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EVOLUTION EQUATIONS AS OPERATOR EQUATIONS IN LATTICES OF HILBERT SPACES

Rainer Picard
Technische Universität Dresden, Germany

Abstract. Evolution equations are considered as operator equations involving a sum of the time-derivative operator ∂_0 regarded as a normal operator in a suitable Hilbert space setting and another fairly arbitrary spatial operator A acting in a Hilbert space H. The initial data are then modeled as H-valued Dirac-δ-type sources located at time 0. A framework to discuss this and more general types of evolution problems is constructed. The solution theory relies on a Fourier-Laplace transform method set in this framework.

1. Introduction

In the current paper we shall pick up an idea developed in [15] based on establishing an initial value problem of the form (∂_0 denoting differentiation with respect to the time parameter)

$$\begin{aligned} \partial_0 U &= 2\pi\mathrm{i}(AU + F) \text{ on } \mathbb{R}^+, \\ U(0+) &= U_0 \end{aligned} \tag{1.1}$$

as an operator equation

$$\partial_0 U = 2\pi\mathrm{i}(AU + F) + \delta \otimes U_0 \text{ on } \mathbb{R}, \tag{1.2}$$

in a suitable Hilbert space frame-work. Here i denotes the imaginary unit and A is a densely defined, closed linear operator on a Hilbert space H_0 with inner product $\langle\cdot|\cdot\rangle_0$ assumed to be linear in the second factor and induced norm $||\cdot||_0$. (The factor $2\pi\mathrm{i}$ has been introduced to conveniently adapt the operator to the Fourier-Laplace transform discussed later.) The initial data $U_0 \in H_0$ and the source term $F : \mathbb{R} \to H_0$ are given. The initial data U_0 appear in the operator equation point of view as additional source term at time $t = 0$.

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For the time being we shall interpret the term $\delta \otimes U_0$ as a mapping acting on $\overset{\circ}{C}_\infty(\mathbb{R}, \mathbb{C})$, i.e., on complex-valued C_∞-functions with compact support in $\mathbb{R}$, in the following way

$$(\delta \otimes U_0)(\varphi) := \varphi(0)\, U_0 \text{ for all } \varphi \in \overset{\circ}{C}_\infty(\mathbb{R}, H). \tag{1.3}$$

Writing formally $D_0 U$ for $\frac{1}{2\pi \mathrm{i}} \partial_0 U$ we are led to the formulation

$$(D_0 - A)U = F + \frac{1}{2\pi \mathrm{i}} \delta \otimes U_0. \tag{1.4}$$

We shall consider the operator on the left-hand side as an operator sum. This idea is close in spirit to some applications considered in [6].

Initially we shall consider $(D_0 - A)$ on the algebraic tensor product $\overset{\circ}{C}_\infty(\mathbb{R}) \otimes_a D(A)$, i.e., on the linear space generated by linear combinations of products of complex-valued C_∞-functions having compact support with elements in $D(A)$, as a densely defined operator in the Hilbert space $H_{\nu,0} \otimes H_0$ obtained as the completion of $\overset{\circ}{C}_\infty(\mathbb{R}) \otimes_a D(A)$ with respect to the norm $|| \cdot ||_{\nu,0,0}$ given by the inner product

$$\langle U, V \rangle_{\nu,0,0} := \int_{\mathbb{R}} \langle U(t) | V(t) \rangle_0 \exp(-4\pi\nu t) dt, \tag{1.5}$$

for $U, V \in \overset{\circ}{C}_\infty(\mathbb{R}) \otimes_a D(A)$ (as an early reference to the concept of a tensor product of Hilbert spaces we refer to [5]). The observation that D_0 considered on $\overset{\circ}{C}_\infty(\mathbb{R}) \otimes_a D(A)$ is in fact an essentially normal operator on this weighted L_2-type space will lead to substantial simplifications of the theory.

As the distributional right-hand side of (1.4) already indicates, we will have to generalize the concept of applying D_0 and A in the spirit of distributions. This can be achieved by extending our considerations to chains of Hilbert spaces associated with D_0 and A. The construction of such chains has been well-studied and applied extensively in the literature and is connected with key expressions like "rigged (or equipped) Hilbert space", "Gelfand triple", "extended Hilbert space", "countably Hilbertian space", scales of Hilbert (or Banach) spaces" etc., see e.g. [8], [14] chapt. 8 & 9, and the survey article [10]. Since from our perspective we have two operators (rather than just one) involved we shall need to consider tensor products of such chains (i.e. linear lattices). These tensor products inherit the lattice structure of $\mathbb{Z}^2$ and are referred to as Sobolev lattices. As a more recent reference for chain constructions and tensor products of Hilbert spaces and operators we refer to [4], for the latter concepts see also [16].

It will turn out that we shall obtain a quite transparent solution theory which is quite elementary and in a sense more general than standard semi-group theory in as much as the ideas of semi-group theory are not needed at all. There are, however, close links to the theory of distributional semi-groups, [12], as well as to the concept of integrated semi-groups (see e.g. [1], [13]) and regularized semi-groups, compare e.g. the detailed account of these concepts in [11]. Indeed, the resolvent conditions required in this paper are comparable to those utilized in those contexts, although, the perspective on evolution equations presented here is conceptually different. Also, the more complex technicalities of [6] are not needed in this approach, since we are only interested in special sums with the time-derivative being one of the terms of the sum. Moreover, we restrict our attention to the Hilbert space case. Since the Fourier-Laplace transform enters as an essential tool we profit from the benefits of Laplace transform methods for evolution equations as emphasized in [2]. The use of transform techniques is at least implicitly a common means in conjunction with evolution equations, see e.g.[7], part 2, section 11, [3], [2]. In the semigroup context the Fourier-Laplace transform usually enters as an operator-valued transform whereas in the present paper the Fourier-Laplace transform will appear as a unitary transformation. Given that the major building blocks of the approach presented, such as chains of Hilbert spaces, sums of commuting operators and (vector-valued) Fourier-Laplace transform, have been around for more than 30 years, it may seem somewhat surprising that our elementary approach to evolution equations has not been discovered earlier. The novelty of the approach first introduced in [15] and further explored here is indeed only a rather subtle change of perspective hinging on the observation that the Fourier-Laplace transform yields a spectral representation of the time differentiation, which in turn is then recognized as a normal operator in a suitably weighted L_2-space.

2. Chains and Lattices of Hilbert Spaces

Although the construction of chains of Hilbert spaces is a well-known procedure, we will proceed to introduce them here in order to keep the presentation fairly self-contained as well as to introduce our basic notational framework. There are also some specific less common features (e.g. the concept of Sobolev lattices) in the following construction which are better explained in a more detailed development.

Let $C : D(C) \subseteq H_0 \to H_0$ be an arbitrary densely defined, closed linear operator on a Hilbert space H_0 (with inner product $\langle \cdot, \cdot \rangle_0$ and norm $|| \cdot ||_0$; all inner products are assumed (as is more common in the physics literature) to be linear in the *second* factor). Assuming that 0 is in the resolvent set $\varrho(C)$ we find $D(C)$ can be regarded as a Hilbert space w.r.t. the inner product

$$\langle u, v \rangle_1 = \langle Cu, Cv \rangle_0 \,, \tag{2.1}$$

for $u, v \in D(C)$. Continuing this idea we now define associated Hilbert spaces

$$H_k(C) := (D((C^k), \langle \cdot, \cdot \rangle_k), \tag{2.2}$$

where

$$\langle u, v\rangle_k \equiv \left\langle C^k u, C^k v\right\rangle, u, v \in D(C^k).$$

That $H_k(C)$ is indeed a Hilbert space follows by induction from the closedness of C, $k \in \mathbb{N}$. Moreover, if we consider the completions

$$H_{-k}(C) := ||\cdot||_{-k} H_0 \tag{2.3}$$

with respect to the norm $||\cdot||_{-k} := ||C^{-k}\cdot||$, $k \in \mathbb{N}$, then we also get Hilbert spaces.

Definition 2.1. *The family* $(H_k(C))_{k\in\mathbb{Z}}$ *of Hilbert spaces will be called the* Sobolev chain *associated with* C.

Lemma 2.2. *For the Sobolev chain* $(H_k(C))_{k\in\mathbb{Z}}$ *we have that the imbedding*

$$H_{k+1}(C) \hookrightarrow H_k(C)$$

is continuous and has dense range for all $k \in \mathbb{Z}$.

