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OPERATOR REPRESENTATIONS OF N_∞^+–FUNCTIONS IN A MODEL KREIN SPACE L_σ^2

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To the memory of Branko Najman

Abstract. We introduce the class N_∞^+ of all complex functions Q such that $Q_+(z) := z \cdot Q(z)$ is a Nevanlinna function. If $0 \in D(Q)$ and $\lim_{y\to\infty} Q(iy) = 0$ we prove an integral representation $Q(z) := \int\limits_{-\infty}^{\infty} \frac{d\sigma(t)}{t-z}$ with a nonmonotonic function σ. If in particular Q_+ is an R_1-function we obtain an operator representation $Q(z) = [(\mathcal{A} - z)^{-1}F_-, F_-]_\sigma$ with a selfadjoint, nonnegative and boundedly invertible multiplication operator $\mathcal{A}$ in the model Krein space $(L_\sigma^2, [.,.]_\sigma)$ and an element $F_- \in L_\sigma^2$. The nonsingularity of the critical point infinity of $\mathcal{A}$ makes this representation unique up to a Krein space isomorphism.

1. Introduction

A Nevanlinna function is a complex function Q having an integral representation of the form

$$Q(z) = \alpha + \beta z + \int\limits_{-\infty}^{\infty} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) d\sigma(t) \tag{1.1}$$

where $\alpha \in \mathbf{R}$, $\beta \geq 0$ and σ is a nondecreasing function satisfying $\int\limits_{-\infty}^{\infty} \frac{d\sigma(t)}{1+t^2} < \infty$. In particular by definition Q belongs to the subclass R_1 of the class of Nevanlinna functions if $\beta = 0$ and $\int\limits_{-\infty}^{\infty} \frac{d\sigma(t)}{1+|t|} < \infty$ (see [KK1]). Now, if a Nevanlinna function Q satisfies $\lim\limits_{y\to\infty} \frac{Q(iy)}{y} = 0$ then for $z_0 \in D(Q)$, the

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domain of holomorphy of Q, the function Q has an operator representation of the form

$$Q(z) = s - i\,\mathrm{Im} z_0\,[v, v] + (z - \bar{z}_0)[(A - z_0)(A - z)^{-1}\ v, v] \tag{1.2}$$

where A is a selfadjoint operator in a Hilbert space $(K, [.,.])$ and $s \in \mathbf{R}$, $v \in K$ such that

$$K = \overline{\mathrm{span}\{(A - z)^{-1}v \mid z \in \mathbf{C}\backslash\mathbf{R}\}}. \tag{1.3}$$

This representation is unique up to an isomorphism. Note that a holomorphic function Q with $\mathbf{C}\backslash\mathbf{R} \subset D(Q)$ is a Nevanlinna function if and only if the Nevanlinna kernel

$$N_Q(z, \zeta) := \frac{Q(z) - \overline{Q(\zeta)}}{z - \overline{\zeta}} \qquad (z, \zeta \in D(Q))$$

is nonnegative, i.e. $\sum\limits_{i,j=1}^{n} N_Q(z_i, z_j)\zeta_i\overline{\zeta}_j \geq 0$ for all $n \in \mathbf{N}$ and all $z_1, ..., z_n \in D(Q)$, $\zeta_1, ..., \zeta_n \in \mathbf{C}$. Krein and Langer generalized the representations (1.1) and (1.2) to N_κ–functions ($\kappa \in \mathbf{N} \cup \{0\}$) allowing κ negative squares of the kernel $N_Q(z, \zeta)$ (see e.g. [KL]). In this case $(K, [.,.])$ is a Pontrjagin space of type Π_κ. In order to obtain representations by *nonnegative* selfadjoint operators in Π_κ the class N_κ^+ was introduced consisting of all N_κ–functions Q such that $Q_+(z) := z \cdot Q(z)$ is a Nevanlinna function (see e.g. [KL, §2]).

In the present note we drop the restrictions to the number of negative squares of the kernel $N_Q(z, \zeta)$ but still assume that $Q_+(z) = z \cdot Q(z)$ is a Nevanlinna function. We denote the class of all such functions Q by N_∞^+. For an N_∞^+–function Q with $0 \in D(Q)$ and $\lim\limits_{y\to\infty} Q(iy) = 0$ we obtain an integral representation

$$Q(z) = \int\limits_{-\infty}^{\infty} \frac{d\sigma(t)}{t - z} \qquad (z \in D(Q))$$

with a *nonmonotonic* function σ. Under the additional condition that Q_+ is an R_1–function we obtain an operator representation of the form

$$Q(z) = [(A - z)^{-1}v, v] \qquad (z \in D(Q))$$

with a selfadjoint, nonnegative and boundedly invertible operator A in a Krein space $(K, [.,.])$ and an element $v \in K$ such that (1.3) is satisfied. Moreover we find a representation of this kind such that infinity is not a singular critical point of A. According to Langer's theory of definitizable operators (see e.g. [L]) this means that the eigenspectral function of A in the Krein space $(K, [.,.])$ is bounded. Additionally we show a uniqueness result: All operator representations of Q such that infinity is not a singular critical point are unitarily equivalent.

