with respect to B, and no subinterval of $(\operatorname{Re} a, \operatorname{Re} b)$ is of definite type.

We shall prove (i-iv) by verifying the following three assertions. In (i-iv) we did not consider all possible choices for a and b. But the following assertions apply also to the remaining cases.

Assertion 1. $\sigma_p(B) = \emptyset$.

Proof. By Lemma 3.1 it is sufficient to prove that, for every $\lambda \in \mathbf{C}$, $u \in \mathcal{D}(H_0)$, $L(\lambda)u = 0$ implies $u = 0$.

Let $u \in \mathcal{D}(H_0)$, $L(\lambda)u = 0$. Then

$$\begin{aligned} &\left\{-\frac{d^2}{dx^2} + m_a\right\} u(x) = 0 \text{ if } x \in (0, \infty), \\ &\left\{-\frac{d^2}{dx^2} + m_b\right\} u(x) = 0 \text{ if } x \in (-\infty, 0), \end{aligned} \tag{4.18}$$

where

$$\begin{aligned} m_a &:= m^2 - (\lambda^2 - 2(\operatorname{Re} a)\lambda + |a|^2), \\ m_b &:= m^2 - (\lambda^2 - 2(\operatorname{Re} b)\lambda + |b|^2). \end{aligned} \tag{4.19}$$

It is easy to see that if $m_a \leq 0$ or $m_b \leq 0$ then there is no nontrivial solution $u \in \mathcal{D}(H_0)$ of (4.18). Assume that $m_a,\, m_b \in \mathbf{C} \setminus (-\infty, 0]$. Let ν_a (ν_b) be the square root of m_a (resp. m_b) with negative (resp. positive) real part. Then a solution $u \in L^2(\mathbf{R})$ of (4.18) has the form

$$u(x) = \begin{cases} c_+ e^{\nu_a x} & \text{if} \quad x \in (0, \infty) \\ c_- e^{\nu_b x} & \text{if} \quad x \in (-\infty, 0). \end{cases}$$

But the only function u of this form which is continuous at 0 and has a continuous derivative at 0 is $u = 0$. This proves Assertion 1. □

Let $H_{0,+}$ ($H_{0,-}$) be the restriction of the maximal operator corresponding to $-\frac{d^2}{dx^2}$ in $L^2(\mathbf{R}_+)$ (resp. $L^2(\mathbf{R}_-)$) by the boundary condition $u(0) = 0$. $H_{0,+}$ and $H_{0,-}$ are selfadjoint operators whose spectra coincide with $[0, \infty)$.

Define an operator B_+ in $L^2(\mathbf{R}_+) \times L^2(\mathbf{R}_+)$ by

$$B_+ := \begin{pmatrix} a & 1 \\ H_{0,+} + m^2 & \bar{a} \end{pmatrix}$$

and an operator B_- in $L^2(\mathbf{R}_-) \times L^2(\mathbf{R}_-)$ by

$$B_- := \begin{pmatrix} b & 1 \\ H_{0,-} + m^2 & \bar{b} \end{pmatrix}.$$

Then by Lemma 3.1 $\lambda \in \sigma(B_+)$ if and only if $0 \in \sigma(H_{0,+} + m_a)$ or, equivalently, $m_a \in (-\infty, 0]$ (see (4.19)). The latter relation holds if and only if

$$\lambda \in \{\operatorname{Re} a \pm \sqrt{m^2 + \nu^2 - (\operatorname{Im} a)^2} :\, \nu \in [0, \infty)\},$$

where the square root may be nonreal. This implies the following: If $|\operatorname{Im} a| \leq m$, then

$$\sigma(B_+) = \mathbf{R} \cap \{z :\, |z - a| \geq m\}. \tag{4.20}$$

The spectrum of B_+ is real. If $|\operatorname{Im} a| > m$ then

$$\sigma(B_+) = \mathbf{R} \cup \{\operatorname{Re} a + it\sqrt{(\operatorname{Im} a)^2 - m^2} :\, t \in [-1, 1]\}. \tag{4.21}$$

In this case $\sigma(B_+)$ contains nonreal points.

Similar relations hold for B_+ and a replaced by B_- and b. By Lemma 3.1 we have $\sigma_p(B_+) = \sigma_p(B_-) = \emptyset$.

We consider the operator

$$C := \begin{pmatrix} a & 0 & 1 & 0 \\ 0 & b & 0 & 1 \\ H_{0,+} + m^2 & 0 & \bar{a} & 0 \\ 0 & H_{0,-} + m^2 & 0 & \bar{b} \end{pmatrix}$$

in $L^2(\mathbf{R}_+) \times L^2(\mathbf{R}_-) \times L^2(\mathbf{R}_+) \times L^2(\mathbf{R}_-)$. Since C is the direct product of B_+ and B_- we have $\sigma(C) = \sigma(B_+) \cup \sigma(B_-)$ and $\sigma_p(C) = \emptyset$.

<u>Assertion 2.</u> $\sigma(B) = \sigma(C)$.

Proof. Let $\lambda_0 \in \rho(B) \cap \rho(C)$ and let C_0 be the restriction of C to the linear set of all elements

$$(u_{1+}, u_{1-}, u_{2+}, u_{2-})^T \in \mathcal{D}(H_{0,+}) \times \mathcal{D}(H_{0,-}) \times L^2(\mathbf{R}_+) \times L^2(\mathbf{R}_-)$$

such that

$$\frac{d}{dx}\, u_{1+}(+0) - \frac{d}{dx}\, u_{1-}(-0) = 0.$$

Then B is an extension of C_0 and $(B-\lambda_0)^{-1}$ and $(C-\lambda_0)^{-1}$ coincide on the range of $C_0 - \lambda_0$ which has codimension one. This implies that

$$(B-\lambda_0)^{-1} - (C-\lambda_0)^{-1}$$

has rank one. Since every point of $\rho(C)$ belongs to a connected component of $\rho(C)$ containing points of $\rho(B)$, it follows by a well-known result on compact perturbations that $\sigma(B) \cap \rho(C) \subset \sigma_p(B)$. Then Assertion 1 gives $\rho(C) \subset \rho(B)$. Similarly, on account of $\sigma_p(C) = \emptyset$ we obtain $\rho(B) \subset \rho(C)$ and, hence, $\rho(B) = \rho(C)$, which proves Assertion 2. □

The relations (4.20), (4.21) (and similar ones for B_+ replaced by B_-) and Assertion 2 imply the description of $\sigma(B)$ in (ii) and (iv).

Assertion 3. A real open interval is of positive (negative) type with respect to B if and only if it is of positive (resp. negative) type with respect to C.

Proof. Let the open interval Δ be of positive type with respect to B. Suppose that there exists a point $t_0 \in \Delta$ such that no open neighbourhood of t_0 is of positive type with respect to C. Then since the difference of the resolvents of B and C is of rank one, on account of [8] t_0 is an eigenvalue of C, which contradicts $\sigma_p(C) = \emptyset$. We remark that under the assumptions of the present paper the discreteness of the nonreal spectrum assumed in the theorem in [8] plays no role in the proof of that theorem. A similar reasoning with B and C interchanged proves the assertion. □

By applying Theorem 3.3 to B_+ and B_- we can easily characterize the intervals of positive and negative type with respect to B_+ and B_- and, hence, with respect to C. Then Assertion 3 implies the sign properties of B mentioned in (iii) and (iv).

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Technische Universität Berlin
Fachbereich Mathematik
MA 6-4, Straße des 17. Juni 136
D-10623 Berlin, Germany
E-mail: `jonas@math.tu-berlin.de`

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