Proof. By construction we have $H_0 \equiv H_0(C)$ dense in $H_{-k}(C)$, $k \in \mathbb{N}$. Therefore, clearly

$$H_{k+1}(C) \hookrightarrow H_k(C)$$

for $k \in \mathbb{Z}^-$. Moreover, by assumption $H_1(C) \equiv D(C)$ dense in $H_0(C) \equiv H_0$. Let now $f \in H_k(C)$, $k \in \mathbb{N}$, then $C^k f \in H_0(C)$. Let $(\varphi_n)_n$ be a sequence in $H_1(C) \equiv D(C)$ approximating $C^k f \in H_0(C)$, then $(C^{-k}\varphi_n)_n$ is indeed a sequence in $H_{k+1}(C)$ approximating $f \in H_k(C)$ in $H_k(C)$. This shows the density of $H_{k+1}(C)$ in $H_k(C)$ also for $k \in \mathbb{N}$. The dense inclusion of $H_{k+1}(C)$ in $H_k(C)$ for $k \in \mathbb{Z}$ shows in particular the density of $H_{|k|+1}(C)$ in $H_k(C)$ for $k \in \mathbb{Z}$. The continuity of the imbedding follows now from a simple calculation. Indeed we have initially

$$||\varphi||_k = ||C^k\varphi||_0 = ||C^{-1}C^{k+1}\varphi||_0 \leq ||C^{-1}||\,||C^{k+1}\varphi||_0 \leq ||C^{-1}||\,||\varphi||_{k+1} \tag{2.4}$$

e.g. for $\varphi \in H_{|k+1|}(C)$, $k \in \mathbb{Z}$. The desired continuity estimate follows now from the density result. ∎

For $k \in \mathbb{N}$ let now $f \in H_{-k}(C)$. We recall that by construction of f as an element of a completion we have that f is indeed an equivalence class of sequences $(\varphi_n)_n$ in H_0 such that $(C^{-k}\varphi_n)_n$ is a Cauchy sequence in H_0. In particular $||f||_{-k} = \lim_{n\to\infty} ||C^{-k}\varphi_n||_0$. Identifying H_0 with its dual space H_0^*, i.e., the set of continuous linear functionals on H_0 equipped with the linear structure

$$(\alpha u + v)(\varphi) := \alpha^* u(\varphi) + v(\varphi) \text{ for } \alpha \in \mathbb{C}, \varphi \in H_0, u, v \in H_0^*,$$

and the usual operator norm as a norm, on the basis of the Riesz representation theorem, we obtain with

$$u(\varphi) = \langle u|\varphi\rangle_0 \text{ for } u, \varphi \in H_0 \equiv H_0^*,$$

that f gives rise to a continuous linear functional $j_k(f)$ on $H_k(C^*)$ by defining

$$j_k(f)(\varphi) := \lim_{n\to\infty} \langle \varphi_n|\varphi\rangle_0 \tag{2.5}$$

for all $\varphi \in H_k(C^*)$.

In the spirit of this correspondence we find

LEMMA 2.3. *For the families of Hilbert spaces* $(H_k(C))_{k\in\mathbb{Z}}$ *and* $(H_k(C^*))_{k\in\mathbb{Z}}$ *we have*

$$H_{-k}(C) = (H_k(C^*))^*, k \in \mathbb{Z},$$

in the sense of the unitary correspondence

$$\begin{aligned} j_k : H_{-k}(C) &\to (H_k(C^*))^* \\ f &\mapsto j_k(f) \end{aligned}$$

defined by $j_k(f)(\varphi) := \lim_{n\to\infty} \langle\varphi_n|\varphi\rangle_0$ *for all* $\varphi \in H_k(C^*)$ *and* $(\varphi_n)_n$ *a representing sequence for* f.

PROOF. It is sufficient to show the equality for $k \in \mathbb{N}$. The rest follows by the reflexivity of Hilbert spaces and interchanging the role of C and C^*.

Thus it remains to show that for $k \in \mathbb{N}$ the above formally defined mapping

$$\begin{aligned} j_k : H_{-k}(C) &\to (H_k(C^*))^* \\ f &\mapsto j_k(f) \end{aligned} \tag{2.6}$$

is a well-defined unitary map. Clearly $j_k(f)$ does not depend on the particular choice of representing Cauchy sequence $(\varphi_n)_n$. This is obvious from the injectivity of the well-defined mapping

$$\begin{aligned} H_{-k}(C) &\to H_0 \\ f &\mapsto \lim_{n\to\infty} C^{-k}\varphi_n \end{aligned}$$

$$\begin{aligned} \lim_{n\to\infty} \langle\varphi_n|\varphi\rangle_0 &= \lim_{n\to\infty} \left\langle \varphi_n|{C^*}^{-k}{C^*}^k\varphi\right\rangle_0 \\ &= \lim_{n\to\infty} \left\langle C^{-k}\varphi_n|{C^*}^k\varphi\right\rangle_0 \\ &= \left\langle \lim_{n\to\infty} C^{-k}\varphi_n|{C^*}^k\varphi\right\rangle_0 \end{aligned}$$

for all $\varphi \in H_k(C^*)$. This proves that j_k is well-defined. The continuity of j_k can be derived from the last calculation. Indeed

$$\begin{aligned}
|j_k(f)(\varphi)| &= |\lim_{n\to\infty} \langle \varphi_n|\varphi\rangle_0| \\
&= \lim_{n\to\infty} |\left\langle C^{-k}\varphi_n|{C^*}^k\varphi\right\rangle_0| \\
&\leq \lim_{n\to\infty} ||C^{-k}\varphi_n||_0||{C^*}^k\varphi||_0 \\
&= ||f||_{-k}||\varphi||_{*,k}
\end{aligned}$$

for all $\varphi \in H_k(C^*)$, where $||\cdot||_{*,k}$ denotes the Norm of $H_k(C^*)$. The latter also shows that

$$||j_k|| \leq 1.$$

Moreover, letting $\varphi = {C^*}^{-k} \lim_{n\to\infty} C^{-k}\varphi_n$ yields

$$\begin{aligned}
j_k(f)(\varphi) &= \lim_{n\to\infty} \langle \varphi_n|\varphi\rangle_0 \\
&= \lim_{n\to\infty} \left\langle C^{-k}\varphi_n|{C^*}^k\varphi\right\rangle_0 \\
&= \left\langle \lim_{n\to\infty} C^{-k}\varphi_n|{C^*}^k\varphi\right\rangle_0 \\
&= \left\langle \lim_{n\to\infty} C^{-k}\varphi_n|\lim_{m\to\infty} C^{-k}\varphi_m\right\rangle_0 \\
&= ||f||_{-k}^2,
\end{aligned}$$

and

$$||\varphi||_{*,k} = ||{C^*}^k{C^*}^{-k}\lim_{n\to\infty} C^{-k}\varphi_n||_0 = ||\lim_{n\to\infty} C^{-k}\varphi_n||_0 = ||f||_{-k}.$$

Thus we have

$$||j_k|| = 1,$$

and j_k is an isometry. It remains to show that j_k is also onto. Let now $F \in (H_k(C^*))^*$ be arbitrary then with the Riesz map

$$R_{*,k} : (H_k(C^*))^* \to H_k(C^*)$$

we find

$$R_{*,k}F \in H_k(C^*),$$

and so

$${C^*}^k R_{*,k}F \in H_0(C^*) \equiv H_0(C) \equiv H_0.$$

Note that by definition of $R_{*,k}$ we have

$$\langle R_{*,k}F|\varphi\rangle_{*,k} = F(\varphi)$$

for all $\varphi \in H_k(C^*)$. Let now $(\psi_n)_n$ be a sequence in H_k(C) such that

$$\lim_{n\to\infty} \psi_n = {C^*}^k R_{*,k}F.$$

That such a sequence exists follows from the density of $H_k(C)$ in H_0. Then $(\varphi_n)_n := (C^k\psi_n)_n$ is a sequence in H_0 such that $(C^{-k}\varphi_n)_n = (\psi_n)_n$ is a Cauchy sequence in H_0.

Let now f denote the corresponding equivalence class in $H_{-k}(C)$. We only need to show $j_k(f) = F$.