Note that this result can also be deduced from general calculations of Jonas for "definitizable functions" recently developed in [J]. However, whereas Jonas' approach to operator representations uses an abstract representation theorem of Azizov [A] in this paper we present a direct construction using the integral representation of Q_+. In this way we obtain the operator representation by means of a multiplication operator in the model Krein space L^2_σ equipped with the inner product $[F,G]_\sigma := \int_{-\infty}^{\infty} F\bar{G}d\sigma$. An earlier approach to operator representations of holomorphic functions Q such that the kernel $N_Q(z,\zeta)$ may have an infinite number of negative squares was given by Dijksma, Langer and de Snoo in [DLS]. In that paper no restrictions to the function $z\cdot Q(z)$ were required. But then critical points could not be studied and uniqueness statements could only be formulated in a "weak" sense.

Functions $Q \in N_\infty^+$ appear e.g. as Titchmarsh–Weyl coefficients of indefinite Sturm–Liouville problems and more generally of indefinite Krein–Feller differential equations of the form $-D_mD_xf = zf$ (see [F1]).

2. Integral and operator representations

Let $Q : D(Q) \to \mathbf{C}$ be a holomorphic function defined on an open set $D(Q) \subset \mathbf{C}$ with $\mathbf{C}\backslash\mathbf{R} \subset D(Q)$ and assume that

$$(2.4)\quad Q(\bar{z}) = \overline{Q(z)} \qquad (z \in D(Q)),$$

$$(2.5)\quad Q_+(z) := z\cdot Q(z) \quad (z \in D(Q)) \quad \text{defines a Nevanlinna function.}$$

In analogy to the notation in [KL] we denote the class of all such functions Q by N_∞^+. From (2.5) we obtain the integral representation

$$(2.6)\quad Q_+(z) = \alpha + \beta z + \int_{-\infty}^{\infty} \left(\frac{1}{t-z} - \frac{t}{1+t^2}\right) d\sigma_+(t) \ (z \in D(Q_+) = D(Q))$$

(see e.g. [KK1]) where $\alpha \in \mathbf{R}$, $\beta \geq 0$ and σ_+ is a nondecreasing leftcontinuous function satisfying

$$(2.7)\qquad \int_{-\infty}^{\infty} \frac{d\sigma_+(t)}{1+t^2} < \infty.$$

If σ_+ is normalized by $\sigma_+(0) = 0$ then α, β and σ_+ are uniquely determined. In particular we have

$$\beta = \lim_{y\to\infty} \frac{Q_+(iy)}{iy}.$$

In the following we additionally assume that

$$(2.8)\qquad 0 \in D(Q),$$

$$(2.9)\qquad Q(iy) \to 0 \qquad (y \to \infty).$$

Then in the representation (2.6) we have $\beta = 0$ and σ_+ is constant in an open neighbourhood of 0. Consequently the integrals

$$\sigma(t) := \int_0^t \frac{d\sigma_+(s)}{s}, \qquad \sigma_-(t) := \int_0^t \frac{d\sigma_+(s)}{s^2} \qquad (t \in \mathbf{R}) \tag{2.10}$$

exist. Note that by (2.7) σ_- is a bounded function. Moreover, σ_+ and σ_- are nondecreasing whereas σ is nonincreasing on $(-\infty, 0)$ and nondecreasing on $(0, \infty)$. Since σ need not be of bounded variation in the following the integration with respect to σ will be understood in the sense of [F1, Appendix A]. Then a function $F : \mathbf{R} \to \mathbf{C}$ is integrable with respect to σ if and only if F is integrable with respect to the nondecreasing function

$$\|\sigma\|(t) := \int_0^t \frac{d\sigma_+(s)}{|s|} \qquad (t \in \mathbf{R})$$

which is called "the total variation of σ"(see [F1]). In this case we have

$$\int_{-\infty}^{\infty} F d\sigma = \int_{-\infty}^{\infty} F \cdot (\chi_{(0,\infty)} - \chi_{(-\infty,0)}) \; d\|\sigma\|.$$

