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A TRANSMISSION PROBLEM FOR ELLIPTIC EQUATIONS INVOLVING A PARAMETER AND A WEIGHT

Melvin Faierman
University of the Witwatersrand, South Africa

Dedicated to the memory of Branko Najman

Abstract. In the course of developing a spectral theory for non–selfadjoint elliptic problems involving an indefinite weight function, there arises a transmission problem which has not previously been dealt with. By reducing our problem to one for ordinary differential equations with the aid of the Fourier transformation, we are able to resolve the problem and to establish a priori estimates for its solutions which we require for the further development of the theory.

1. Introduction

A particular problem which arises in the course of developing a spectral theory for non–selfadjoint elliptic boundary value problems involving an indefinite weight function is that of obtaining a priori estimates in a neighbourhood of the origin in $\mathbb{R}^n$, $n \geq 2$, for solutions of equations of the form

$$\begin{aligned} L^{(1)}(x,D)u - q^{2m}\omega(x)u &= f_1 \quad \text{in} \quad \mathbb{R}^n_+, \\ L^{(2)}(x,D)u &= f_2 \quad \text{in} \quad \mathbb{R}^n_-. \end{aligned} \tag{1.1}$$

Here $x = (x_1, \dots, x_n) = (x', x_n)$ denotes a generic point in $\mathbb{R}^n$, $D = (D_1, \dots, D_n)$, $D_j = \partial/\partial x_j$ for $j = 1, \dots, n$, $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n \mid x_n > 0\}$, $\mathbb{R}^n_- = \{x \in \mathbb{R}^n \mid x_n < 0\}$, $L^{(1)}$ (resp. $L^{(2)}$) is a linear differential operator of order $2m$ defined in $\overline{\mathbb{R}^n_+}$ (resp. $\overline{\mathbb{R}^n_-}$), where $^-$ denotes closure, q is a complex parameter varying in a closed sector Σ of $\mathbb{C}$ with vertex at the origin, and $\omega(x)$ is a real–valued function in $L^\infty(\mathbb{R}^n_+)$ such that ω is uniformly continuous and

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$|\omega(x)|$ has a positive infimum in that subset of $\mathbb{R}^n_+$ for which $|x| < 1$. Our assumptions concerning (1.1) will be made precise in §2.

For $j = 1, 2$, let $L_0^{(j)}(x, D)$ denote the principal part of $L^{(j)}(x, D)$, let $L_{00}^{(j)}(D) = L_0^{(j)}(0, D)$, and let ω_0 denote the limit as $x \to 0$, $x \in \mathbb{R}^n_+$, of $\omega(x)$. Then by appealing to a well known method, we can reduce our problem concerning (1.1) to that of obtaining a priori estimates for solutions of equations of the form

$$\begin{aligned} L_{00}^{(1)}(D)u - q^{2m}\omega_0 u = f_1 \quad \text{in} \quad \mathbb{R}^n_+, \\ L_{00}^{(2)}(D)u = f_2 \quad \text{in} \quad \mathbb{R}^n_-. \end{aligned} \tag{1.2}$$

Furthermore, by employing the method of [1] (this involves eliminating the parameter q by introducing a new space variable t), we can reduce our problem concerning (1.2) to a more standard one, namely to that of obtaining a priori estimates for solutions of equations of the form

$$\begin{aligned} A_1 u = f_1 \quad \text{in} \quad \mathbb{R}^{n+1}_+, \\ A_2 u = f_2 \quad \text{in} \quad \mathbb{R}^{n+1}_-, \end{aligned} \tag{1.3}$$

where $A_1 = A_1(D, D_t) = L_{00}^{(1)}(D) - (-1)^m \omega_0 e^{i\theta} D_t^{2m}$, $D_t = \partial/\partial t$, $\theta = \arg q^{2m}$, $A_2(D) = L_{00}^{(2)}(D)$, $\mathbb{R}^{n+1}_+ = \{(x,t) \in \mathbb{R}^{n+1} \mid x_n > 0\}$, and $\mathbb{R}^{n+1}_- = \{(x,t) \in \mathbb{R}^{n+1} \mid x_n < 0\}$. Hence, since in the sequel we shall impose conditions which ensure that A_1 and A_2 are elliptic, and since transmission problems for elliptic equations have been the subject of much investigation (we refer to [5] and [20] for the relevant references), it is very tempting at this stage to subsume (1.3) by the more general transmission problem

$$\begin{aligned} A_1 u_1 = f_1 \quad \text{in} \quad \mathbb{R}^{n+1}_+, \\ A_2 u_2 = f_2 \quad \text{in} \quad \mathbb{R}^{n+1}_-, \end{aligned} \tag{1.4}$$

$$D_n^j u_1 - D_n^j u_2 = 0 \quad \text{on} \quad x_n = 0 \quad \text{for} \quad j = 0, \ldots, (2m-1),$$

and then arrive at the required a priori estimates by appealing to the literature. Unfortunately, though, the problem (1.4) falls outside the scope of the investigations cited above. Indeed, to clarify this last statement, let us remark that the usual method for obtaining a priori estimates for solutions of (1.4) (see [20]) is to map the closure of $\mathbb{R}^{n+1}_-$ onto the closure of $\mathbb{R}^{n+1}_+$, and in this way reduce the problem (1.4) to a boundary value problem for an elliptic system acting in $\mathbb{R}^{n+1}_+$; the required a priori estimates can then be directly obtained from [4]. However, if one applies this method to the problem (1.4), then one arrives at a system which is not elliptic, and hence the results of [4] cannot be used. Thus in order to resolve our problem, new methods must be introduced.

One finds in the literature many papers devoted to the spectral theory for general selfadjoint elliptic boundary value problems involving an indefinite weight function (cf. [14], [15], and [19]). However for non–selfadjoint problems, most of the literature to date is restricted to the case where the elliptic operator involved is of the second order, i.e., $m = 1$ (cf. [9], [10], [11], and [13]). To show the connection of our work with the works just cited, let us point out that for the case $m = 1$, a priori estimates for solutions of a less general problem than (1.1) were established in [9] and these were used in [10], [11], and [13] to derive information concerning the completeness of the principal vectors in certain function spaces and the angular and asymptotic distribution of the eigenvalues of the operator induced in an appropriate Hilbert space by the non–selfadjoint elliptic problem. Analogous results were also derived in [6], [8], and [12] for the case $m > 1$, but only under the assumption that the reciprocal of the weight function was essentially bounded in the space concerned. Hence as a consequence of the a priori estimates established in this paper, we are now able to extend these results to the case where the weight function vanishes on a set of positive measure, and indeed this topic will be treated in a subsequent paper.

In §2 of this paper we introduce some further assumptions and state our main result (see Theorem 2.1). §3 is devoted to some technical results which we require for the proof of Theorem 2.1 and these are used in §4 to prove the theorem.

2. The main result

In this section we are going to introduce some further assumptions concerning (1.1) and state the main result of this paper (see Theorem 2.1 below). Accordingly, in conjunction with the notation given in §1, we introduce the further notation $D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$, where $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi–index whose length $\sum_{j=1}^n \alpha_j$ is denoted by $|\alpha|$. Furthermore, for $0 \le s < \infty$ and G an open set in $\mathbb{R}^p (p \in \mathbb{N})$, we let $H^s(G)$ denote the usual Sobolev–Slobodeckii space of order s related to $L^2(G)$ and denote by $(. , .)_{s,G}$ and $\| \|_{s,G}$ the inner product and norm, respectively, in this space, while for $s < 0$, we let $\| \|_{s,\mathbb{R}^{n-1}}$ denote the norm in the Bessel–potential space $H_2^s(\mathbb{R}^{n-1})$ (see [21, p.177]). Also for $d > 0$ we let B_d denote the open ball in $\mathbb{R}^n$ with centre at the origin and radius d. Turning now to (1.1), we henceforth suppose that

Assumption 2.1.

(1) $L^{(1)}(x,D) = \sum_{|\alpha|\le 2m} a_\alpha^{(1)}(x)D^\alpha$, where the $a_\alpha^{(1)}(x)$ are complex–valued functions in $L^\infty(\overline{\mathbb{R}^n_+})$ such that $a_\alpha^{(1)} \in C^0(\overline{\mathbb{R}^n_+}\cap B_1)$ for $|\alpha| = 2m$;

(2) $L^{(2)}(x,D) = \sum_{|\alpha|\le 2m} a_\alpha^{(2)}(x)D^\alpha$, where the $a_\alpha^{(2)}(x)$ are complex–valued functions in $L^\infty(\overline{\mathbb{R}^n_-})$ such that $a_\alpha^{(2)}$ is of class $C^m(\overline{\mathbb{R}^n_-}\cap B_1)$ if $m < |\alpha| < 2m$ and of class $C^{2m}(\overline{\mathbb{R}^n_-}\cap B_1)$ if $|\alpha| = 2m$;

(3) $(-1)^m L_{00}^{(1)}(\xi) - \omega_0 q^{2m} \neq 0$ for all $\xi = (\xi_1,\dots,\xi_n) = (\xi',\xi_n) \in \mathbb{R}^n$ and $q \in \Sigma$ satisfying $|\xi| + |q| \neq 0$;

(4) $L^{(2)}(0,D)$ is properly elliptic.