Indeed,

$$\begin{aligned} j_k(f)(\varphi) &= \lim_{n\to\infty} \langle \varphi_n|\varphi\rangle_0 \\ &= \lim_{n\to\infty} \left\langle \psi_n|C^{*^k}\varphi\right\rangle_0 \\ &= \left\langle \lim_{n\to\infty} \psi_n|C^{*^k}\varphi\right\rangle_0 \\ &= \left\langle C^{*^k}R_{*,k}F|C^{*^k}\varphi\right\rangle_0 \\ &= \langle R_{*,k}F|\varphi\rangle_{*,k} \\ &= F(\varphi) \end{aligned}$$

for all $\varphi \in H_k(C^*)$. ∎

The last lemma will motivate for us to identify $H_{-k}(C)$ with $(H_k(C^*))^*$ for $k \in \mathbb{Z}$. We also notice that the inner product $\langle\cdot|\cdot\rangle_0$ can apparently be continuously extended to

$$\begin{aligned} \langle\cdot|\cdot\rangle_0 : H_k(C^*) \times H_{-k}(C) &\to \mathbb{C} \\ (u, v) &\mapsto \langle u|v\rangle_0 \end{aligned}$$

for $k \in \mathbb{Z}$ by letting

$$\langle u|v\rangle_0 := \lim_{n\to\infty} \left\langle C^{*^k}\varphi_n|C^{-k}\psi_n\right\rangle_0 \tag{2.7}$$

where $(\varphi_n)_n$ and $(\psi_n)_n$ are sequences in $H_{|k|}(C^*)$ and $H_{|k|}(C)$, respectively, with $\varphi_n \to u$ in $H_k(C^*)$ and $\psi_n \to v$ in $H_{-k}(C)$ as $n \to \infty$, $k \in \mathbb{Z}$. This extension is well-defined. Let $(\widetilde{\varphi}_n)_n$ and $(\widetilde{\psi}_n)_n$ be two other sequences in $\mathrm{H}_{|k|}(\mathrm{C}^*)$ and $\mathrm{H}_{|k|}(\mathrm{C})$, respectively, with $\widetilde{\varphi}_n \to u$ in $H_k(C^*)$ and $\widetilde{\psi}_n \to v$ in $H_{-k}(C)$ as $n \to \infty$, then

$$\begin{aligned} &\lim_{n\to\infty} \left\langle C^{*^k}\varphi_n|C^{-k}\psi_n\right\rangle_0 - \lim_{n\to\infty} \left\langle C^{*^k}\widetilde{\varphi}_n|C^{-k}\widetilde{\psi}_n\right\rangle_0 \\ &= \lim_{n\to\infty}(\left\langle C^{*^k}\varphi_n|C^{-k}\psi_n\right\rangle_0 - \left\langle C^{*^k}\widetilde{\varphi}_n|C^{-k}\widetilde{\psi}_n\right\rangle_0) \\ &= \lim_{n\to\infty} \left\langle C^{*^k}\varphi_n - C^{*^k}\widetilde{\varphi}_n|C^{-k}\psi_n\right\rangle_0 + \lim_{n\to\infty} \left\langle C^{*^k}\widetilde{\varphi}_n|C^{-k}\psi_n - C^{-k}\widetilde{\psi}_n\right\rangle_0 \\ &\leq \lim_{n\to\infty} ||C^{*^k}\varphi_n - C^{*^k}\widetilde{\varphi}_n||_0||C^{-k}\psi_n||_0 \\ &\qquad + \lim_{n\to\infty} ||C^{*^k}\widetilde{\varphi}_n||_0||C^{-k}\psi_n - C^{-k}\widetilde{\psi}_n||_0 \\ &= \lim_{n\to\infty} ||C^{*^k}\varphi_n - C^{*^k}\widetilde{\varphi}_n||_0 \lim_{n\to\infty} ||C^{-k}\psi_n||_0 + \\ &\qquad + \lim_{n\to\infty} ||C^{*^k}\widetilde{\varphi}_n||_0 \lim_{n\to\infty} ||C^{-k}\psi_n - C^{-k}\widetilde{\psi}_n||_0 \end{aligned}$$

$$= \lim_{n\to\infty} ||\varphi_n - \widetilde{\varphi}_n||_{*,k}||v||_{-k} + ||u||_{*,k} \lim_{n\to\infty} ||\psi_n - \widetilde{\psi}_n||_{-k}$$
$$= 0.$$

The desired continuity estimate follows

$$\begin{aligned} |\langle u|v\rangle_0| &= \lim_{n\to\infty} |\left\langle C^{*^k}\varphi_n|C^{-k}\psi_n\right\rangle_0| \\ &\leq \lim_{n\to\infty} ||C^{*^k}\varphi_n||_0||C^{-k}\psi_n||_0 \\ &= \lim_{n\to\infty} ||C^{*^k}\varphi_n||_0 \lim_{n\to\infty} ||C^{-k}\psi_n||_0 \\ &= ||u||_{*,k}||v||_{-k}. \end{aligned}$$

Noting that, moreover,

$$\langle u|v\rangle_0 = \lim_{n\to\infty} \left\langle C^{*^k}\varphi_n|C^{-k}\psi_n\right\rangle_0 = \lim_{n\to\infty} \langle\varphi_n|\psi_n\rangle_0$$

motivates the continued use of the inner product symbol $\langle\cdot|\cdot\rangle_0$ in this more general sense. In the case of $k \in \mathbb{N}$ we have thus motivated and introduced the suggestive notation $\langle u|v\rangle_0^* = \langle v|u\rangle_0$ for the application of the functional $v \in H_{-k}(C)$ to $u \in H_k(C^*)$.

A chain is a linear lattice which indeed $(H_k(C))_{k\in\mathbb{Z}}$ is with respect to the dense and continuous imbedding "$\hookrightarrow$" as order relation. Obviously, we have since C is assumed to be unbounded

(2.8) $$H_k(C) \hookrightarrow H_j(C) \text{ if and only if } k \geq j, k, j \in \mathbb{Z}.$$

Thus, the chain $(H_k(C))_{k\in\mathbb{Z}}$ with respect to "$\hookrightarrow$" corresponds to $\mathbb{Z}$ with "$\geq$". In particular, we observe the lattice structure with

(2.9)
$$\sup(H_k(C), H_j(C)) := H_{\min(k,j)}(C), \inf(H_k(C), H_j(C)) := H_{\max(k,j)}(C),$$

for $k, j \in \mathbb{Z}$. This justifies the name Sobolev chain introduced in definition 2.1.

We summarize our findings for later reference.

Theorem 2.4. *Let $C : D(C) \subseteq H_0 \to H_0$ be an unbounded, densely defined, closed, linear operator with $0 \in \varrho(C)$. Here H_0 denotes a Hilbert space with norm $||\cdot||_0$ and inner product $\langle\cdot|\cdot\rangle_0$ (as always, assumed to be linear in the second factor). Then we have that the family of Hilbert spaces $(H_k(C))_{k\in\mathbb{Z}}$ has the property*

$$H_{k+1}(C) \hookrightarrow H_k(C)$$

is a continuous and dense imbedding for any $k \in \mathbb{Z}$.

By construction of the Sobolev chain associated with the operator C we see that

$$\begin{aligned} H_{|k|+1}(C) \subseteq H_{k+1}(C) &\to H_k(C) \\ \varphi &\mapsto C\varphi \end{aligned}$$

has a continuous extension, which we shall denote by

$$C_{k+1,k} : H_{k+1}(C) \to H_k(C), k \in \mathbb{Z}. \tag{2.10}$$

We find

LEMMA 2.5. *The mapping* $C_{k+1,k} : H_{k+1}(C) \to H_k(C)$ *is unitary for any* $k \in \mathbb{Z}$.

PROOF. For $\varphi \in H_{|k|+1}(C)$ we see (compare the reasoning in (2.4)

$$||\varphi||_{k+1} = ||C^{k+1}\varphi||_0 = ||C^k C\varphi||_0 = ||C\varphi||_k, k \in \mathbb{Z}.$$

By taking limits this shows that $C_{k+1,k}$ is indeed isometric. Moreover, let $\psi \in H_k(C)$ then there is a sequence $(\psi_n)_n$ in $H_{|k|}(C)$ such that $(C^k\psi_n)_n \equiv (C^{k+1}C^{-1}\psi_n)_n$ converges in H_0. Then $\varphi := \lim_{n\to\infty} C^{-1}\psi_n$ exists in $H_{k+1}(C)$ and $(C^{-1}\psi_n)_n$ is actually a sequence in $H_{|k|+1}(C)$. Now, we have $\varphi := \lim_{n\to\infty} C^{-1}\psi_n \in H_{k+1}(C)$ and $\lim_{n\to\infty} \psi_n = \lim_{n\to\infty} CC^{-1}\psi_n = \lim_{n\to\infty} C_{k+1,k}C^{-1}\psi_n$ exists in $H_k(C)$, $k \in \mathbb{Z}$. Consequently, we find

$$C_{k+1,k}\varphi = \lim_{n\to\infty} C_{k+1,k}C^{-1}\psi_n = \lim_{n\to\infty} \psi_n = \psi$$

where the limits are taken in $H_k(C)$, $k \in \mathbb{Z}$. Since $\psi \in H_k(C)$ was arbitrary, this finally shows the unitarity of $C_{k+1,k}$. □