Now writing equation (2.6) by means of σ we obtain

$$\begin{aligned} Q_+(z) &= \alpha + \int_{-\infty}^{\infty} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) t \; d\sigma(t) \\ &= \alpha + \int_{-\infty}^{\infty} \frac{d\sigma(t)}{1+t^2} + \int_{-\infty}^{\infty} \frac{z}{t-z} \; d\sigma(t) = z \cdot \int_{-\infty}^{\infty} \frac{d\sigma(t)}{t-z} \end{aligned} \tag{2.11}$$

since by (2.5) and (2.8) we have $Q_+(0) = 0$. This implies the integral representation

$$Q(z) = \int_{-\infty}^{\infty} \frac{d\sigma(t)}{t-z} \qquad (z \in D(Q)). \tag{2.12}$$

In order to interprete (2.11) and (2.12) as operator representations with multiplication operators we consider the model spaces $L^2_{\sigma_+}$, L^2_σ and $L^2_{\sigma_-}$ of all (equivalence classes of) functions $F : \mathbf{R} \to \mathbf{C}$ which are measurable and square integrable with respect to $\sigma_+, \|\sigma\|$ and σ_- respectively. We put

$$\begin{aligned}(F,G)_{\sigma_+} &:= \int_{-\infty}^{\infty} F\bar{G}d\sigma_+ \qquad (F,G\in L^2_{\sigma_+}),\\ (F,G)_{\sigma_-} &:= \int_{-\infty}^{\infty} F\bar{G}d\sigma_- \qquad (F,G\in L^2_{\sigma_-}),\\ [F,G]_{\sigma} &:= \int_{-\infty}^{\infty} F\bar{G}d\sigma, \qquad (F,G)_\sigma := \int_{-\infty}^{\infty} F\bar{G}d\|\sigma\| \qquad (F,G\in L^2_{\sigma}).\end{aligned}$$

Then $(L^2_{\sigma_+},(.,.)_{\sigma_+})$, $(L^2_{\sigma_-},(.,.)_{\sigma_-})$ and $(L^2_{\sigma},(.,.)_{\sigma})$ are Hilbert spaces and $(L^2_\sigma,[.,.]_\sigma)$ is a Krein space (see e.g. [F1, Appendix C]). Moreover

$$\mathcal{J}F := (\chi_{(0,\infty)} - \chi_{(-\infty,0)})\cdot F \qquad (F\in L^2_\sigma)$$

is a fundamental symmetry and $(.,.)_\sigma$ is the corresponding positive definite inner product. For $F\in L^2_{\sigma_+},\ G\in L^2_\sigma$ we have

$$\begin{aligned}(F,F)_{\sigma_+} &= \int_{-\infty}^{\infty} |F|^2 d\sigma_+ = \int_{-\infty}^{\infty} |F(t)|^2\ |t|\ d\|\sigma\|(t) \ge c\int_{-\infty}^{\infty} |F|^2\ d\|\sigma\|\\ (G,G)_{\sigma} &= \int_{-\infty}^{\infty} |G|^2\ d\|\sigma\| = \int_{-\infty}^{\infty} |G(t)|^2\ |t|\ d\sigma_-(t) \ge c\int_{-\infty}^{\infty} |G|^2\ d\sigma_-\end{aligned}$$

with a constant $c>0$. Therefore we obtain

$$L^2_{\sigma_+}\subset L^2_\sigma \subset L^2_{\sigma_-} \tag{2.13}$$

and both imbeddings are continuous. Moreover with the set of all step functions also $L^2_{\sigma_+}$ is dense in L^2_σ and L^2_σ is dense in $L^2_{\sigma_-}$. In the space triplet (2.13) we consider the operators of multiplication with the function $a(t) := t$ ($t\in\mathbf{R}$), i.e. the operators $\mathcal{A}_+,\ \mathcal{A}$ and $\mathcal{A}_-$, given by

$$D(\mathcal{A}_\pm) := \{F\in L^2_{\sigma_\pm}\mid a\cdot F\in L^2_{\sigma_\pm}\}, \qquad \mathcal{A}_\pm F := a\cdot F. \tag{2.14}$$

Then in particular we have $\mathcal{J}\mathcal{A}F = |a|F$ ($F\in D(\mathcal{A})$). Therefore the operators $\mathcal{A}_+,\ \mathcal{J}\mathcal{A}$ and $\mathcal{A}_-$ are selfadjoint in the corresponding Hilbert spaces $(L^2_{\sigma(\pm)},\ (.,.)_{\sigma(\pm)})$ and $\mathcal{J}\mathcal{A}$ is nonnegative. Consequently $\mathcal{A}$ is selfadjoint and nonnegative in the Krein space $(L^2_\sigma,\ [.,.]_\sigma)$. Moreover, since $\sigma_+,\ \sigma$ and σ_- are constant in an open neighbourhood of 0 the operators $\mathcal{A}_+,\ \mathcal{A}$ and $\mathcal{A}_-$ are boundedly invertible. Therefore $\mathcal{A}$ is definitizable with $\sigma(A)\subset\mathbf{R}\backslash\{0\}$ and infinity is the only possible critical point of $\mathcal{A}$ (see [L]).