For $u \in H^{2m}(\mathbb{R}^n_+)$ and $q \in \Sigma\backslash\{0\}$ let us introduce the norm

$$|||u|||_q^+ = \|u\|_{2m,\mathbb{R}^n_+} + |q|^{2m}\|u\|_{0,\mathbb{R}^n_+},$$

while for $u \in H^{2m}(\mathbb{R}^n_-)$ and $q \in \Sigma\backslash\{0\}$ we introduce the norm

$$\begin{aligned} |||u|||_q^- &= \|u\|_{2m,\mathbb{R}^n_-} + |q|^{m-1/2}\|u\|_{m+1/2,\mathbb{R}^n_-} \\ &\quad + |q|^{m-1/2}\left(\int_{-\infty}^{0} \left\|(D_n^{2m}u)(\,.\,,x_n)\right\|^2_{-(m-1/2),\mathbb{R}^{n-1}} dx_n\right)^{1/2}. \end{aligned}$$

Then our main result is contained in the following theorem (here we write *supp* for support).

Theorem 2.1. *Given any $\epsilon > 0$ there exists a $\delta > 0$ such that for any $u \in H^{2m}(\mathbb{R}^n)$ with $\operatorname{supp} u \subset B_\delta$ and any $q \in \Sigma$ with $|q| \ge \epsilon$, we have*

$$\begin{aligned} |||u|||_q^+ + |||u|||_q^- \le c\Bigg[&\left\|\left(L^{(1)} - q^{2m}\omega(x)\right)u\right\|_{0,\mathbb{R}^n_+} + \|L^{(2)}u\|_{0,\mathbb{R}^n_-} \\ &+ |q|^{m-1/2}\left(\int_{-\infty}^{0} \left\|(L^{(2)}u)(\,.\,,x_n)\right\|^2_{-(m-1/2),\mathbb{R}^{n-1}} dx_n\right)^{1/2}\Bigg], \end{aligned}$$

where the constant c depends upon the $a_\alpha^{(j)}(x)$ and their derivatives, $\omega(x)$, Σ, ϵ, m, and n, but not upon u or q.

3. Preliminaries

In this section we are going to derive some results which are needed for the proof of Theorem 2.1 Accordingly, for $|q| \ge \epsilon$ let

$$L_-(D) = L_{00}^{(2)}(D),\ L_+(D,q) = L_{00}^{(1)}(D) - q^{2m}\omega_0,$$

and for $u \in C^\infty(\mathbb{R}^n)$ with $supp\, u \subset B_d,\ 0 < d < 1$, let

$$\big(a_-^{-1} L_-(D)u\big)(x) = f_-(x) \quad \text{for} \quad x \in \mathbb{R}^n_-, \tag{3.1}$$

$$\big(a_+^{-1} L_+(D,q)u\big)(x) = f_+(x,q) \quad \text{for} \quad x \in \mathbb{R}^n_+, \tag{3.2}$$

where $a_\pm$ denotes the coefficient of D_n^{2m} in the operators $L_\pm$. Then

Lemma 3.1. For $0 \le r < m$ we have

$$\big\|(D_n^r u)(\,.\,,0)\big\|_{2m-r-1/2,\mathbb{R}^{n-1}} \le c\Big[\|L_+u\|_{0,\mathbb{R}^n_+} + \|L_-u\|_{0,\mathbb{R}^n_-} + \Phi(d)\big\|(D_n^r u)(\,.\,,0)\big\|_{0,\mathbb{R}^{n-1}}\Big],$$

$$|q|^{2m-r-1/2}\big\|(D_n^r u)(\,.\,,0)\big\|_{0,\mathbb{R}^{n-1}} \le c\Bigg[\|L_+u\|_{0,\mathbb{R}^n_+} + \sum_{j=m}^{2m-1} |q|^{2m-j-1/2}\bigg(\int_{-\infty}^{0} \big\|(L_-u)(\,.\,,x_n)\big\|^2_{-(2m-j-1/2),\mathbb{R}^{n-1}} dx_n\bigg)^{1/2} + \Phi(d)|q|^{2m-r-1/2}\big\|(D_n^r u)(\,.\,,0)\big\|_{0,\mathbb{R}^{n-1}}\Bigg],$$

where $\Phi(d) = 0$ if $m = 1$, $\Phi(d) = d^{(m-1/2)(n-1)/2(m-1)}$ if $m > 1$, and the constant c depends only upon the $a_\alpha^{(j)}(0)\,(|\alpha| = 2m)$, ω_0, Σ, ϵ, m, and n.

Proof. Writing t for x_n, let

$$U(\xi',t) = (\mathcal{F}u)(\xi',t),\ F_+(\xi',t,q) = (\mathcal{F}f_+)(\xi',t,q),\ F_-(\xi',t) = (\mathcal{F}f_-)(\xi',t),$$

where $\mathcal{F}v$ denotes the Fourier transformation of v with respect to $x'(x' \to \xi')$, and let us also introduce the notation $g^{(r)}(t) = d^r g(t)/dt^r$ for $r \ge 0$. Then it follows from (3.1–2) that for each pair (ξ', q), $U(\xi',t)$ is the unique solution of each of the initial value problems

$$\begin{aligned} a_-^{-1} L_-(i\xi', d/dt)y &= F_-(\xi',t) \quad \text{in} \quad -d \le t \le 0, \\ y^{(r)}(-d) &= 0 \quad \text{for} \quad r = 0,\dots,(2m-1), \end{aligned}$$

(3.3) and

$$\begin{aligned} a_+^{-1} L_+(i\xi', d/dt, q)y &= F_+(\xi',t,q) \quad \text{in} \quad 0 \le t \le d, \\ y^{(r)}(d) &= 0 \quad \text{for} \quad r = 0,\dots,(2m-1). \end{aligned}$$

Let us now hold $\xi' \ne 0$ and q fixed and write $y(t)$, $h_+(t)$, and $h_-(t)$ for $U(\xi',t)$, $F_+(\xi',t,q)$, and $F_-(\xi',t)$, respectively. Then as a consequence of Assumption 2.1. and [7, Proposition 2.2], we know that in the μ–plane, $L_+(i\xi', i\mu, q)$ and $L_-(i\xi', i\mu)$ each have precisely m zeros, counted according to multiplicity having positive (resp. negative) imaginary parts. Hence it follows from (3.3) and the arguments of [3, §1] that for $r = 0, \dots, (2m-1)$,

$$\begin{aligned} y^{(r)}(t) &= \sum_{j=1}^{2} Y_{jr}^{-}(t) \quad \text{in} \quad -d \le t \le 0, \\ &= -\sum_{j=1}^{2} Y_{jr}^{+}(t) \quad \text{in} \quad 0 \le t \le d, \end{aligned}$$

where for $j = 1, 2$,

$$\begin{aligned} & Y_{jr}^{-}(t) = \int_{-d}^{t} I_{jr}^{-}(t,\tau) h_{-}(\tau) d\tau, \\ & Y_{jr}^{+}(t) = \int_{t}^{d} I_{jr}^{+}(t,\tau) h_{+}(\tau) d\tau, \\ & I_{jr}^{-}(t,\tau) = (2\pi)^{-1} \int_{\gamma_j^-} (i\mu)^r e^{i(t-\tau)\mu} \left[a_{-}^{-1} L_{-}(i\xi', i\mu)\right]^{-1} d\mu, \\ & I_{jr}^{+}(t,\tau) = (2\pi)^{-1} \int_{\gamma_j^+} (i\mu)^r e^{i(t-\tau)\mu} \left[a_{+}^{-1} L_{+}(i\xi', i\mu, q)\right]^{-1} d\mu, \end{aligned} \tag{3.4}$$

and γ_1^+ and γ_1^- are closed contours lying in the half–plane $Im\,\mu > 0$ which enclose all the zeros of $L_+(i\xi', i\mu, q)$ and $L_-(i\xi', i\mu)$, respectively, having positive imaginary parts, while γ_2^+ and γ_2^- are closed contours lying in the half–plane $Im\,\mu < 0$ which enclose all the zeros of $L_+(i\xi', i\mu, q)$ and $L_-(i\xi', i\mu)$, respectively, having negative imaginary parts. Thus we have

$$y^{(r)}(0) = \textstyle\sum_{j=1}^{2} Y_{jr}^{-}(0) \ \text{ and } \ y^{(r)}(0) = -\sum_{j=1}^{2} Y_{jr}^{+}(0) \tag{3.5}$$

for $r = 0, \dots, (2m-1)$. The equations (3.5) are not adequate for our purposes in that the integrands $I_{2r}^{-}(0,\tau)$ and $I_{1r}^{+}(0,\tau)$ may become exponentially large. Consequently, in order to eliminate these terms let us observe from (3.5) that

$$Y_{1r}^{+}(0) + Y_{2r}^{-}(0) = -Y_{2r}^{+}(0) - Y_{1r}^{-}(0) \quad \text{for} \quad r = 0, \dots, (2m-1). \tag{3.6}$$