Since $(H_k(C))_{k\in\mathbb{Z}}$ is the Sobolev chain associated with C, we also find

$$C_{k,k-1} \subseteq C_{k+1,k}$$

for all $k \in \mathbb{Z}$. Recalling that $C_{k+1,k}$ is by the fact that it is a mapping indeed a specific subset of $H_{k+1}(C) \oplus H_k(C)$, we find that indeed that $C_{k+1,k}$ is a closed subspace of $H_{k+1}(C) \oplus H_k(C)$. Moreover, the induced continuous and dense imbedding of

$$H_k(C) \oplus H_{k-1}(C) \hookrightarrow H_{k+1}(C) \oplus H_k(C)$$

implies that also

$$C_{k,k-1} \hookrightarrow C_{k+1,k}$$

for all $k \in \mathbb{Z}$. Therefore we may define

$$\begin{aligned} C : \bigcup_{k\in\mathbb{Z}} H_k(C) &\to \bigcup_{k\in\mathbb{Z}} H_k(C) \\ \varphi &\mapsto C\varphi \end{aligned} \tag{2.11}$$

with

$$C\varphi := C_{k,k-1}\varphi \tag{2.12}$$

for $\varphi \in H_k(C)$ and all $k \in \mathbb{Z}$. In other words, we obtain a mapping, which we will for simplicity of notation again denote by C,

$$C = \bigcup_{k\in\mathbb{Z}} C_{k,k-1},$$

such that $C|_{H_k} = C_{k,k-1}$ for all $k \in \mathbb{Z}$. Here "$|_{\ldots}$" should be read as "restricted to …". In $H_{-\infty}(C) := \bigcup_{k\in\mathbb{Z}} H_k(C)$ we may define a natural concept of convergence by saying $\varphi_n \to \varphi$ in $H_{-\infty}(C)$ as $n \to \infty$ if

$$\varphi_n \to \varphi \text{ in } H_k(C) \text{ as } n \to \infty \text{ for some } k \in \mathbb{Z}.$$

In this sense,

$$\begin{aligned} C : H_{-\infty}(C) &\to H_{-\infty}(C) \\ \varphi &\mapsto C\varphi \end{aligned}$$

is now continuous.

In general, we will call a mapping

$$\begin{aligned} G : H_{-\infty}(C) &\to H_{-\infty}(C) \\ \varphi &\mapsto G\varphi \end{aligned}$$

continuous, if for every $k \in \mathbb{Z}$ there is a $j \in \mathbb{Z}$ such that

$$\begin{aligned} H_k(C) &\to H_j(C) \\ \varphi &\mapsto G\varphi \end{aligned}$$

is continuous. Due to the structure of a Sobolev chain it is apparently sufficient to have this for all sufficienly large negative indices k in order to show continuity of G.

The intermediate operators $C_{k+1,k}$ give rise to unbounded operators

$$\begin{aligned} C_{k,k} : H_{k+1}(C) \subseteq H_k(C) &\to H_k(C) \\ \varphi &\mapsto C_{k+1,k}\varphi \end{aligned} \tag{2.13}$$

for $k \in \mathbb{Z}$. The connection of the generalized C to the original operator C, which is clearly just $C_{0,0}$, is contained in the trivial correspondence

$$C\varphi = C_{0,0}\varphi \text{ for all } \varphi \in H_k(C), k \in \mathbb{Z}^+,$$

and less trivially by

$$\langle\psi|C\varphi\rangle_0 = \langle C^*\psi|\varphi\rangle_0 \text{ for all } \varphi \in H_{-k}(C), \psi \in H_{k+1}(C^*) \text{ for } k \in \mathbb{N},$$

where the inner product notation $\langle\cdot|\cdot\rangle_0$ is used in the sense of the above extension to $H_{k+1}(C^*) \times H_{-k-1}(C)$ and $H_k(C^*) \times H_{-k}(C)$, respectively. The operator C^* is the analogously constructed extension of $C^*_{0,0}$ rather than $C_{0,0}$

as for the extension C. The operator C also yields a simple description of the norm in H_k(C)

$$||\varphi||_k = ||C^k\varphi||_0 \text{ for all } \varphi \in H_k(C), k \in \mathbb{Z}.$$

DEFINITION 2.6. *Let $(H_k(C))_{k\in\mathbb{Z}}$ be a Sobolev chain associated with the operator C. Then the continuous operator $\bigcup_{k\in\mathbb{Z}} C_{k,k-1}$ on $H_{-\infty}(C)$ will be called the extension of C to the Sobolev chain $(H_k(C))_{k\in\mathbb{Z}}$ (and usually denoted by the same name).*

For our purposes we also need the construction of tensor products of Sobolev chains.

DEFINITION 2.7. *Let $(H_k(C))_{k\in\mathbb{Z}}$ and $(H_k(B))_{k\in\mathbb{Z}}$ two Sobolev chains associated with operators B and C, respectively. Then $(H_j(B)\otimes H_k(C))_{(j,k)\in\mathbb{Z}^2}$ is also a lattice with respect to the dense and continuous imbedding "$\hookrightarrow$". Such a lattice will be called* Sobolev lattice*.*

REMARK 2.8. We note here that the concept of a Sobolev lattice associated with two operators B, C clearly extends to several factors (i.e., more than two operators). It is a rather natural generalization of the concept of a Sobolev chain.

Indeed, we find

$$H_j(B)\otimes H_k(C) \hookrightarrow H_u(B)\otimes H_v(C)$$

if and only if $j \geq u$ and $k \geq v$, for j, k, u, $v \in \mathbb{Z}$, and we realize an immediate correspondence to $\mathbb{Z}^2$ with "$\geq$" component-wise. In particular, we observe the lattice structure with

$$\sup(H_j(B)\otimes H_k(C), H_u(B)\otimes H_v(C)) := H_{\min(u,j)}(B)\otimes H_{\min(k,v)}(C),$$

$$\inf(H_j(B)\otimes H_k(C), H_u(B)\otimes H_v(C)) := H_{\max(u,j)}(B)\otimes H_{\max(k,v)}(C),$$

for k,$j \in \mathbb{Z}$. By analogy to the case of a single chain we denote

$$H_{-\infty}(B)\otimes H_{-\infty}(C) := \bigcup_{j,k\in\mathbb{Z}} H_j(B)\otimes H_k(C),$$

and use the analogous convergence concept:

$$\varphi_n \to \varphi \text{ in } H_{-\infty}(B)\otimes H_{-\infty}(C) \text{ as } n\to\infty \text{ if}$$
$$\varphi_n \to \varphi \text{ in } H_j(B)\otimes H_k(C) \text{ as } n\to\infty \text{ for some } k,j\in\mathbb{Z}.$$

Denoting the continuous extension (i.e., the closure) of the densely defined mapping

$$H_j(B)\otimes_a H_{|k|+1}(C) \subseteq H_j(B)\otimes H_{k+1}(C) \to H_j(B)\otimes H_k(C)$$
$$\psi\otimes\varphi \mapsto \psi\otimes C\varphi$$

by $C_{j,k+1,k}$. We find

$$C_{j,k+1,k} : H_j(B) \otimes H_{k+1}(C) \quad \to \quad H_j(B) \otimes H_k(C)$$
$$\varphi \mapsto C_{j,k+1,k}\varphi$$

again to be unitary for all $j, k \in \mathbb{Z}$.

Similarly as above we observe that

$$C_{j,k,k-1} \subset C_{j,k+1,k} \text{ for } j, k \in \mathbb{Z}.$$

Therefore we may define

$$C : H_{-\infty}(B) \otimes H_{-\infty}(C) \to H_{-\infty}(B) \otimes H_{-\infty}(C)$$
$$\varphi \mapsto C\varphi$$

with

$$C\varphi := C_{j,k+1,k}\varphi \text{ for all } \varphi \in H_j(B) \otimes H_{k+1}(C), j, k \in \mathbb{Z}.$$

In other words, we obtain a mapping

$$C = \bigcup_{j,k \in \mathbb{Z}} C_{j,k+1,k},$$

such that

$$C|_{H_j \otimes H_{k+1}} = C_{j,k+1,k} \text{ for all } j, k \in \mathbb{Z}.$$

In the above sense of convergence

$$C : H_{-\infty}(B) \otimes H_{-\infty}(C) \to H_{-\infty}(B) \otimes H_{-\infty}(C)$$
$$\varphi \mapsto C\varphi$$

and by an analogous construction for another operator B ($B|_{H_{j+1} \otimes H_k} = B_{j+1,j,k}$ for all $j, k \in \mathbb{Z}$)

$$B : H_{-\infty}(B) \otimes H_{-\infty}(C) \to H_{-\infty}(B) \otimes H_{-\infty}(C)$$
$$\varphi \mapsto B\varphi$$

are now again linear and continuous. Moreover, they are commuting in the obvious sense.