PROPOSITION 2.1. *Infinity is not a singular critical point of $\mathcal{A}$.*

Proof. If $|\mathcal{A}_-|$ denotes the absolute value of the operator $\mathcal{A}_-$ in $(L^2_{\sigma_-}, (.,.)_{\sigma_-})$ then we have $|\mathcal{A}_-|F = |a| \cdot F \quad (F \in D(\mathcal{A}_-))$. This implies $D(|\mathcal{A}_-|^{\frac{1}{2}}) = \{F \in L^2_{\sigma_-} \mid |a|^{\frac{1}{2}} F \in L^2_{\sigma_-}\} = L^2_{\sigma}$. Then the proposition follows e.g. from [F2, Theorem 3]. ∎

Further properties of the space triplet and of the multiplication operators can be adapted from [F1, Section 4.4].

Now, in order to return to representations of Q_+ and Q we consider the functions

$$F_+(t) := \frac{1}{t}\ , \quad F_-(t) := 1 \qquad (t \in \mathbf{R}).$$

Then by (2.7) we have $F_+ \in L^2_{\sigma_+},\ F_- \in L^2_{\sigma_-}$. Therefore (2.11) can be written as

$$Q_+(t) = z \cdot \int_{-\infty}^{\infty} \frac{t}{t-z}\ \frac{d\sigma_+(t)}{t^2}\ = z \cdot (\mathcal{A}_+(\mathcal{A}_+ - z)^{-1}\ F_+, F_+)_{\sigma_+}$$

which is an operator representation of Q_+ according to [KL, Satz 1.4]. A representation of Q by means of $\mathcal{A}$ and F_- can be obtained from (2.12) whenever $F_- \in L^2_{\sigma}$. This is true if and only if σ or $\|\sigma\|$, respectively, is bounded. Further this is equivalent to

$$\int_{-\infty}^{\infty} \frac{d\sigma_+(t)}{|t|}\ < \infty, \tag{2.15}$$

i.e. Q_+ is an R_1–function. Now under condition (2.15) we obtain from (2.12)

$$Q(z) = [(\mathcal{A} - z)^{-1}\ F_-,\ F_-]_{\sigma}. \tag{2.16}$$

Note that without condition (2.15) equation (2.12) can also be interpreted as an operator representation of the form (2.16) with $\mathcal{A}$ replaced by $\mathcal{A}_-$ and with $[\,.\,,\,.\,]_{\sigma}$ interpreted as the duality between $L^2_{\sigma_+}$ and $L^2_{\sigma_-}$.

Lemma 2.2. *If Q_+ is an R_1-function the set* $\operatorname{span}\{(\mathcal{A}-z)^{-1}\ F_- | z \in \mathbf{C}\backslash\mathbf{R}\}$ *is dense in $(L^2_{\sigma}, (.,.)_{\sigma})$.*

Proof. Let $F \in L^2_{\sigma}$ such that $0 = ((\mathcal{A} - z)^{-1}\ F_-,\ F)_{\sigma} = \int_{-\infty}^{\infty} \frac{\bar{F}}{t-z}\ d\|\sigma\|$ for all $z \in \mathbf{C}\backslash\mathbf{R}$. If F is real and nonnegative then by $\mu(t) := \int_0^t F d\|\sigma\|$ $(t \in \mathbf{R})$ a bounded leftcontinuous nondecreasing function is defined with $\int_{-\infty}^{\infty} \frac{d\mu(t)}{t-z} = 0 \quad (z \in \mathbf{C}\backslash\mathbf{R})$. Then the inversion formula of Stieltjes–Livšic (see e.g. [F1, Proposition A.10]) applied to μ yields $0 = \int_{\Delta} d\mu = \int_{\Delta} F\ d\|\sigma\|$ for all bounded intervals Δ. For an arbitrary $F \in L^2_{\sigma}$ consider a decomposition $\bar{F} = F_1 - F_2 - i(F_3 - F_4)$ with nonnegative functions $F_1, F_2, F_3, F_4 \in L^2_{\sigma}$ and apply the inversion formula for the four functions separately. Then again we obtain $\int\limits_\Delta \bar F \, d\|\sigma\| = 0$ for all bounded intervals Δ. This implies $F = 0$ $\|\sigma\|$-a.e. □