Now let $\{\mu_k^+\}_1^{s^+}$ (resp. $\{\mu_k^-\}_1^{s^-}$) denote the distinct zeros of $L_+(i\xi', i\mu, q)$ (resp. $L_-(i\xi', i\mu)$) and suppose that $\mu_k^{\pm}$ has multiplicity $m_k^{\pm}$ for $k = 1, \dots, s^{\pm}$ and that $Im\,\mu_k^{\pm}$ is positive (resp. negative) for $k = 1, \dots, n^{\pm}$ (resp. $k = (n^{\pm} + 1), \dots, s^{\pm}$). Then it follows from the residue theorem that (3.6) can be written in the form

$$\sum_{k=1}^{n^+}\sum_{p=0}^{m_k^+-1}\frac{r!}{\Gamma(r-p+1)}(i\mu_k^+)^{r-p}J_{kp}^+ + \sum_{k=n^-+1}^{s^-}\sum_{p=0}^{m_k^- -1}\frac{r!}{\Gamma(r-p+1)}(i\mu_k^-)^{r-p}J_{kp}^- = -Y_{2r}^+(0)-Y_{1r}^-(0)\quad\text{for}\quad r=0,\dots,(2m-1), \tag{3.7}$$

where $\Gamma(z)$ denotes the Gamma function,

$$J_{kp}^{\pm}=\pm\int_0^{\pm d}c^{-i\tau\mu_k^{\pm}}P_{kp}^{\pm}(\tau)h_{\pm}(\tau)d\tau,$$

and $P_{kp}^{\pm}(\tau)$ is a polynomial in τ of degree $m_k^{\pm}-1-p$ whose coefficients are polynomials in the $(\mu_k^{\pm}-\mu_j^{\pm})^{-1}$, $j\neq k$. We may view (3.7) as a simultaneous system of $2m$ linear equations in the "unknowns" $J_{kp}^{\pm}$, and we let $\mathcal{A}$ denote the $2m\times 2m$ matrix constructed from the coefficients on the left side of (3.7). Then it is not difficult to verify that

$$\det\mathcal{A}=\Big(\prod_{k=1}^{n^\dagger}\prod_{j=1}^{m_k-1}j!\Big)\prod_{\substack{j,k=1\\ j>k}}^{n^\dagger}(\nu_j-\nu_k)^{m_jm_k}\neq 0, \tag{3.8}$$

where $m_k=m_k^+$, $\nu_k=i\mu_k^+$ for $k=1,\dots,n^+$ and $m_{n^++k}=m_{n^-+k}^-$, $\nu_{n^++k}=i\mu_{n^-+k}^-$ for $k=1,\dots,(s^--n^-)$, while $n^\dagger=n^++s^--n^-$. Hence if we solve (3.7) for the $J_{kp}^{\pm}$ and bear in mind that the first (resp. second) double sum on the left side of (3.7) is $Y_{1r}^+(0)$ (resp. $Y_{2r}^-(0)$), then we see that

$$\begin{aligned} Y_{2r}^-(0)&=-\textstyle\sum_{j=0}^{2m-1}Q_{rj}^-\big(Y_{2j}^+(0)+Y_{1j}^-(0)\big)\\ Y_{1r}^+(0)&=-\textstyle\sum_{j=0}^{2m-1}Q_{rj}^+\big(Y_{2j}^+(0)+Y_{1j}^-(0)\big)\end{aligned} \tag{3.9}$$

for $r=0,\dots,(2m-1)$, where $Q_{rj}^-=\det\mathcal{A}_{rj}^-/\det\mathcal{A}$, $Q_{rj}^+=\det\mathcal{A}_{rj}^+/\det\mathcal{A}$, and with $\mathcal{A}=(a_{ks})$, $1\le k,\ s\le 2m$, $\mathcal{A}_{rj}^-$ (resp. $\mathcal{A}_{rj}^+$) is the matrix obtained from $\mathcal{A}$ by replacing the $(j+1)$–th row of $\mathcal{A}$ by $(0\dots 0a_{r+1,m+1}\dots a_{r+1,2m})$ (resp. $(a_{r+1,1}\dots a_{r+1,m}0\dots 0)$).

Let us next fix our attention upon a pair r, j and for $z\in\mathbb{C}$ and σ an integer satisfying $0\le\sigma\le 2m-1$ let $f_\sigma(z)=z^\sigma$. Then employing the notation of (3.8), we assert that

$$\Big(\prod_{k=1}^{n^+}\prod_{\ell=n^++1}^{n^\dagger}(\nu_\ell-\nu_k)^{m_\ell m_k}\Big)Q_{rj}^+$$

is a determinant of order $2m$ whose entry in the $(\sigma+1)$–th row and $\left(\sum_{k=1}^{s-1} m_k + p + 1\right)$–th column ($1 \le s \le n^+$, $0 \le p \le m_s - 1$) is

$$\frac{1}{2\pi i}\int\limits_{\gamma_1} \frac{f_\sigma(z)}{\left(\prod_{k=1}^{s-1}(z-\nu_k)^{m_k}\right)(z-\nu_s)^{p+1}}\,dz,$$

where γ_1 is a closed contour lying in the left–half of the complex z–plane enclosing all the ν_k for which $1 \le k \le n^+$, and we are to replace $f_\sigma(z)$ by $f_r(z)$ if $\sigma = j$, while its entry in the $(\sigma+1)$–th row and $\left(m + \sum_{k=n^++1}^{s-1} m_k + p + 1\right)$–th column ($n^+ + 1 \le s \le n^\dagger$, $0 \le p \le m_s - 1$) is

$$\frac{1}{2\pi i}\int\limits_{\gamma_2} \frac{f_\sigma(z)}{\left(\prod_{k=n^++1}^{s-1}(z-\nu_k)^{m_k}\right)(z-\nu_s)^{p+1}}\,dz,$$

where γ_2 is a closed contour lying in the right–half of the complex z–plane enclosing all the ν_k for which $(n^+ + 1) \le k \le n^\dagger$, and we are to replace $f_\sigma(z)$ by 0 if $\sigma = j$. To establish the assertion, let us firstly suppose that the $\mu_k^\pm$ of (3.7) are all simple zeros. Then by successive subtractions of the first m columns and of the last m columns of $Q_{rj}^+ = \kappa^{-1}\det\mathcal{A}_{rj}^+$ ($\kappa = \det\mathcal{A}$) and by appealing to the results of [16, p.231] concerning the calculus of differences, we immediately obtain the validity of the assertion for this case. Turning to the case where the $\mu_k^\pm$ are not all simple, let us replace $L_+(\mu)$ and $L_-(\mu)$ by $L_+(\mu)+\zeta$ and $L_-(\mu)+\zeta$, respectively, where for brevity we have written $L_+(\mu)$ for $L_+(i\xi', i\mu, q)$ and $L_-(\mu)$ for $L_-(i\xi', i\mu)$, and where $\zeta \in \mathbb{C}\setminus\{0\}$ is small in modulus. Let us also denote the analogue of Q_{rj}^+ for the perturbed polynomials by $Q_{rj}^+(\zeta)$. Since the zeros of $L_\pm(\mu)+\zeta$ are all simple, the assertion is certainly true for $Q_{rj}^+(\zeta)$. On the other hand, by successive subtractions of the columns of $Q_{rj}^+(\zeta)$ corresponding to those zeros of $L_\pm(\mu)+\zeta$ which tend to a common zero of $L_\pm(\mu)$ as $\zeta \to 0$, by appealing to the results of [16], and by making use of the Taylor series expansion of $f_\sigma(z)$ about the points ν_k, it is not difficult to show with the aid of the residue theorem that $Q_{rj}^+(\zeta) \to Q_{rj}^+$ as $\zeta \to 0$. In light of these facts, we need only let $\zeta \to 0$ to arrive at the assertion for the general case. Hence if we put $\rho = \left(|\xi'|^2 + |q|^2\right)^{1/2}$, then it follows from what we have just said about Q_{rj}^+ and from the Laplace method for expanding a determinant that

$$\left|Q_{rj}^+(\xi', q)\right| \le c\rho^{r-j}\left(|\xi'|/\rho\right)^{\max\{m-j,0\}},$$

where the constant c does not depend upon r, j, ξ', or q. Similarly, we can show that

$$\left|Q_{rj}^-(\xi', q)\right| \le c|\xi'|^{r-j}\left(|\xi'|/\rho\right)^{\max\{j-m+1,0\}}.$$