We will call a general mapping

$$G : H_{-\infty}(B) \otimes H_{-\infty}(C) \to H_{-\infty}(B) \otimes H_{-\infty}(C)$$
$$\varphi \mapsto G\varphi$$

continuous, if for every $(u, v) \in \mathbb{Z}^2$ there is a $(j, k) \in \mathbb{Z}^2$ such that

$$H_u(B) \otimes H_v(C) \to H_j(B) \otimes H_k(C)$$
$$\varphi \mapsto G\varphi$$

is continuous. The construction clearly extends to several factors. The connection to the operator $I \otimes C_{0,0}$ is given by

$$C\varphi = I \otimes C\varphi \text{ for all } \varphi \in H_j(B) \otimes H_k(C), j \in \mathbb{Z}, k \in \mathbb{Z}^+,$$

and by

$$\langle \psi | C\varphi \rangle_{\otimes} = \langle (I \otimes C_{0,0})^* \psi | \varphi \rangle_{\otimes}$$

for all, $\varphi \in H_{-j}(B) \otimes H_{-k}(C)$, $\psi \in H_j(B^*) \otimes H_{k+1}(C^*)$ for $j, k \in \mathbb{N}$, where the inner product notation $\langle \cdot | \cdot \rangle_{\otimes}$ is used in the sense of the extension to the Cartesian products

$$(H_j(B^*) \otimes H_{k+1}(C^*)) \times (H_{-j}(B) \otimes H_{-k-1}(C))$$

and

$$(H_j(B^*) \otimes H_k(C^*)) \times (H_{-j}(B) \otimes H_{-k}(C)),$$

respectively. For the operator $B \otimes I$ we have the analogous properties. The operators B and C also yield a simple description of the norm in in $H_j(B) \otimes H_k(C)$

$$||\varphi||_{\otimes, jk} = ||B^j C^k \varphi||_{\otimes,0,0} = ||C^k B^j \varphi||_{\otimes,0,0}$$

for all $\varphi \in H_j(B) \otimes H_k(C), j, k \in \mathbb{Z}$.

Definition 2.9. *Let $(H_j(B) \otimes H_k(C))_{j,k \in \mathbb{Z}}$ be a Sobolev lattice associated with the operators B and C. Then the continuous operator $\bigcup_{j,k \in \mathbb{Z}} C_{j,k,k-1}$ and the analogously constructed $\bigcup_{j,k \in \mathbb{Z}} B_{j,j-1,k}$ acting on $H_{-\infty}(B) \otimes H_{-\infty}(C)$ will be called the extension of C and B to the Sobolev lattice $(H_j(B) \otimes H_k(C))_{j,k \in \mathbb{Z}}$ (and usually denoted again by the same names).*

The following examples are of importance in our approach.

2.1. *Examples.*

Example 1. For $\nu \in \mathbb{R}$ the mapping

$$\begin{aligned} \mathring{C}_\infty(\mathbb{R}) \subseteq L_2(\mathbb{R}, e^{-4\pi\nu t} dt) &\to L_2(\mathbb{R}, e^{-4\pi\nu t} dt) \\ \varphi &\mapsto (2\pi \mathrm{i})^{-1} \varphi' + \mathrm{i}\nu\varphi \end{aligned}$$

is essentially selfadjoint, i.e., its closure is a selfadjoint operator, which will be denoted by D_ν. Here the Hilbert space $L_2(\mathbb{R}, \mathrm{e}^{-4\pi\nu t} dt)$ is just the completion of $\mathring{C}_\infty(\mathbb{R})$ with respect to the norm $|| \cdot ||_{\nu,0}$ induced by the inner product

$$\langle \varphi | \psi \rangle_{\nu,0} := \int_{\mathbb{R}} \varphi(t)^* \psi(t) \mathrm{e}^{-4\pi\nu t} dt \text{ for } \varphi, \psi \in \mathring{C}_\infty(\mathbb{R}).$$

For convenience we define for later use

$$D_0 := D_\nu - \mathrm{i}\nu,$$

and note that

$$D_0 \varphi = (2\pi \mathrm{i})^{-1} \varphi'$$

for all $\varphi \in \overset{\circ}{C}_\infty(\mathbb{R})$. The dependence of D_0 on the parameter ν has to be deduced from the context. Since D_ν is selfadjoint, we have $\pm\mathrm{i} \in \varrho(D_\nu)$. So we can construct the associated Sobolev chains

$$(H_k(D_\nu - \mathrm{i}))_{k\in\mathbb{Z}},$$

which we shall also refer to as the chain of exponentially weighted Sobolev spaces.

In particular, we have

$$H_0(D_\nu - \mathrm{i}) = L_2(\mathbb{R}, \mathrm{e}^{-4\pi\nu t}dt).$$

The norm and inner product in $H_k(D_\nu - \mathrm{i})$ will be labelled as $||\cdot||_{\nu,k}$ and $\langle\cdot|\cdot\rangle_{\nu,k}$, respectively. Note that

$$||\varphi||_{\nu,k} \le ||\varphi||_{\nu,k+1} \text{ for all } \varphi \in H_{k+1}(D_\nu - \mathrm{i}),$$

since

$$||(D_\nu - \mathrm{i})^{-1}|_{L_2(\mathbb{R}),\exp(-4\pi\nu t)dt)}|| = 1.$$

We note that by a standard cut-off and smoothing procedure it can easily be shown that

(2.14) $$\overset{\circ}{C}_\infty(\mathbb{R}) \text{ is dense in } H_k(D_\nu - \mathrm{i}) \text{ for } k \in \mathbb{Z}.$$

Since $\overset{\circ}{C}_\infty(\mathbb{R}) \subset \bigcap_k H_k(D_\nu - \mathrm{i}) =: H_\infty(D_\nu - \mathrm{i})$, the density result also shows that

(2.15) $$H_\infty(D_\nu - \mathrm{i}) \text{ dense in } H_k(D_\nu - \mathrm{i}) \text{ for } k \in \mathbb{Z}.$$

Next we construct a companion chain as another example needed in our context.

EXAMPLE 2. The mapping

$$\begin{aligned} \overset{\circ}{C}_\infty(\mathbb{R}) \subseteq L_2(\mathbb{R}) &\to L_2(\mathbb{R}) \\ \varphi &\mapsto m\varphi \end{aligned}$$

with

$$(m\varphi)(t) := t\varphi(t) \text{ for all } \varphi \in \overset{\circ}{C}_\infty(\mathbb{R}),$$

can be easily seen to have a selfadjoint operator as closure, which will be denoted again by m. Since m is selfadjoint, we have e.g. $\pm\mathrm{i} \in \varrho(m)$. So we can construct the associated Sobolev chains

$$(H_k(m - \mathrm{i}))_{k\in\mathbb{Z}}.$$

The norm and inner product in $H_k(m-\mathrm{i})$ will be labelled as $||\cdot||_k$ and $\langle\cdot|\cdot\rangle_{\nu,k}$, respectively.

The latter two Sobolev chains are connected by the Fourier-Laplace transform $\mathcal{L}_\nu$ defined by

$$\begin{aligned}(\mathcal{L}_\nu\varphi)(x) :&= \int_{\mathbb{R}} \exp(-2\pi \mathrm{i}(x-\mathrm{i}\nu)y)\varphi(y)dy,\\ &= \int_{\mathbb{R}} \exp(-2\pi \mathrm{i}xy)\exp(-2\pi\nu y)\varphi(y)dy,\\ &= (\mathcal{L}_0 \exp(-2\pi\nu m)\varphi)(x)\end{aligned}$$

for all $\varphi \in \mathring{C}_\infty(\mathbb{R})$.

The Fourier-Laplace transform $\mathcal{L}_\nu$ extends by continuity and by the density of $\mathring{C}_\infty(\mathbb{R})$ in $H_k(D_\nu - \mathrm{i})$ for $k \in \mathbb{Z}$ to a continuous bijection

$$\mathcal{L}_\nu : H_{-\infty}(D_\nu - \mathrm{i}) \to H_{-\infty}(m - \mathrm{i}),$$

where we re-utilize the same name for the extension (compare [15]). Indeed, $\mathcal{L}_\nu|_{H_k} : H_k(D_\nu - \mathrm{i}) \to H_k(m - \mathrm{i})$ is unitary for any $k \in \mathbb{Z}$.

3. Space-Time Evolution Equations

Let now $A : D(A) \subseteq H \to H$ be an arbitrary densely defined, closed linear operator on a Hilbert space H with nonempty resolvent set $\varrho(A)$ say $\lambda_0 \in \varrho(A)$. Consider the Sobolev lattice $(H_j(D_\nu - \mathrm{i}) \otimes H_k(\lambda_0 - A))_{j,k\in\mathbb{Z}}$, the associated topological vector space $H_{-\infty}(D_\nu - \mathrm{i}) \otimes H_{-\infty}(\lambda_0 - A)$ and the extensions of $D_\nu \equiv D_\nu \otimes I_H$ and $A \equiv I \otimes A$ to the Sobolev lattice $(H_j(D_\nu - \mathrm{i}) \otimes H_k(\lambda_0 - A))_{j,k\in\mathbb{Z}}$, which we shall denote for ease of notation again by D_ν and $A(I : H_0(D_\nu - \mathrm{i}) \to H_0(D_\nu - \mathrm{i}), I_H : H \to H$ denoting the respective identities). Note that the choice of a different point in $\varrho(A)$ yields the same linear space with an equivalent norm.