Summing up we have proved

THEOREM 2.3. *Let $Q \in N_\infty^+$ with $0 \in D(Q)$ such that $Q_+(z) = z \cdot Q(z)$ $\quad (z \in D(Q))$ is an R_1–function (which implies (2.9)). Then with the function σ, defined in (2.10), we obtain the representations*

$$Q(z) = \int_{-\infty}^{\infty} \frac{d\sigma(t)}{t-z} = [(\mathcal{A}-z)^{-1}F_-, F_-]_\sigma \quad (z \in D(Q) = \rho(\mathcal{A}))$$

where $\mathcal{A}$ is the selfadjoint nonnegative and boundedly invertible multiplication operator in the Krein space $(L^2_\sigma, [.,.]_\sigma)$, given by (2.14), and $F_- = 1 \in L^2_\sigma$. Moreover, it holds that

$$L^2_\sigma = \overline{\operatorname{span}\{(\mathcal{A}-z)^{-1}F_- | z \in \mathbf{C}\backslash\mathbf{R}\}}$$

and infinity is not a singular critical point of $\mathcal{A}$. □

A linear bijective mapping $\mathcal{F}$ from one Kreine space $(K_1, [.,.]_1)$ to another Krein space $(K_2, [.,.]_2)$ is called a Krein space isomorphism if $[\mathcal{F}f, \mathcal{F}g]_2 = [f,g]_1$ for all $f, g \in K_1$. With this definition we conversely obtain

THEOREM 2.4. *Let Q satisfy the same conditions as in Theorem 2.3 and let σ, $\mathcal{A}$ and F_- be given as in Theorem 2.3.*

(i) Assume that $\tau : \mathbf{R} \to \mathbf{R}$ is a leftcontinuous function of bounded variation normalized by $\tau(0) = 0$ such that

$$Q(z) = \int_{-\infty}^{\infty} \frac{d\tau(t)}{t-z} \qquad (z \in \mathbf{C}\backslash\mathbf{R}).$$

Then $\tau = \sigma$.

(ii) Assume that $(K, [.,.])$ is a Krein space, A is a selfadjoint, nonnegative and boundedly invertible operator in $(K, [.,.])$ and $f_- \in K$ such that

$$Q(z) \;=\; [(A-z)^{-1}f_-, f_-] \qquad (z \in \mathbf{C}\backslash\mathbf{R}),$$

$$K \;=\; \overline{\operatorname{span}\{(A-z)^{-1}f_- | z \in \mathbf{C}\backslash\mathbf{R}\}}. \tag{2.17}$$

Then infinity is not a singular critical point of A if and only if A and $\mathcal{A}$ are unitarily equivalent, i.e.

$$A = \mathcal{F}^{-1}\, \mathcal{A}\mathcal{F}$$

with a Krein space isomorphism $\mathcal{F}$ from $(K, [.,.])$ to $(L^2_\sigma, [.,.]_\sigma)$. In this case the isomorphism $\mathcal{F}$ can be chosen such that

$$\mathcal{F}(f_-) = F_-.$$

Proof. (i) Consider decompositions $\tau = \tau^+ - \tau^-$, $\sigma = \sigma^+ - \sigma^-$ with bounded leftcontinuous nondecreasing functions $\tau^+, \tau^-, \sigma^+, \sigma^- : \mathbf{R} \to \mathbf{R}$ (see e.g. [F1, Appendix A]). Then the statement follows from the inversion formula of Stieltjes–Livšic (see e.g. [F1, Proposition A.10]) applied to $\tau^+, \tau^-, \sigma^+, \sigma^-$ separately. Here we use the fact that

$$\int_{-\infty}^{\infty} \frac{d\tau^+(t)}{t-z} - \int_{-\infty}^{\infty} \frac{d\tau^-(t)}{t-z} = \int_{-\infty}^{\infty} \frac{d\sigma^+(t)}{t-z} - \int_{-\infty}^{\infty} \frac{d\sigma^-(t)}{t-z} \quad (z \in \mathbf{C}\backslash\mathbf{R})$$

(ii) If A and $\mathcal{A}$ are unitarily equivalent then infinity is not a singular critical point of A since a Krein space isomorphism preserves this property.

Now assume that infinity is not a singular critical point of A. In order to construct the required isomorphism $\mathcal{F} : K \to L^2_\sigma$ we consider the inner product

$$\{f, g\}_- := [A^{-1}f, g] \qquad (f, g \in K). \tag{2.18}$$

If J is a fundamental symmetry of $(K, [.,.])$ and $(f, g) = [Jf, g] \quad (f, g \in K)$ is the corresponding positive definite inner product then JA is selfadjoint, nonnegative and boundedly invertible in $(K, (.,.))$. Therefore for $f \in K$ we have

$$\{f, f\}_- = (A^{-1}f, JA\, A^{-1}f) \geq c \cdot (A^{-1}f, A^{-1}f), \tag{2.19}$$

$$\{f, f\}_-^2 = [A^{-1}f, f]^2 \leq (A^{-1}f, A^{-1}f) \cdot (f, f) \leq d \cdot (f, f)^2 \tag{2.20}$$