We conclude immediately from (3.4) and the second equations of (3.5) and (3.9) that for $0 \le r < m$,

$$\begin{aligned}&\big\|(D_n^r u)(\,.\,,0)\big\|^2_{2m-r-1/2,\mathbb{R}^{n-1}}\\ &\quad\le\ c\Big[\|L_+u\|^2_{0,\mathbb{R}^n_+}+\|L_-u\|^2_{0,\mathbb{R}^n_-}+\big\|(\mathcal{F}^{-1}\chi)\star(D_n^r u)(\,.\,,0)\big\|^2_{0,\mathbb{R}^{n-1}}\Big],\end{aligned}$$

$$\begin{aligned}&|q|^{2(2m-r-1/2)}\big\|(D_n^r u)(\,.\,,0)\big\|^2_{0,\mathbb{R}^{n-1}}\le c\Big[\|L_+u\|^2_{0,\mathbb{R}^n_+}\\ &\qquad+\sum_{j=m}^{2m-1}|q|^{2(2m-j-1/2)}\int_{-\infty}^{0}\big\|(L_-u)(\,.\,,x_n)\big\|^2_{-(2m-j-1/2),\mathbb{R}^{n-1}}dx_n\\ &\qquad+|q|^{2(2m-r-1/2)}\big\|(\mathcal{F}^{-1}\chi)\star(D_n^r u)(\,.\,,0)\big\|^2_{0,\mathbb{R}^{n-1}}\Big],\end{aligned}$$

where $\chi = \chi(\xi')$ denotes the characteristic function of the set $\{\xi' \in \mathbb{R}^{n-1}\,\big|\,|\xi_j| < \delta = d^{1/2(m-1)}$ for $j = 1,\dots,(n-1)\}$ if $m > 1$ and is zero otherwise, $\star$ denotes convolution, and the constant c depends only upon the $a_\alpha^{(j)}(0)$ $\big(|\alpha| = 2m\big)$, ω_0, Σ, ϵ, m, and n. Observing from [21, Lemma 2.2.4, p.167] that $(\mathcal{F}^{-1}\chi)(x') = (2/\pi)^{(n-1)/2}\prod_{j=1}^{n-1}\big(\sin\{\delta x_j\}\big)/x_j$, the assertions of the lemma now follow from an argument similar to that used in the proof of Young's inequality for convolution integrals. ∎

Turning to the next result of this section, let $f \in C^\infty(\mathbb{R}^n)$ such that $supp\, f \subset B_1$ and let us introduce in $\mathbb{R}^n$ the function $g(x)$ by putting

$$\begin{aligned}g(x',x_n) &= f(x',x_n) \quad\text{for}\quad x_n \le 0,\\ &= \sum_{j=1}^{\ell} c_j f(x',-x_n/j) \quad\text{for}\quad x_n > 0,\end{aligned}$$

where $\ell = 2m+2n+3$ and $\sum_{j=1}^{\ell}(-j)^r c_j = 1$ for $r = -(n+1),\dots,(2m+n+1)$. It is important to observe that $g \in C^{n+1}(\mathbb{R}^n)$ and $supp\, g \in B_\ell$. For $0 \ne \xi \in \mathbb{R}^n$, let $U_1(\xi) = G(\xi)/L_-(i\xi)$, where $G(\xi) = (Fg)(\xi)$ and F denotes the Fourier transformation in $\mathbb{R}^n$ with respect to x $(x \to \xi)$, and put $u_1 = F^{-1}U_1$.

Lemma 3.2. It is the case that $u_1 \in H^{2m}(\mathbb{R}^n) \cap C^{2m}(\mathbb{R}^n)$ and $L_-(D)u_1 = g$ in $\mathbb{R}^n$. Furthermore, $\big(1+|x|\big)^{n+1}(D^\alpha u_1)(x)$ is bounded in $\mathbb{R}^n$ for any multi–index α satisfying $|\alpha| \le 2m$. Finally,

$$\|u_1\|_{m+1/2,\mathbb{R}^n} + \Big(\int_{-\infty}^{\infty} \big\|(D_n^{2m}u_1)(\,.\,,x_n)\big\|^2_{-(m-1/2),\mathbb{R}^{n-1}} dx_n\Big)^{1/2}$$
$$\le \quad c\Big(\int_{-\infty}^{\infty} \big\|(f(\,.\,,x_n)\big\|^2_{-(m-1/2),\mathbb{R}^{n-1}} dx_n\Big)^{1/2},$$

where the constant c depends only upon the $a_\alpha^{(2)}(0)$ $\big(|\alpha| = 2m\big)$, m, and n, but not upon f.

Proof. For multi–index $\alpha = (\alpha_1, \ldots, \alpha_n)$ let $\mathcal{D}^\alpha = \mathcal{D}_1^{\alpha_1} \cdots \mathcal{D}_n^{\alpha_n}$, where $\mathcal{D}_j = \partial/\partial\xi_j$ for $j = 1, \ldots, n$. Then it follows from the definitions that $\big(1 + |\xi|\big)^{n+1}(\mathcal{D}^\alpha G)(\xi)$ is bounded in $\mathbb{R}^n$ and that $U_1(\xi) = (f, h)_{0,\mathbb{R}^n_-}$, where

$$\overline{h} = \overline{h}(x,\xi) \quad = \quad \left(\frac{(-i)^{n+1}}{(2\pi)^{n/2}}\right)\left(\frac{\xi_n^{2m+n+1}}{L_-(\xi)}\right) x_n^{2m+n+1} e^{-i\xi'\cdot x'}$$
$$\times \sum_{r=0}^{\infty} \frac{(-i\xi_n x_n)^r}{(r+2m+n+1)!}\Big[1 - \sum_{j=1}^{\ell}(-j)^{r+2m+n+2}c_j\Big],$$

$\overline{h}$ denotes complex conjugation of h, and $\cdot$ denotes the inner product in $\mathbb{R}^{n-1}$. We conclude from these facts that $\xi^\alpha U_1(\xi) \in L^2(\mathbb{R}_n) \cap L^1(\mathbb{R}^n)$ for $|\alpha| \le 2m$, and hence it follows from an argument similar to that used in the proof of Lemma 2.10 of [18, p.72] that $u_1 \in H^{2m}(\mathbb{R}^n) \cap C^{2m}(\mathbb{R}^n)$. That $L_-(D)u_1 = g$ in $\mathbb{R}^n$ is now an immediate consequence of these results and the definition of u_1. Moreover, if α is the multi–index given in the statement of the lemma and $\beta = (\beta_1, \ldots, \beta_n)$ is any multi–index satisfying $|\beta| \le n+1$, then it is clear from what has been shown above that $\mathcal{D}^\beta\big(\xi^\alpha U_1(\xi)\big) \in L^1(\mathbb{R}_n)$, and hence $x^\beta(D^\alpha u_1)(x)$ is bounded in $\mathbb{R}^n$. It follows from this fact that $\big(1+|x|\big)^{n+1}(D^\alpha u_1)(x)$ is bounded in $\mathbb{R}^n$, and thus all the assertions of the lemma, except the last, have now been proved.