Then with $D_0 := D_\nu - \mathrm{i}\nu$ we have

$$D_0 - A : H_{-\infty}(D_\nu - \mathrm{i}) \otimes H_{-\infty}(\lambda_0 - A) \to H_{-\infty}(D_\nu - \mathrm{i}) \otimes H_{-\infty}(\lambda_0 - A) \tag{3.1}$$

as a continuous operator. Our aim is to investigate equations of the form

$$(D_0 - A)u = f \in H_{-\infty}(D_\nu - \mathrm{i}) \otimes H_{-\infty}(\lambda_0 - A).$$

As a first result we have the following solution theory of such equations.

Theorem 3.1. *Let $A : D(A) \subseteq H \to H$ be a densely defined, closed linear operator on a Hilbert space H such that there is a $\nu \in \mathbb{R}$ with*

$$\mathbb{R} - \mathrm{i}\nu \subset \varrho(A), \tag{3.2}$$

and

$$\sup_{\lambda\in\mathbb{R}} ||(\lambda - \mathrm{i})^{-k}(\lambda - \mathrm{i}\nu - A)^{-1}|| < \infty \text{ for some } k \in \mathbb{N}. \tag{3.3}$$

Then the extended operator

$$D_0 - A : H_{-\infty}(D_\nu - \mathrm{i}) \otimes H_{-\infty}(\mathrm{i}\nu + A) \to H_{-\infty}(D_\nu - \mathrm{i}) \otimes H_{-\infty}(\mathrm{i}\nu + A)$$

is a continuous bijection.

PROOF. Let $f \in H_s(D_\nu - \mathrm{i}) \otimes H_v(\mathrm{i}\nu + A)$, $s, v \in \mathbb{Z}$, be given. The extended Fourier-Laplace transform

$$\mathcal{L}_\nu : H_{-\infty}(D_\nu - \mathrm{i}) \otimes H_{-\infty}(\mathrm{i}\nu + A) \to H_{-\infty}(m - \mathrm{i}) \otimes H_{-\infty}(\mathrm{i}\nu + A),$$

determined by letting

$$\mathcal{L}_\nu(\varphi \otimes w) := \mathcal{L}_\nu \varphi \otimes w$$

for $\varphi \in \overset{\circ}{C}_\infty(\mathbb{R})$, $w \in H_{-\infty}(\mathrm{i}\nu + A)$, can now be utilized to give a solution of the equation

$$(D_0 - A)u = f. \tag{3.4}$$

Since (3.2) implies that $(\lambda - \mathrm{i}\nu - A)^{-1}$ extends to a continuous operator on $H_v(\mathrm{i}\nu + A)$ and because of (3.3) even to a bounded operator from the Hilbert space $H_s(m - \mathrm{i}) \otimes H_v(\mathrm{i}\nu + A)$ to the Hilbert space $H_{s-k}(m - \mathrm{i}) \otimes H_v(\mathrm{i}\nu + A)$, we have indeed hat

$$(m - \mathrm{i}\nu - A)^{-1}\mathcal{L}_\nu f \in H_{s-k}(m - \mathrm{i}) \otimes H_v(\mathrm{i}\nu + A),$$

and then

$$u = \mathcal{L}_\nu^*(m - \mathrm{i}\nu - A)^{-1}\mathcal{L}_\nu f$$

yields a solution of (3.4) in $H_{-\infty}(D_\nu - \mathrm{i}) \otimes H_{-\infty}(\mathrm{i}\nu + A)$. Uniqueness follows again by applying the Fourier-Laplace transform. Indeed, let $f = 0$ in (3.4), then

$$(m - \mathrm{i}\nu - A)\mathcal{L}_\nu u = 0. \tag{3.5}$$

By the density of $\overset{\circ}{C}_\infty(\mathbb{R})$ in $H_s(m - \mathrm{i})$ and the definition of the tensor product we have a sequence $(\varphi_n)_n$ in the algebraic tensor product $\overset{\circ}{C}_\infty(\mathbb{R}) \otimes_a H_{|v|+1}(\mathrm{i}\nu + A)$ approximating $\mathcal{L}_\nu u$. Since

$$(m - \mathrm{i}\nu - A) : H_{-\infty}(m - \mathrm{i}) \otimes H_{-\infty}(\mathrm{i}\nu + A) \to H_{-\infty}(m - \mathrm{i}) \otimes H_{-\infty}(\mathrm{i}\nu + A)$$

is continuous, we have

$$(m - \mathrm{i}\nu - A)\varphi_n \to (m - \mathrm{i}\nu - A)\mathcal{L}_\nu u = 0$$

in $H_s(m - \mathrm{i}) \otimes H_v(\mathrm{i}\nu + A)$ as $n \to \infty$ for some $j, m \in \mathbb{Z}$. It follows that

$$(m - \mathrm{i})^s(\mathrm{i}\nu + A)^v(m - \mathrm{i}\nu - A)\varphi_n \to 0$$

in $H_0(m - \mathrm{i}) \otimes H_0(\mathrm{i}\nu + A)$ as $n \to \infty$, or

$$\int_{\mathbb{R}} ||(t - \mathrm{i})^s(\mathrm{i}\nu + A)^v(t - \mathrm{i}\nu - A)\varphi_n(t)||_0^2 dt \to 0$$

as $n \to \infty$. Using assumption (3.3) we get with

$$C_1 := sup_{\lambda \in \mathbb{R}}||(\lambda - \mathrm{i})^{-k}(\lambda - \mathrm{i}\nu - A)^{-1}||$$

the estimate

$$\begin{aligned}(3.6)\qquad &\int_{\mathbb{R}} ||(t-\mathrm{i})^{s-k}(\mathrm{i}\nu+A)^{v}\varphi_n(t)||_0^2 dt \\ &\qquad \le C_1 \int_{\mathbb{R}} ||(t-\mathrm{i})^{s}(i\nu+A)^{v}(t-\mathrm{i}\nu-A)\varphi_n(t)||_0^2 dt,\end{aligned}$$

and so

$$\int_{\mathbb{R}} ||(t-\mathrm{i})^{s-k}(\mathrm{i}\nu+A)^{v}\varphi_n(t)||_0^2 dt \to 0$$

as $n \to \infty$. The latter shows that

$$\varphi_n \to 0 \text{ in } H_{-\infty}(m-\mathrm{i}) \otimes H_{-\infty}(\mathrm{i}\nu+A) \text{ as } n \to \infty.$$

Since on the other hand $(\varphi_n)_n$ was taken such that

$$\varphi_n \to \mathcal{L}_\nu u \text{ in } H_{-\infty}(m-\mathrm{i}) \otimes H_{-\infty}(\mathrm{i}\nu+A) \text{ as } n \to \infty,$$

we obtain

$$\mathcal{L}_\nu u = 0,$$

and so as desired

$$u = 0.$$

Finally, continuity of the solution operator also follows from the estimate (3.6) and the density of the algebraic tensor product $\overset{\circ}{C}_\infty(\mathbb{R}) \otimes_a H_{|v|+1}(\mathrm{i}\nu+A)$ in $H_s(m-\mathrm{i}) \otimes H_v(\mathrm{i}\nu+A)$. In fact, this way we obtain

$$\begin{aligned}(3.7)\qquad ||(D_0-A)^{-1}f||_{\nu,s-k,v} &= ||(m-\mathrm{i}\nu-A)^{-1}\mathcal{L}_\nu f||_{s-k,v} \\ &\le C_1||\mathcal{L}_\nu f||_{s,v} = C_1||f||_{\nu,s,v}\end{aligned}$$

for all $f \in H_s(D_\nu-\mathrm{i}) \otimes H_v(\mathrm{i}\nu+A)$, $s, v \in \mathbb{Z}$. ◻

Lemma 3.2. *The condition (3.3) in theorem 3.1, is equivalent to the existence of $u, v, k \in \mathbb{N}$ with $u+v=k$ and*

$$(3.8)\qquad \sup_{\lambda\in\mathbb{R}} ||(\lambda-\mathrm{i})^{-u}(\mathrm{i}\nu-A)^{-v}(\lambda-\mathrm{i}\nu-A)^{-1}|| < \infty.$$

Moreover, if condition (3.3) holds then (3.8) holds for all $u, v \in \mathbb{N}$ with $u+v = k$, where k is the natural number whose existence is assured by (3.3).