with some constants $c, d > 0$. Consequently $\{.,.\}_-$ is a positive definite inner product and $(K, (.,.))$ is continuously imbedded in the completion of K with respect to $\{.,.\}_-$ which will be denoted by $(K_-, \{.,.\}_-)$. From (2.18), (2.19), (2.20) we obtain for $f, g \in K$

$$\{A^{-1}f,\ g\}_- = [A^{-1}f,\ A^{-1}g] = \{f,\ A^{-1}g\}_-,$$

$$\{A^{-1}f,\ A^{-1}f\}_- \leq \sqrt{d}\,(A^{-1}f,\ A^{-1}f) \leq \frac{\sqrt{d}}{c}\,\{f, f\}_-.$$

Therefore A^{-1} can be uniqueley extended to a bounded selfadjoint and injective operator B_- in $(K_-, \{.,.\}_-)$. Then $A_- := B_-^{-1}$ is a selfadjoint and boundedly invertible extension of the operator A in $(K_-, \{.,.\}_-)$. Moreover, estimate (2.19) implies $D(A_-) = R(B_-) \subset K$ and equation (2.18) extends to $\{f, g\}_- = [A_-^{-1}f,\ g]$ for all $f \in K_-, g \in K$. Consequently for all $f \in D(A_-), g \in K$ we have $\{A_-f,\ g\}_- = [A_-^{-1}\ A_-f,\ g] = [f, g]$ and hence, the operator A_- is associated to the sesquilinear form $[.,.]$ in the sense of the First Representation Theorem [F2, Theorem 1]. Then by the Second Representation Theorem [F2, Theorem 3] the nonsingularity of the critical point infinity of A implies $K = D(|A_-|^{\frac{1}{2}})$ where $|A_-|$ denotes the absolute value of the operator A_- in $(K_-, \{.,.\}_-)$. Moreover, from [F2, Theorem 3] it follows that $(.,.)$ and $\{|A_-|^{\frac{1}{2}}\,.\,, |A_-|^{\frac{1}{2}}\,.\,\}_-$ induce equivalent norms on $D(|A_-|^{\frac{1}{2}})$.

Now let E_- denote the leftcontinuous resolution of the identitiy of A_- in $(K_-, \{.,.\}_-)$. Then we have

$$\begin{aligned} Q(z) &= [(A-z)^{-1}f_-,\ f_-] = \{A_-(A_- - z)^{-1}f_-,\ f_-\}_- \\ &= \int\limits_{-\infty}^{\infty} \frac{t}{t-z}\, d\{E_-(t)f_-, f_-\}_- = \int\limits_{-\infty}^{\infty} \frac{d\tau(t)}{t-z} \qquad (z \in \mathbf{C}\backslash\mathbf{R}) \end{aligned}$$

where $\tau(t) := \int\limits_0^t s\, d\{E_-(s)f_-, f_-\}_-$ $(t \in \mathbf{R})$. Since $f_- \in D(|A_-|^{\frac{1}{2}})$ it holds that
$\int\limits_{-\infty}^{\infty} |s| d\{E_-(s)f_-,\ f_-\}_- < \infty$ and hence τ is of bounded variation. Then (i) implies $\tau = \sigma$ and consequently $\sigma_-(t) = \int\limits_0^t \frac{d\sigma(s)}{s} = \{E_-(t)f_-,\ f_-\}_- - \{E_-(0)f_-,\ f_-\}_-$ $(t \in \mathbf{R})$. From property (2.17) it follows that A_- has simple spectrum with generating element f_- in the sense of [AG, Nr. 83]. This means that span$\{E_-(\Delta)f_- \mid \Delta \subset \mathbf{R},\ \Delta \text{ interval}\}$ is dense in $(K_-, \{.,.\}_-)$. Indeed let $f \in K_-$ with $\{E_-(\Delta)f_-,\ f\}_- = 0$ for all intervals Δ. Then we also have $\{E_-(t)f_-,\ f\}_- = 0$ for all $t \in \mathbf{R}$ and hence $\{(A-z)^{-1}f_-,\ f\}_- = \int\limits_{-\infty}^{\infty} \frac{1}{t-z}\, d\{E_-(t)f_-,\ f\}_- = 0$. By (2.17) this implies $\{g, f\}_- = 0$ for all $g \in K$ and consequently $f = 0$. Therefore, by [AG, Nr. 83, Satz 2] the formula

$$\Phi(F) := \int\limits_{-\infty}^{\infty} F(t) dE_-(t) f_- \qquad (F \in L^2_{\sigma_-})$$

defines an isometric and bijective linear mapping Φ from $(L^2_{\sigma_-}, (.,.)_{\sigma_-})$ to $(K_-, \{.,.\}_-)$ such that $A_- = \Phi\ \mathcal{A}_- \Phi^{-1}$. Consequently with A_- and $\mathcal{A}_-$ also the operators $|A_-|^{\frac{1}{2}}$ and $|\mathcal{A}_-|^{\frac{1}{2}}$ are unitarily equivalent (by means of Φ), i.e. $|A_-|^{\frac{1}{2}} = \Phi\ |\mathcal{A}_-|^{\frac{1}{2}} \Phi^{-1}$. In particular this implies

$$K = D(|A_-|^{\frac{1}{2}}) = \Phi(D(|\mathcal{A}_-|^{\frac{1}{2}})) = \Phi(L^2_\sigma)$$