In proving the last assertion, we shall make use of the fact that

$$\int_{-\infty}^{\infty} \big\|(D_n^{2m}u_1)(\,.\,,x_n)\big\|^2_{-(m-1/2),\mathbb{R}^{n-1}} dx_n = \int_{\mathbb{R}^n} \big(1+|\xi'|^2\big)^{-(m-1/2)}\xi_n^{4m}\big|U_1(\xi)\big|^2 d\xi.$$

Then fixing our attention firstly upon the case $0 < |\xi| < 1$, we have

$$\begin{aligned}\big(1+|\xi|^2\big)^{(m+1/2)/2}U_1(\xi) &= \int_{-\infty}^{0} \big(f(\,.\,,x_n), v_1(\,.\,,x_n,\xi)\big)_{0,\mathbb{R}^{n-1}}dx_n,\\ \big(1+|\xi'|^2\big)^{-(m-1/2)/2}\xi_n^{2m}U_1(\xi) &= \int_{-\infty}^{0} \big(f(\,.\,,x_n), v_2(\,.\,,x_n,\xi)\big)_{0,\mathbb{R}^{n-1}}dx_n,\end{aligned}$$

where $\overline{v}_1(x',x_n,\xi) = \chi(x)\big(1+|\xi|^2\big)^{(m+1/2)/2}\overline{h}(x,\xi)$, $\overline{v}_2(x',x_n,\xi) = \chi(x)\times \big(1+|\xi'|^2\big)^{-(m-1/2)/2}\xi_n^{2m}\overline{h}(x,\xi)$, and $\chi(x)\in C^\infty(\mathbb{R}^n)$, $0\le\chi(x)\le 1$, $\chi(x)=1$ for $x\in B_1$, and $supp\,\chi\subset B_2$. Hence if we let $\langle\,.\,,\,.\,\rangle$ denote the pairing between $H_2^{-(m-1/2)}(\mathbb{R}^{n-1})$ and its dual $H^{m-1/2}(\mathbb{R}^{n-1})$, then for $-\infty < x_n < 0$ we have

$$\begin{aligned}\Big|\big(f(\,.\,,x_n), v_j(\,.\,,x_n,\xi)\big)_{0,\mathbb{R}^{n-1}}\Big| &= \Big|\big\langle f(\,.\,,x_n), \overline{v}_j(\,.\,,x_n,\xi)\big\rangle\Big|\\ &\le \big\|f(\,.\,,x_n)\big\|_{-(m-1/2),\mathbb{R}^{n-1}}\big\|\overline{v}_j(\,.\,,x_n,\xi)\big\|_{m-1/2,\mathbb{R}^{n-1}}\end{aligned}$$

for $j=1,2$, and so we conclude that

$$\begin{aligned}I_0 &= \int_{|\xi|<1} \big(1+|\xi|^2\big)^{m+1/2}|U_1(\xi)|^2d\xi + \\ &\qquad \int_{|\xi|<1} \big(1+|\xi'|^2\big)^{-(m-1/2)}\xi_n^{4m}\big|U_1(\xi)\big|^2d\xi \qquad (3.10)\\ &\le c\int_{-\infty}^{0} \big\|f(\,.\,,x_n)\big\|^2_{-(m-1/2),\mathbb{R}^{n-1}}dx_n,\end{aligned}$$

where here and below c denotes a generic constant which may vary from inequality to inequality and which only depends upon the $a_\alpha^{(2)}(0)$ ($|\alpha|=2m$), m, and n. On the other hand,

$$\begin{aligned}I_\infty &= \int_{|\xi|\ge1} \big(1+|\xi|^2\big)^{m+1/2}\big|U_1(\xi)\big|^2d\xi + \\ &\qquad \int_{|\xi|\ge1} \big(1+|\xi'|^2\big)^{-(m-1/2)}\xi_n^{4m}\big|U(\xi)\big|^2d\xi\\ &\le c\|g\|^2_{-(m-1/2),\mathbb{R}^n} \le c\int_{-\infty}^{\infty} \big\|g(\,.\,,x_n)\big\|^2_{-(m-1/2),\mathbb{R}^{n-1}}dx_n,\end{aligned}$$

and hence it follows from the definition of g that (3.10) remains valid when I_0 there is replaced by I_∞. This completes the proof of the lemma. ∎

We come now to the final results of this section; and in proving these results we shall make use of the extension operator $E: C^{2m}(\overline{\mathbb{R}^n_-}) \to C^{2m}(\mathbb{R}^n)$ defined by

$$\begin{aligned}(Ef)(x',x_n) &= f(x',x_n) \quad \text{if} \quad x_n \le 0,\\ &= \sum_{j=1}^{2m+1} c_j f(x',-x_n/j) \quad \text{if} \quad x_n > 0\end{aligned}$$

for $f \in C^{2m}(\overline{\mathbb{R}^n_-})$, where $\sum_{j=1}^{2m+1}(-1/j)^r c_j = 1$ for $r = 0,\dots,2m$.

Lemma 3.3. *Let $u \in C^\infty(\mathbb{R}^n)$ such that $\operatorname{supp} u \subset B_1$ and let $b \in L^\infty(\overline{\mathbb{R}^n_-})$ such that b is of class C^{2m} in some subset of $\overline{\mathbb{R}^n_-} \cap B_1$ containing $\operatorname{supp} u \cap \overline{\mathbb{R}^n_-}$. Let $\alpha = (\alpha_1,\dots,\alpha_n)$ be a multi–index satisfying $|\alpha| = 2m$ and let $\gamma > 0$. Then*

$$\begin{aligned}\int_{-\infty}^{0} \big\|\big(D^\alpha(bu)\big)(\,.\,,x_n)\big\|^2_{-(m-1/2),\mathbb{R}^{n-1}} dx_n \le c\Big[\gamma^{\alpha_n}\|bu\|_{m+1/2,\mathbb{R}^n_-}\\ +\gamma^{\alpha_n-2m}\int_{-\infty}^{0}\big\|\big(D_n^{2m}(bu)\big)(\,.\,,x_n)\big\|^2_{-(m-1/2),\mathbb{R}^{n-1}}dx_n\Big],\end{aligned} \tag{3.11}$$

where the constant c depends only upon m and n.

Proof. We have

$$\begin{aligned} I &= \int_{-\infty}^{0} \big\|\big(D^\alpha(bu)\big)(\,.\,,x_n)\big\|^2_{-(m-1/2),\mathbb{R}^{n-1}} dx_n\\ &\le \int_{-\infty}^{0}\bigg(\int_{\mathbb{R}^{n-1}} \big(1+|\xi'|^2\big)^{m-\alpha_n+1/2}\big|\big(D_n^{\alpha_n}\mathcal{F}(bu)\big)(\xi',x_n)\big|^2 d\xi'\bigg)dx_n\\ &\le \int_{\mathbb{R}^n}\big(1+|\xi'|^2\big)^{m-\alpha_n+1/2}|\xi_n|^{2\alpha_n}\big|\big(FE(bu)\big)(\xi)\big|^2 d\xi,\end{aligned}$$

where $\mathcal{F}$ and F are the Fourier transformations in $\mathbb{R}^{n-1}$ and $\mathbb{R}^n$, respectively, introduced above. Hence if in this last integral we decompose the domain of integration into the sets $\Omega = \{\xi \in \mathbb{R}^n \mid \xi_n^2/\big(1+|\xi|^2\big) \le \gamma$ and $\mathbb{R}^n\backslash\Omega$, then we obtain

$$\begin{aligned} I &\le \gamma^{\alpha_n}\big\|E(bu)\big\|_{m+1/2,\mathbb{R}^n}\\ &+ \gamma^{\alpha_n-2m}\int_{\mathbb{R}^{n-1}}\big(1+|\xi'|^2\big)^{-(m-1/2)}\bigg(\int_{-\infty}^{\infty}\big|\big(\mathcal{F}D_n^{2m}E(bu)\big)(\xi',x_n)\big|^2dx_n\bigg)d\xi',\end{aligned}$$

and the assertion of the lemma follows immediately from the definition of E and some standard interpolation results (see Theorem 5.1, p.27, Theorem 7.1, p.30, and Theorem 9.1, p.40 of [17]). ▯

A scrutiny of the proof of Lemma 3.3 shows that

Corollary 3.4. *Let u and b satisfy the hypotheses of Lemma 3.3 and let $r \in \mathbb{Z}$ satisfy $0 \le r \le 2m$. Then the inequality (3.11) remains valid when the expression on the left side of this inequality is replaced by*

$$\int_{\mathbb{R}^{n-1}} \left(1+|\xi'|^2\right)^{m-r+1/2} \left(\int_{-\infty}^{0} \left| \left(D_n^r \mathcal{F}(bu)\right)(\xi', x_n)\right|^2 dx_n \right) d\xi'.$$