Proof. We first have by the resolvent equality

$$\begin{aligned}(\lambda-\mathrm{i})(\mathrm{i}\nu+A)^{-1}(\lambda-\mathrm{i}\nu-A)^{-1} &= (\mathrm{i}\nu+A)^{-1} - (\lambda-\mathrm{i}\nu-A)^{-1} + \\ &-\mathrm{i}(\mathrm{i}\nu+A)^{-1}(\lambda-\mathrm{i}\nu-A)^{-1}.\end{aligned}$$

This implies (with $v \geq 1$)

$$\begin{aligned} E(u,v) &:= (\lambda - \mathrm{i})^{-u}(\mathrm{i}\nu + A)^{-v}(\lambda - \mathrm{i}\nu - A)^{-1} \\ &= -(\lambda - \mathrm{i})^{-u-1}(\mathrm{i}\nu + A)^{-v} \\ &\quad - (\lambda - \mathrm{i})^{-u-1}(\mathrm{i}\nu + A)^{-(v-1)}(\lambda - \mathrm{i}\nu - A)^{-1} \\ &\quad + \mathrm{i}(\lambda - \mathrm{i})^{-u-1}(\mathrm{i}\nu + A)^{-v}(\lambda - \mathrm{i}\nu - A)^{-1} \\ &= (\lambda - \mathrm{i})^{-u-1}(\mathrm{i}\nu + A)^{-v} - E(u+1, v-1) + E(u+1, v). \end{aligned}$$

From this calculation we obtain (noting that $||E(u+1,v)|| \leq ||E(u,v)||$ and $||E(u+1,v)|| \leq ||(\mathrm{i}\nu + A)^{-1}||\ ||E(u+1,v-1)||$) the two estimates

$$||E(u,v)|| \leq ||(\lambda - \mathrm{i})^{-u-1}(\mathrm{i}\nu + A)^{-v}|| + (1 + ||(\mathrm{i}\nu + A)^{-1}||)||E(u+1, v-1)||,$$

$$||E(u+1, v-1)|| \leq ||(\lambda - \mathrm{i})^{-u-1}(\mathrm{i}\nu + A)^{-v}|| + 2||E(u,v)||.$$

From these the claim follows by induction. ∎

The smallest such number $k \in \mathbb{N}$ occurring in assumption (3.3) will in lieu of (3.7) be referred to as *regularity defect*. There is one other aspect of a solution theory associated with the operator in (3.1). We would want to have *causal* solutions. We first need the concept of time-support.

Definition 3.3. *Let $g \in H_{-k}(C)$, $k \in \mathbb{Z}$. The* time-support *of g is defined as*

$$\operatorname{supp}_0 g := \mathbb{R} - \bigcup \{ I | I \text{ open interval } \& \langle g | \varphi \otimes \psi \rangle_{\nu,0,0} = 0 \\ \text{for all } \varphi \in \overset{\circ}{C}_\infty(I), \psi \in H_k(C) \}.$$

In our context this concept leads to the following definition of *causality*.

Definition 3.4. *Let $C : D(C) \subseteq H \to H$ be a densely defined, closed linear operator on a Hilbert space H with $0 \in \varrho(C)$ and $\nu \in \mathbb{R}$ fixed. Then a continuous mapping*

$$G : H_{-\infty}(D_\nu - \mathrm{i}) \otimes H_{-\infty}(C) \to H_{-\infty}(D_\nu - \mathrm{i}) \otimes H_{-\infty}(C)$$

will be called causal *if*

$$\inf \operatorname{supp}_0 Gf \geq \inf \operatorname{supp}_0 f \in \mathbb{R} \cup \{-\infty\} \text{ for all } f \in H_{-\infty}(D_\nu - \mathrm{i}) \otimes H_{-\infty}(C).$$

With the concept of causality we obtain the following refinement of our solution theory.

Theorem 3.5. *Let $A : D(A) \subseteq H \to H$ be a densely defined, closed linear operator on a Hilbert space H such that there is a $\nu_0 \in \mathbb{R}^+$ with*

$$\mathbb{R} - \mathrm{i}\nu \subset \varrho(A) \text{ for all } \nu \geq \nu_0, \tag{3.9}$$

and

$$\sup_{\lambda\in\mathbb{C}^-} ||(\lambda-\mathrm{i})^{-k}(\lambda-\mathrm{i}\nu_0-A)^{-1}|| < \infty \text{ for some } k\in\mathbb{N}. \tag{3.10}$$

Then the extended operator

$$D_0 - A : H_{-\infty}(D_\nu - \mathrm{i})\otimes H_{-\infty}(\mathrm{i}\nu + A) \to H_{-\infty}(D_\nu - \mathrm{i})\otimes H_{-\infty}(\mathrm{i}\nu + A)$$

is a continuous and causal bijection for all $\nu \geq \nu_0$*.*

PROOF. Let $f \in H_s(D_\nu - \mathrm{i})\otimes H_v(\mathrm{i}\nu + A)$, $s, v\in\mathbb{Z}$, be given. Then according to the proof of theorem 3.1 the solution u of (3.4) is given by

$$u = (D_0 - A)^{-1} f = \mathcal{L}_\nu (m - \mathrm{i}\nu - A)^{-1}\mathcal{L}_\nu f \in H_{s-k}(D_\nu - \mathrm{i})\otimes H_v(\mathrm{i}\nu + A).$$

Assuming now that $\operatorname{supp}_0 f \subseteq [0,\infty[$, we need to show that $\operatorname{supp}_0 u \subseteq [0,\infty[$. We first observe that

$$z \mapsto (\mathcal{L}_{\nu-\mathfrak{Im}(z)}(D_\nu - \mathrm{i})^s(\mathrm{i}\nu + A)^v f)(\mathfrak{Re}(z))$$

is analytic in $\mathbb{C}^- := \mathbb{R} - \mathrm{i}\mathbb{R}^+$, indeed

$$z \mapsto \left\langle w | (\mathcal{L}_{\nu-\mathfrak{Im}(z)}(D_\nu - \mathrm{i})^s(\mathrm{i}\nu + A)^v f)(\mathfrak{Re}(z))\right\rangle_0$$

satisfies the assumptions of the Paley-Wiener theorem (see e.g. [9]) for any $w \in H = H_0(\mathrm{i}\nu + A)$. From the analyticity of the resolvent it follows that

$$z \mapsto (z-\mathrm{i})^{-k}(z - \mathrm{i}\nu - A)^{-1}(\mathcal{L}_{\nu-\mathfrak{Im}(z)}(D_\nu - \mathrm{i})^s(\mathrm{i}\nu + A)^v f)(\mathfrak{Re}(z))$$

is also analytic in $\mathbb{C}^-$. Consequently, also the function ζ_w given by

$$z \mapsto \left\langle w | (z-\mathrm{i})^{-k}(z - \mathrm{i}\nu - A)^{-1}(\mathcal{L}_{\nu-\mathfrak{Im}(z)}(D_\nu - \mathrm{i})^s(\mathrm{i}\nu + A)^v f)(\mathfrak{Re}(z))\right\rangle_0$$

is analytic in $\mathbb{C}^-$. Moreover, we have

$$\zeta_w(\cdot - \mathrm{i}\varepsilon) \in L_2(\mathbb{R})$$

for any $\varepsilon \in \mathbb{R}^+$, and

$$\begin{aligned}
&\int_{\mathbb{R}} |\zeta_w(\lambda - \mathrm{i}\varepsilon)|^2 d\lambda = \\
&\qquad = \int_{\mathbb{R}} |\left\langle w|(\lambda - \mathrm{i}(1+\varepsilon))^{-k}(\lambda - \mathrm{i}(\nu+\varepsilon) - A)^{-1}\right. \\
&\qquad\qquad\qquad \left.(\mathcal{L}_{\nu+\varepsilon}(D_\nu - \mathrm{i})^s(\mathrm{i}\nu + A)^v f)\ (\lambda)\right\rangle_0 |^2 d\lambda \\
&\qquad \leq \int_{\mathbb{R}} ||w||_0^2 ||(\lambda - \mathrm{i}(1+\varepsilon))^{-k}(\lambda - \mathrm{i}(\nu+\varepsilon) - A)^{-1} \\
&\qquad\qquad\qquad (\mathcal{L}_{\nu+\varepsilon}(D_\nu - \mathrm{i})^s(\mathrm{i}\nu + A)^v f)(\lambda)||_0 d\lambda \\
&\qquad \leq C_1^2 ||w||_0^2 \int_{\mathbb{R}} ||(\mathcal{L}_{\nu+\varepsilon}(D_\nu - \mathrm{i})^s(\mathrm{i}\nu + A)^v f)(\lambda)||_0 d\lambda \\
&\qquad \leq C_1^2 ||w||_0^2 ||(D_\nu - \mathrm{i})^s(\mathrm{i}\nu + A)^v f||^2_{\nu+\varepsilon,0} \\
&\qquad \leq C_1^2 ||w||_0^2 ||(D_\nu - \mathrm{i})^s(\mathrm{i}\nu + A)^v f||^2_{\nu,0}.
\end{aligned}$$