(compare the proof of Proposition 1). Then we obtain

$$A = A_-|_{A_-^{-1}(K)} = \Phi \mathcal{A}_-|_{\mathcal{A}_-^{-1}(L^2_\sigma)} \Phi^{-1} = \Phi\ \mathcal{A}\Phi^{-1}.$$

Moreover, Φ is an isomorphism between the Hilbert spaces $(L^2_\sigma, (.,.)_\sigma)$ and $(K, \{|A_-|^{\frac{1}{2}}\, .\, ,\ |A_-|^{\frac{1}{2}}\, .\, \}_-)$ since for $F,\ G \in L^2_\sigma$ we have

$$\{|A_-|^{\frac12}\Phi F,\ |A_-|^{\frac12}\Phi G\}_- = (|\mathcal{A}_-|^{\frac12}F, |\mathcal{A}_-|^{\frac12}G)_{\sigma_-} = \int_{-\infty}^{\infty} F(t)\overline{G(t)}|t|\,d\sigma_-(t) = (F,G)_\sigma .$$

Therefore the equation

$$[F,G]_\sigma = \int_{-\infty}^{\infty} F(t)\overline{G(t)}t\,d\sigma_-(t) = (\mathcal{A}_-F,G)_{\sigma_-} = \{A_-\Phi F, \Phi G\}_- = [\Phi F, \Phi G]$$

which is valid for $F \in D(\mathcal{A}_-),\ G \in L^2_\sigma$ extends by continuity to all $F, G \in L^2_\sigma$. Then the statement follows with $\mathcal{F} := \Phi^{-1}$. In particular note that $\Phi(F_-) = \int_{-\infty}^{\infty} dE_-(t)f_- = f_-$. ∎

Now let us consider some examples. First let $Q(z) := \sin^2\alpha(\cot\alpha - \Omega(z))\quad (z \in D(\Omega))$ where $\alpha \in (0, \frac{\pi}{2})$ and $\Omega(z)$ is the Titchmarsh-Weyl coefficient of the indefinite Sturm–Liouville problem

(2.21)

$$\begin{cases} -f'' &= z\,\mathrm{sgn}(x-x_0)|x-x_0|^\nu\ f \qquad \text{on } [0,1],\\ \sin\alpha f'(0) - \cos\alpha f(0) &= z\,\sin\alpha f(0),\\ f'(1) &= z\,f(1) \end{cases}$$

with $x_0 \in (0,1)$ and $\nu > -1$. Then Q satisfies the conditions of Theorem 2.3. Moreover the operator realization of problem (2.21) in the Krein space $L^2_\rho \times \mathbf{C}^2$ with $\rho(x) = \mathrm{sgn}(x-x_0)|x-x_0|^\nu$ allows an operator representation of Q as described in Theorem 2.4 (ii) with $f_- := (0,-1,0)$ and infinity is not a singular critical point. In this case $\mathcal{F}$ can be chosen as a generalized Fourier transformation. This is described in detail in [F1]. In particular the statements mentioned above follow from [F1, Example 2.4, Example 2.15, Proposition 4.2, Section 4.2, Corollary 3.9, Theorem 4.20]. According to [F1] the example of problem (2.21) can be generalized to certain indefinite Krein–Feller differential equations of the form $-D_mD_xf = zf$ where the function m starts with a jump at the left endpoint. Of this form is also the following example.