4. Proof of Theorem 2.1

Let us suppose firstly that $u \in C^\infty(\mathbb{R}^n)$ such that $supp\, u \subset B_d,\ 0 < d < 1$. Then referring to the beginning of §3 for terminology and assuming henceforth that $|q| \ge \epsilon$, it follows from [7, §3] that

$$\begin{aligned} |||u|||_q^+ \quad \le \quad & c\Bigg[\|L_+ u\|_{0,\mathbb{R}^n_+} + \sum_{r=0}^{m-1} \Big(\big\|(D_n^r u)(\,.\,,0)\big\|_{2m-r-1/2,\mathbb{R}^{n-1}} \\ & \qquad + |q|^{2m-r-1/2} \big\|(D_n^r u)(\,.\,,0)\big\|_{0,\mathbb{R}^{n-1}} \Big) \Bigg], \end{aligned}$$

where here and below c denotes a generic constant which may vary from inequality to inequality and in each case it can only depend upon some or all of the quantities $a_\alpha^{(j)}(0)$ $\big(|\alpha| = 2m\big)$, ω_0, Σ, ϵ, m, and n. Hence in light of Lemma 3.1 and the interpolation and trace inequalities of [7, §1], we obtain

$$\begin{aligned} |||u|||_q^+ \quad \le \quad & c\Bigg[\|L_+ u\|_{0,\mathbb{R}^n_+} + \|L_- u\|_{0,\mathbb{R}^n_-} \\ & + \sum_{j=m}^{2m-1} |q|^{2m-j-1/2} \left(\int_{-\infty}^{0} \big\|(L_- u)(\,.\,,x_n)\big\|^2_{-(2m-j-1/2),\mathbb{R}^{n-1}} dx_n \right)^{1/2} \\ & + \Phi(d) |||u|||_q^+ \Bigg]. \end{aligned}$$

We conclude immediately from this last inequality and a minor modification of the interpolation inequality of [7, §1] that

$$\begin{aligned} |||u|||_q^+ \le c\Bigg[& \|L_+ u\|_{0,\mathbb{R}^n_+} + \|L_- u\|_{0,\mathbb{R}^n_-} \\ & + |q|^{m-1/2} \left(\int_{-\infty}^{0} \big\|(L_- u)(\,.\,,x_n)\big\|^2_{-(m-1/2),\mathbb{R}^{n-1}} dx_n \right)^{1/2} + \Phi(d) |||u|||_q^+ \Bigg]. \end{aligned} \tag{4.1}$$

Likewise it follows from [3, Theorem 14.1] that

$$\|u\|_{2m,\mathbb{R}^n_-} \le c\Big[\|L_-u\|_{0,\mathbb{R}^n_-} + \sum_{r=0}^{m-1} \big\|(D_n^r u)(\,.\,,0)\big\|_{2m-r-1/2,\mathbb{R}^{n-1}}\Big],$$

and hence by arguing as in the previous case, we obtain

$$\|u\|_{2m,\mathbb{R}^n_-} \le c\Big[\|L_+u\|_{0,\mathbb{R}^n_+} + \|L_-u\|_{0,\mathbb{R}^n_-} + \Phi(d)\||u|\|_q^+\Big]. \tag{4.2}$$

Turning now to estimates for

$$\|u\|_{m+1/2,\mathbb{R}^n_-} \quad \text{and} \quad \Bigg(\int_{-\infty}^{0} \big\|(D_n^{2m}u)(\,.\,,x_n)\big\|^2_{-(m-1/2),\mathbb{R}^{n-1}} dx_n\Bigg)^{1/2},$$

let $g_r(x') = (D_n^r u)(x',0)$ for $r = 0,\dots,(m-1)$, $f(x) = \big(L_-(D)u\big)(x)$, let u_1 denote the function of Lemma 3.2 constructed from the f just defined, and for $r = 0,\dots,(m-1)$ let $h_r(x') = (D_n^r u_1)(x',0)$. If $u_2 = u - u_1$ and $v = u_2 \mid \overline{\mathbb{R}^n_-}$, then v is a solution of the boundary value problem:

$$L_-(D)y = 0 \text{ in } \mathbb{R}^n_-, \quad D_n^r y = g_r(x') - h_r(x') \text{ on } x_n = 0 \text{ for } r = 0,\dots,(m-1).$$

Hence if we write t for x_n, let

$$V(\xi',t) = (\mathcal{F}v)(\xi',t),\ G_r(\xi') = (\mathcal{F}g_r)(\xi'),\ H_r(\xi') = (\mathcal{F}h_r)(\xi'),$$

where $\mathcal{F}$ is the Fourier transformation in $\mathbb{R}^{n-1}$ introduced in §3, and observe that for each $\xi' \in \mathbb{R}^{n-1}$, $V(\xi',t)$, as a function of t, is in $H^{2m}\big((-\infty,0)\big)$, then it follows from [3, §1] (see also [7, §3]) that for $\xi' \ne 0$ and $t < 0$,

$$V(\xi',t) = \sum_{r=0}^{m-1} \big(G_r(\xi') - H_r(\xi')\big)\Omega_r(\xi',t), \tag{4.3}$$

where

$$\Omega_r(\xi',t) = \int_\gamma e^{it\mu}\frac{N_r(\xi',\mu)}{M^-(\xi',\mu)}d\mu, \tag{4.4}$$

γ is a closed contour lying in the half–plane $Im\,\mu < 0$ which enclosed all the zeros $\big\{\mu_j(\xi')\big\}_1^m$ of $L_-(\xi',\mu)$ having negative imaginary parts, $M^-(\xi',\mu) = \prod_{j=1}^m \big(\mu - \mu_j(\xi')\big)$, and $N_r(\xi',\mu)$ is a polynomial in μ whose coefficients are infinitely differentiable functions of ξ' for $\xi' \in \mathbb{R}^{n-1}\backslash\{0\}$ and which is positive homogeneous of degree $m - r - 1$ in all its arguments.

Next with E denoting the extension operator and F the Fourier transformation in $\mathbb{R}^n$ introduced in §3 and bearing in mind (4.3–4), we have for

$\xi' \neq 0$,

$$(2\pi)^{1/2}(FEv)(\xi) = \int_{-\infty}^{\infty} e^{-i\xi_n t}(\mathcal{F}Ev)(\xi', t)dt$$

$$= \sum_{r=0}^{m-1} \big(G_r(\xi') - H_r(\xi')\big) \int_{\gamma} \frac{N_r(\xi', \mu)}{M^-(\xi', \mu)} \Big[\int_{-\infty}^{0} e^{it(\mu - \xi_n)} dt + \sum_{j=1}^{2m+1} jc_j \int_{-\infty}^{0} e^{it(\mu + j\xi_n)} dt \Big] d\mu. \tag{4.5}$$

Observing that the expression in the square bracket on the right side of (4.5) is just

$$-i\Big[\frac{1}{\mu - \xi_n} + \sum_{j=1}^{2m+1} \frac{jc_j}{\mu + j\xi_n}\Big] = -i\Big(\prod_{j=1}^{2m+1}(1+j)\Big)\mu^{2m+1}\Big/(\mu-\xi_n)\prod_{j=1}^{2m+1}(\mu+j\xi_n),$$

we conclude that

$$\big|(FEv)(\xi)\big| \leq c \sum_{r=0}^{m-1} \big|G_r(\xi') - H_r(\xi')\big| \Big/ |\xi'|^{r+1}\big(1 + |\xi_n|/|\xi'|\big)^{2m+2}.$$

Thus if we put

$$W(\phi) = \|\phi\|_{m+1/2,\mathbb{R}^n_-} + \Big(\int_{-\infty}^{0} \big\|(D_n^{2m}\phi)(\,.\,,t)\big\|^2_{-(m-1/2),\mathbb{R}^{n-1}} dt\Big)^{1/2},$$

then it is not difficult to deduce from this last inequality that

$$W(v) \leq \|Ev\|_{m+1/2,\mathbb{R}^n} + \Big(\int_{\mathbb{R}^n} \big(1 + |\xi'|^2\big)^{-m+1/2} \xi_n^{4m} \big|(FEv)(\xi)\big|^2 d\xi\Big)^{1/2} \leq$$

$$c\Big[\|v\|_{0,\mathbb{R}^n_-} + \sum_{r=0}^{m-1} \Big(\big\|(D_n^r u)(\,.\,,0)\big\|_{m-r,\mathbb{R}^{n-1}} + \big\|(D_n^r u_1)(\,.\,,0)\big\|_{m-r,\mathbb{R}^{n-1}}\Big)\Big],$$

and hence it follows from a standard trace theorem that

$$W(v) \leq c\Big[\|u\|_{0,\mathbb{R}^n_-} + \|u\|_{m+1/2,\mathbb{R}^n_+} + \|u_1\|_{m+1/2,\mathbb{R}^n_-}\Big]. \tag{4.6}$$