3. A Counterexample

In this section we will see that in Theorem 2.4 the operators A and $\mathcal{A}$ need not necessarily be unitarily equivalent. For $n \in \mathbf{N}$ let $m_n > 0$ such that $\sum_{n=1}^{\infty} m_n < \infty$ and put

$$t_n := \frac{1}{n!}\ ,\ m_{-n} := -m_n\ ,\ t_{-n} := -t_n \qquad (n \in \mathbf{N}).$$

Then with an $\alpha \in (0, \frac{\pi}{2})$ we consider for $z \in \mathbf{C}$ the difference equation system

(3.22)
$$\frac{(t_{n+1}-t_n)f_{n-1}-(t_{n+1}-t_{n-1})f_n+(t_n-t_{n-1})f_{n+1}}{m_n(t_{n-1}-t_n)(t_n-t_{n+1})} = z\ f_n \qquad (n \in \mathbf{Z}\backslash\{-1,0,1\}),$$

(3.23)
$$\frac{1}{m_1}\ \frac{f_1-f_2}{t_1-t_2} \quad = \quad z\ f_1,$$

(3.24)
$$\frac{1}{m_{-1}}(\cot\alpha f_{-1}-\frac{f_{-2}-f_{-1}}{t_{-2}-t_{-1}}) \quad = \quad z\ f_{-1}+\frac{1}{m_{-1}}$$

equipped with the "interface conditions"

(3.25)
$$\lim_{n\to\infty} f_n = \lim_{n\to-\infty} f_n, \qquad \lim_{n\to\infty}\frac{f_n-f_{n+1}}{t_n-t_{n+1}} = \lim_{n\to-\infty}\frac{f_n-f_{n+1}}{t_n-t_{n+1}}.$$

Problems of this kind are studied in detail in [F1]. In the following we will present some consequences of [F1, Example 2.5, Example 2.16, Proposition 4.2, Section 4.2]: For all $z \in \mathbf{C}\backslash\mathbf{R}$ and all z in an open neighbourhood of 0 this system has a unique solution $(f_n^z)_{n\in\mathbf{Z}\backslash\{0\}}$. For that solution we put

$$Q(z) := f_{-1}^z.$$

This function Q is holomorphic and satisfies (2.4), (2.5). Moreover in the integral representation (2.6) of $Q_+(z) = z\cdot Q(z)$ we have $\beta = 0$ and $\int_{-\infty}^{\infty} d\sigma_+ < \infty$. Then Q_+ is an R_1–function and hence Q satisfies the assumptions of Theorem 2.3. Now by

$$l^2(m_n) \quad := \quad \{(f_n)_{n\in\mathbf{Z}\backslash\{0\}} | \sum_{n\in\mathbf{Z}\backslash\{0\}} |f_n|^2\ |m_n| < \infty\},$$

$$[f,g] \quad := \quad \sum_{n\in\mathbf{Z}\backslash\{0\}} f_n\overline{g_n}m_n \qquad (f=(f_n),\ g=(g_n)\in l^2(m_n))$$

we define the Krein space $(l^2(m_n), [\,.\,,\,.\,])$. In this Krein space we consider the operator A with domain

$$D(A) := \{\ f=(f_n)\in l^2(m_n) \quad |\ f \text{ satisfies (3.25) },$$
$$\sum_{n\in\mathbf{Z}\backslash\{-1,0,1\}} \frac{|(t_{n+1}-t_n)f_{n-1}-(t_{n+1}-t_{n-1})f_n+(t_n-t_{n-1})f_{n+1}|^2}{|m_n|(t_{n-1}-t_n)^2(t_n-t_{n+1})^2} < \infty\}$$

and defined for each coordinate $n \in \mathbf{Z}\backslash\{0\}$ by the expression on the left hand side of formula (3.22) if $n \in \mathbf{Z}\backslash\{-1,0,1\}$ and of formula (3.23) if $n = 1$ and of formula (3.24) if $n = -1$. Then A is selfadjoint, nonnegative and boundedly invertible in $l^2(m_n)$. Moreover for the sequence $\delta = (\delta_n)_{n\in\mathbf{Z}\setminus\{0\}}$ with $\delta_{-1} = \frac{1}{m_{-1}}$, $\delta_n = 0$ $(n \in \mathbf{Z}\setminus\{-1, 0\})$ we have

$$Q(z) = [(A - z)^{-1}\delta, \delta] \qquad (z \in D(Q)),$$

$$l^2(m_n) = \overline{\text{span}\{(A - z)^{-1}\delta | z \in \mathbf{C}\setminus\mathbf{R}\}}.$$

Therefore A satisfies the assumptions of Theorem 2.4 (ii). However by [F1, Theorem 3.12] infinity is a singular critical point of A and hence the model operators A and $\mathcal{A}$ of Q are not unitarily equivalent. A more abstract "counterexample" of this kind can be found in [J, Section 2.3].

Finally the author wants to rise the following question: "Does there exist a class of (second order Krein-Feller type) differential operators, such that the functions of the class N_∞^+ are the Titchmarsh-Weyl coefficients of this class?" In the definite case of the class N_0^+ according to Krein's inverse spectral theory we can take a class of differential operators determined by vibrating strings on the positive half axis (see e.g. [KK2, Theorem 11.2]). Recently in [LW] this class of strings was generalized in order to obtain all N_κ^+–functions ($\kappa \geq 1$). However the N_∞^+–situation seems still to be open.

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