Observing from [9, Proposition 3.2] that

$$\|u\|_{0,\mathbb{R}^n_-} \leq \|Eu\|_{0,\mathbb{R}^n} \leq cd^{m+1/2}\|Eu\|_{m+1/2,\mathbb{R}^n},$$

we conclude from (4.6) and a minor modification of the interpolation inequality of [7, §1] that

$$|q|^{m-1/2}W(u) \leq c\Big[d^{m+1/2}|q|^{m-1/2}\|u\|_{m+1/2,\mathbb{R}^n_-} + \||u|\|^+_q + |q|^{m-1/2}W(u_1)\Big],$$

and hence in view of Lemma 3.2 and (4.1–2) we finally obtain

$$Y(u) = |||u|||_q^+ + |||u|||_q^- \leq c\Bigg[\|L_+u\|_{0,\mathbb{R}^n_+ + \|L_-u\|_{0,\mathbb{R}^n_-}} +$$

$$m|q|^{m-1/2}\Bigg(\int_{-\infty}^{0} \big\|(L_-u)(\,.\,,t)\big\|^2_{-(m-1/2),\mathbb{R}^{n-1}}dt\Bigg)^{1/2} + \big(\Phi(d) + d^{m+1/2}\big)Y(u)\Bigg].$$

It follows from this last inequality that if we choose d_0, $0 < d_0 \leq 1/3$, small enough so that $c\big(\Phi(d_0) + d_0^{m+1/2}\big) \leq 1/2$, then

$$\begin{aligned} Y(u) \quad &\leq \quad c\Bigg[\|L_+u\|_{0,\mathbb{R}^n_+} + \|L_-u\|_{0,\mathbb{R}^n_-} \\ &\qquad + |q|^{m-1/2}\Bigg(\int_{-\infty}^{0} \big\|(L_-u)(\,.\,,t)\big\|^2_{-(m-1/2),\mathbb{R}^{n-1}}dt\Bigg)^{1/2}\Bigg] \end{aligned} \tag{4.7}$$

for $d \leq d_0$. We shall suppose henceforth that d_0 is chosen small enough so that the coefficient of D_n^{2m} in $L^{(2)}(x,D)$ does not vanish in $\overline{\Omega}$, where $\Omega = \{x \in B_{5^{1/2}d_0} \mid x_n < 0\}$, and also that $d \leq d_0$.

Let $\chi(d)$ denote the maximum of the expressions $\chi^{(0)}(d)$, $\chi_\alpha^{(j)}(d)$ $(1 \leq j \leq 2,\ |\alpha| = 2m)$, where $\chi^{(0)}(d)$ denotes the supremum of $\big|\omega(x) - \omega_0\big|$ in the set $B_d \cap \mathbb{R}^n_+$ and $\chi_\alpha^{(j)}(d)$ denotes the supremum of $\big|a_\alpha^{(j)}(x) - a_\alpha^{(j)}(0)\big|$ in the set $B_d \cap \mathbb{R}^n_+$ if $j = 1$ and in the set $B_d \cap \mathbb{R}^n_-$ if $j = 2$. Then a standard argument involving the Poincaré inequality shows that the sum of the first two terms in the bracket on the right side of (4.7) does not exceed

$$\big\|\big(L^{(1)} - q^{2m}\omega(x)\big)u\big\|_{0,\mathbb{R}^n_+} + \|L^{(2)}u\|_{0,\mathbb{R}^n_-} + c_1\big(\chi(d) + d\big)\big(|||u|||_q^+ + \|u\|_{2m,\mathbb{R}^n_-}\big),$$

where here and below c_1 denotes a generic constant which may vary from inequality to inequality and in each case it can only depend upon some or all of the quantities $(D^\beta a_\alpha^{(j)})(x)$, $\omega(x)$, Σ, ϵ, m, and n. Turning now to the last term, let us observe that for $-\infty < t < 0$,

$$\big\|(L_-u)(\,.\,,t)\big\|^2_{-(m-1/2),\mathbb{R}^{n-1}} \leq 9\Big[I_1(t) + I_2(t) + \big\|(L^{(2)}u)(\,.\,,t)\big\|^2_{-(m-1/2),\mathbb{R}^{n-1}}\Big],$$

where

$$\begin{aligned} I_1(t) \quad &= \quad \big\|\big((L^{(2)} - L_0^{(2)})u\big)(\,.\,,t)\big\|^2_{-(m-1/2),\mathbb{R}^{n-1}}, \\ I_2(t) \quad &= \quad \big\|\big(L_0^{(2)} - L_-\big)u\big)(\,.\,,t)\big\|^2_{-(m-1/2),\mathbb{R}^{n-1}}. \end{aligned}$$

Fixing our attention firstly upon $I_1(t)$, let $\phi \in C^\infty(\mathbb{R}^{n-1})$ such that $0 \leq \phi(x') \leq 1$, $\phi(x') = 1$ for $|x'| < d_0$, $\phi(x') = 0$ for $|x'| > 2d_0$, and let us consider a typical term $a_\alpha^{(2)} D^\alpha$ appearing in $L^{(2)} - L_0^{(2)}$. Then for $|\alpha| \le m$ we have

$$\begin{aligned} J_1 &= \int_{-\infty}^{0} \big\|(a_\alpha^{(2)} D^\alpha u)(\,.\,,t)\big\|^2_{-(m-1/2),\mathbb{R}^{n-1}} dt \le \int_{-\infty}^{0} \big\|(a_\alpha^{(2)} D^\alpha u)(\,.\,,t)\big\|_{0,\mathbb{R}^{n-1}} dt \\ &\le c_1 \int_{-\infty}^{0} \big\|(D^\alpha u)(\,.\,,t)\big\|^2_{0,\mathbb{R}^{n-1}} dt \le c_1 \|u\|^2_{m,\mathbb{R}^n_-} \le c_1 \|Eu\|_{m,\mathbb{R}^n}, \end{aligned}$$

and hence in view of [9, Proposition 3.2] and some standard interpolation results we conclude that $J_1 \le c_1 d \|u\|^2_{m+1/2,\mathbb{R}^n_-}$. Turning to the case $|\alpha| > m$, let us observe that if $\langle\,.\,,\,.\,\rangle$ denotes the pairing between $H_2^{-(m-1/2)}(\mathbb{R}^{n-1})$ and its dual $H^{m-1/2}(\mathbb{R}^{n-1})$ and $v \in C_0^\infty(\mathbb{R}^{n-1})$, then

$$\begin{aligned} \Big|\big\langle (a_\alpha^{(2)} D^\alpha u)(\,.\,,t), v\big\rangle\Big| &= \Big|\big((a_\alpha^{(2)} D^\alpha u)(\,.\,,t), \overline{v}\big)_{0,\mathbb{R}^{n-1}}\Big| \\ &= \Big|\big\langle (D^\alpha u)(\,.\,,t), a_\alpha^{(2)}(\,.\,,t)\phi v\big\rangle\Big| \\ &\le c_1 \big\|(D^\alpha u)(\,.\,,t)\big\|_{-(m-1/2),\mathbb{R}^{n-1}} \|v\|_{m-1/2,\mathbb{R}^{n-1}}, \end{aligned}$$

and hence $\big\|(a_\alpha^{(2)} D^\alpha u)(\,.\,,t)\big\|_{-(m-1/2),\mathbb{R}^{n-1}} \le c_1 \big\|(D^\alpha u)(\,.\,,t)\big\|_{-(m-1/2),\mathbb{R}^{n-1}}$ and

$$J_1 \le c_1 \int_{\mathbb{R}^{n-1}} \big(1+|\xi'|^2\big)^{|\alpha'|-m+1/2} \bigg(\int_{-\infty}^{0} \big|(D_n^{\alpha_n} \mathcal{F}u)(\xi',t)\big|^2 dt \bigg) d\xi',$$

where we have written $\alpha = (\alpha', \alpha_n)$. Thus it follows from an argument similar to that used in the proof of Lemma 7.3 of [2, p.73] that

$$\begin{aligned} J_1 &\le c_1 d^2 \int_{\mathbb{R}^{n-1}} \big(1+|\xi'|^2\big)^{-m+1/2} \bigg(\int_{-\infty}^{0} \big|(D_n^{2m} \mathcal{F}u)(\xi',t)\big|^2 dt \bigg) d\xi' \\ &\le c_1 d^2 \int_{-\infty}^{0} \big\|(D_n^{2m} u)(\,.\,,t)\big\|^2_{-(m-1/2),\mathbb{R}^{n-1}} dt \end{aligned}$$

if $|\alpha'| = 0$,

$$J_1 \le c_1 d^2 \int_{\mathbb{R}^{n-1}} \big(1+|\xi'|^2\big)^{m-\beta_n+1/2} \bigg(\int_{-\infty}^{0} \big|(D_n^{\beta_n} \mathcal{F}u)(\xi',t)\big|^2 dt \bigg) d\xi'$$

if $|\alpha'| > 0$, where $\beta_n = \alpha_n + 1 < 2m$, and so we conclude from Corollary 3.4 (with $\gamma = 1$, $b(x) = 1$) that $J_1 \le c_1 d^2 Z(u)$, where $Z(u) = \|u\|^2_{m+1/2,\mathbb{R}^n_-} + \int_{-\infty}^{0} \big\|(D_n^{2m} u)(\,.\,,t)\big\|^2_{-(m-1/2),\mathbb{R}^{n-1}} dt$. It has consequently been shown that

$\int_{-\infty}^{0} I_1(t)dt \le c_1 dZ(u)$, and furthermore, the same arguments and the Leibnitz formula show that if α is a multi–index with $|\alpha| = 2m$ and $b(x) = a_\alpha^{(2)}(x) - a_\alpha^{(2)}(0)$, then

$$\Big| \int_{-\infty}^{0} \big\|(bD^\alpha u)(\,.\,,t)\big\|^2_{-(m-1/2),\mathbb{R}^{n-1}} dt - \int_{-\infty}^{0} \big\|\big(D^\alpha(bu)\big)(\,.\,,t)\big\|^2_{-(m-1/2),\mathbb{R}^{n-1}} dt\Big| \le c_1 dZ(u). \tag{4.8}$$

Fixing our attention secondly upon $I_2(t)$, let $\psi \in C^\infty(\mathbb{R}^n)$ such that $0 \le \psi(x) \le 1,\ \psi(x) = 1$ for $|x| < 1,\ \psi(x) = 0$ for $|x| > 2$, let $\psi_d(x) = \psi(x/d)$, and let us consider a typical term $b_\alpha(x)D^\alpha$ appearing in $L_0^{(2)} - L_-$ for which $\alpha_n < 2m$, where $b_\alpha(x) = a_\alpha^{(2)}(x) - a_\alpha^{(2)}(0)$. Then it follows from Lemma 3.3 (with $\gamma = d^{-1/(2m-1)},\ b(x) = b_\alpha(x)$) and (4.8) that

$$J_2 = \int_{-\infty}^{0} \big\|(b_\alpha D^\alpha u)(\,.\,,t)\big\|^2_{-(m-1/2),\mathbb{R}^{n-1}} dt \le c_1\Big[d^{-\alpha_n/(2m-1)}\|b_\alpha u\|_{m+1/2,\mathbb{R}^n_-} + d^{(2m-\alpha_n)/(2m-1)} \int_{-\infty}^{0} \big\|\big(D_n^{2m}(b_\alpha u)\big)(\,.\,,t)\big\|^2_{-(m-1/2),\mathbb{R}^{n-1}} dt + dZ(u)\Big]. \tag{4.9}$$

Now let us observe that $b_\alpha D^\alpha u = b_\alpha \psi_d D^\alpha u$ and that for any multi–index β, with $|\beta| \le 2m$, we have $\big|\big(D^\beta(b_\alpha\psi_d)\big)(x)\big| \le c_1 d^{1-|\beta|}$ for $x \in \mathbb{R}^n_-$. Furthermore, if γ is any multi–index satisfying $|\gamma| \le m$, then we can appeal to [9, Proposition 3.2] and argue with the extension operator E as we did with J_1 above to show that

$$\|D^\gamma u\|^2_{0,\mathbb{R}^n_-} \le c d^{2\left(m+1/2-|\gamma|\right)}\|u\|_{m+1/2,\mathbb{R}^n_-},$$

$$\int_{\mathbb{R}^n_-\times\mathbb{R}^n_-} \frac{\big|(D^\gamma u)(x) - (D^\gamma u)(y)\big|^2}{|x-y|^{n+1}} dxdy \le c_1 d^{2\left(m-|\gamma|\right)}\|u\|_{m+1/2,\mathbb{R}^n_-}.$$

Hence it follows from the definition of $\|b_\alpha u\|_{m+1/2,\mathbb{R}^n_-}$ (see [9, Eq. (2.3)]) that $\|b_\alpha u\|^2_{m+1/2,\mathbb{R}^n_-} \le c_1 d^2 \|u\|^2_{m+1/2,\mathbb{R}^n_-}$, and so we conclude from (4.8–9) that

$$J_2 \le c_1\Big[dZ(u) + d^{1/(2m-1)} \int_{-\infty}^{0} \big\|(bD_n^{2m}u)(\,.\,,t)\big\|^2_{-(m-1/2),\mathbb{R}^{n-1}} dt\Big].$$

On the other hand, we can argue with the pairing $\langle\,.\,,\,.\,\rangle$ between $H_2^{-(m-1/2)}(\mathbb{R}^{n-1})$ and its dual as we did above when dealing with J_1 to show that

$$\int_{-\infty}^{0} \big\|(bD_n^{2m}u)(\,.\,,t)\big\|^2_{-(m-1/2),\mathbb{R}^{n-1}}dt \leq c_1 \int_{-\infty}^{0} \big\|(D_n^{2m}u)(\,.\,,t)\big\|^2_{-(m-1/2),\mathbb{R}^{n-1}}dt,$$

and hence it follows that $J_2 \leq c_1 d^{1/(2m-1)}Z(u)$. Moreover, if we let $a(x) = a^{(2)}_{2me_n}(x)$ and $b(x) = a(x) - a(0)$, where e_n is the unit vector in $\mathbb{R}^n$ whose last component is 1 and all other components are 0, then we have

$$b(x)(D_n^{2m}u)(x) = \big(b(x)/a(x)\big)\Bigg[(L^{(2)}u)(x) - {\sum_{|\alpha|\leq 2m}}' a^{(2)}_\alpha(x)(D^\alpha u)(x)\Bigg] \quad \text{in} \quad \overline{\mathbb{R}^n_-},$$

where we define $b(x)/a(x)$ to be zero in $\overline{\mathbb{R}^n_-}\backslash\overline{\Omega}$ and $\sum'$ indiates that the summation is over those α for which $\alpha \neq 2me_n$. Hence if we argue as we did above with J_1 (replacing $a^{(2)}_\alpha(x)$ there by $b(x)a^{(2)}_\alpha(x)/a(x)$) and with J_2 (replacing $b_\alpha(x)$ there by $b(x)a^{(2)}_\alpha(x)/a(x)$ and observing that (4.8) also holds with the $b(x)$ there replaced by this latter term), then it is not difficult to verify that

$$\begin{aligned}\int_{-\infty}^{0} &\big\|(bD_n^{2m}u)(\,.\,,t)\big\|^2_{-(m-1/2),\mathbb{R}^{n-1}}dt \\ &\leq c_1\Bigg[\int_{-\infty}^{0} \big\|(L^{(2)}u)(\,.\,,t)\big\|^2_{-(m-1/2),\mathbb{R}^{n-1}}dt + d^{1/(2m-1)}Z(u)\Bigg].\end{aligned}$$

Thus we have shown that

$$\int_{-\infty}^{0} I_2(t)dt \leq c_1\Bigg[\int_{-\infty}^{0} \big\|(L^{(2)}u)(\,.\,,t)\big\|^2_{-(m-1/2),\mathbb{R}^{n-1}}dt + d^{1/(2m-1)}Z(u)\Bigg].$$

As a consequence of the foregoing estimates, it follows from (4.7) that

$$\begin{aligned}Y(u) \quad \leq \quad c_1\Bigg[&\big\|\big(L^{(1)} - q^{2m}\omega(x)\big)u\big\|_{0,\mathbb{R}^n_+} + \|L^{(2)}u\|_{0,\mathbb{R}^n_-} \\ &+|q|^{m-1/2}\Bigg(\int_{-\infty}^{0} \big\|(L^{(2)}u)(\,.\,,t)\big\|^2_{-(m-1/2),\mathbb{R}^{n-1}}dt\Bigg)^{1/2} \\ &+\big(\chi(d) + d^{1/2(2m-1)}\big)Y(u)\Bigg],\end{aligned}$$

and hence if we choose δ, $0 < \delta \leq d_0$, small enough so that $c_1\big(\chi(\delta) + \delta^{1/2(2m-1)}\big) \leq 1/2$, then the proof of the theorem is complete for the case of u smooth. The proof of the theorem for the general case now follows from a standard approximation procedure.

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Department of Mathematics
University of the Witwatersrand
P.O. WITS 2050
South Africa
E-mail: 036MEF@cosmos.wits.ac.za

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