The last estimate follows from $\exp(-4\pi\varepsilon t) \leq 1$ for $t \in [0, \infty[$ and since

$$(\mathrm{i}\nu + A) : H_{-\infty}(D_\nu - \mathrm{i}) \otimes H_{-\infty}(\mathrm{i}\nu + A) \to H_{-\infty}(D_\nu - \mathrm{i}) \otimes H_{-\infty}(\mathrm{i}\nu + A)$$

and

$$(D_\nu - \mathrm{i}) : H_{-\infty}(D_\nu - \mathrm{i}) \otimes H_{-\infty}(\mathrm{i}\nu + A) \to H_{-\infty}(D_\nu - \mathrm{i}) \otimes H_{-\infty}(\mathrm{i}\nu + A)$$

as well as their inverses are causal. Since it is obvious that $(\mathrm{i}\nu + A)$, $(\mathrm{i}\nu + A)^{-1}$ and $(D_\nu - \mathrm{i})$ cannot extend the time-support beyond $[0, \infty[$, to show this we only need to prove that the time-support will stay contained in $[0, \infty[$ if

$$(D_\nu - \mathrm{i})^{-1} : H_{-\infty}(D_\nu - \mathrm{i}) \otimes H_{-\infty}(\mathrm{i}\nu + A) \to H_{-\infty}(D_\nu - \mathrm{i}) \otimes H_{-\infty}(\mathrm{i}\nu + A)$$

is applied. Indeed, let $g \in H_{-\infty}(D_\nu - \mathrm{i}) \otimes H_{-\infty}(\mathrm{i}\nu + A)$ be such that

$$\mathrm{supp}_0\, g \subseteq [0, \infty[,$$

i.e.,

$$\langle g | \varphi \otimes \psi \rangle_{\nu,0,0} = 0$$

for all $\varphi \in \overset{\circ}{C}_\infty(\mathbb{R}^-)$ and $\psi \in H_j(\mathrm{i}\nu + A)$ for a sufficiently large $j \in \mathbb{N}$. We have

$$\begin{aligned} \left\langle (D_\nu - \mathrm{i})^{-1} g | \varphi \otimes \psi \right\rangle_{\nu,0,0} &= \left\langle g | (D_\nu + \mathrm{i})^{-1} \varphi \otimes \psi \right\rangle_{\nu,0,0} \\ &= \left\langle g | ((D_\nu + \mathrm{i})^{-1} \varphi) \otimes \psi \right\rangle_{\nu,0,0}. \end{aligned}$$

Now, if $(D_\nu + \mathrm{i})^{-1}\varphi$ can be approximated by elements in $\overset{\circ}{C}_\infty(\mathbb{R}^-)$ in any $H_j(D_\nu - \mathrm{i})$, $j \in \mathbb{Z}$, the claim follows. Apparently it suffices to have this approximation property for $j \in \mathbb{N}$ sufficiently large. Noting the implied boundary conditions at $\pm\infty$, an elementary calculation shows that

$$((D_\nu + \mathrm{i})^{-1}\varphi)(t) = -2\pi\mathrm{i} \exp(2\pi(\nu + 1)t) \int_t^\infty \exp(-2\pi(\nu + 1)s)\varphi(s)ds.$$

Clearly,

$$\mathrm{supp}_0 (D_\nu + \mathrm{i})^{-1}\varphi \subset \mathbb{R}^-,$$

and

$$(D_\nu + \mathrm{i})^{-1}\varphi \in C_\infty(\mathbb{R}) \cap H_\infty(D_\nu - \mathrm{i}).$$

Thus a simple cut-off yields the desired approximation by $\overset{\circ}{C}_\infty(\mathbb{R}^-)$-functions. This finally concludes the proof of the above estimate

$$||\zeta_w(\cdot - \mathrm{i}\varepsilon)||_{0,0} \leq C_1 ||w||_0 ||f||_{\nu,s,v}.$$

Thus, applying the Paley-Wiener theorem yields that

$$\zeta_w(\cdot - \mathrm{i}0+) = \left\langle w | (\cdot - \mathrm{i})^{-k} (\cdot - \mathrm{i}\nu - A)^{-1} \mathcal{L}_\nu (D_\nu - \mathrm{i})^s (\mathrm{i}\nu + A)^v f \right\rangle_0$$

has an inverse Fourier transform with support in $[0, \infty[$. Since pre-multiplication with $\exp(2\pi\nu m)$ does not change the support we have

$$\begin{aligned} 0 &= \langle \mathcal{L}_\nu^* \zeta_w(\cdot - \mathrm{i}0+)|\varphi\rangle_{\nu,0} \\ &= \langle \exp(2\pi\nu m)\mathcal{L}_0^* \zeta_w(\cdot - \mathrm{i}0+)|\varphi\rangle_{\nu,0} = \langle \langle w|u\rangle_0 \, |\varphi\rangle_{\nu,0} \\ &= \langle u|\varphi \otimes w\rangle_{\nu,0} \end{aligned}$$

for all $\varphi \in \overset{\circ}{C}_\infty(\mathbb{R}^-)$ and arbitrary $w \in H_j(\mathrm{i}\nu + A)$ for a sufficiently large $j \in \mathbb{N}$. In other words, we have

$$\mathrm{supp}_0\, u \subseteq [0, \infty[.$$

The same result now follows for arbitrary $\inf \mathrm{supp}_0 f \in \mathbb{R}$ by the time-shift invariance of $(D_0 - A)$. In the limit case $\inf \mathrm{supp}_0 f = -\infty$ nothing needs to be shown. ☐

An analogous statement to lemma 3.2 holds in this case.

LEMMA 3.6. *The condition (3.10) in theorem 3.5, is equivalent to the existence of* $u, v, k \in \mathbb{N}$ *with* $u + v = k$ *and*

$$\sup_{\lambda \in \mathbb{C}^-} ||(\lambda - \mathrm{i})^{-u}(\mathrm{i}\nu + A)^{-v}(\lambda - \mathrm{i}\nu - A)^{-1}|| < \infty.$$

PROOF. The result follows by the same reasoning as in the proof of lemma 3.2. ☐

REMARK 3.7. Re-translating lemma 3.6 by the inverse Fourier-Laplace transform into the time-dependent realm we have also shown that if condition (3.10) of theorem 3.5 holds then we also have the continuity of

$$\begin{aligned} &(D_\nu - \mathrm{i})^{-u}(\mathrm{i}\nu_0 - A)^{-v}(D_\nu - A)^{-1} : \\ &\qquad H_j(D_\nu - \mathrm{i}) \otimes H_m(\mathrm{i}\nu + A) \to H_j(D_\nu - \mathrm{i}) \otimes H_m(\mathrm{i}\nu + A) \end{aligned}$$

for all $j, m \in \mathbb{Z}$ and $u, v \in \mathbb{N}$ with $u + v = k$. Observing that the pure initial value problem

$$(D_0 - A)u = \frac{1}{2\pi \mathrm{i}} \delta \otimes u_0$$

is solved by

$$U u_0 := (D_0 - A)^{-1} \frac{1}{2\pi \mathrm{i}} \delta \otimes u_0,$$

we realize that the result of lemma 3.6 is indeed somewhat akin of the equivalence of integrated semi-groups (see e.g. [1]) and regularized semi-groups (see [11] for a detailed study).

4. Initial Value Problems as Space-Time Evolution Problems

We shall now investigate the relationship between (3.4) and the associated (formal) initial value problem

$$(\partial_0 u)(t) = 2\pi\mathrm{i}(Au(t) + f(t)),\ t > 0, \tag{4.1}$$

with initial condition

$$u(0+) = u_0 \tag{4.2}$$

with given say H-valued f. The assumptions of theorem 3.5 on A are assumed to be satisfied throughout this chapter. In order to compare with the above results we have to make (4.1) and (4.2) more precise. We shall interpret (4.1) as

$$(D_0 - A)u = f \text{ on } \mathbb{R}^+, \tag{4.3}$$

i.e., $\operatorname{supp}((D_0 - A)u - f) \subseteq]-\infty, 0]$, where $f \in H_0(D_\nu - \mathrm{i}) \otimes H_{-\infty}(\mathrm{i}\nu + A)$. W.l.o.g. we may and shall assume that $\operatorname{supp}_0 f \subseteq [0, \infty[$ and $\operatorname{supp}_0 u \subseteq [0, \infty[$. The initial condition is imposed in the sense of convergence in $H_{-\infty}(\mathrm{i}\nu + A)$:

$$u(0+) = u_0 \text{ in } H_{-\infty}(\mathrm{i}\nu + A). \tag{4.4}$$

In order to make the point-wise limit meaningful we will be looking for a solution $u \in \chi_{\mathbb{R}^+}(m)H_1(D_\nu - \mathrm{i}) \otimes H_{-\infty}(\mathrm{i}\nu + A)$. Thus the precise form of the initial value problem is:

IVP: For any given $f \in \chi_{\mathbb{R}^+}(m)H_0(D_\nu - \mathrm{i}) \otimes H_{-\infty}(\mathrm{i}\nu + A)$ *and* $u_0 \in H_{-\infty}(\mathrm{i}\nu + A)$ *find* $u \in \chi_{\mathbb{R}^+}(m)H_1(D_\nu - \mathrm{i}) \otimes H_{-\infty}(\mathrm{i}\nu + A)$ *such that (4.3) and (4.4) hold.*

In order to establish the link between the initial value problem and a corresponding operator equation we need the following regularity statement.

Lemma 4.1. *Let* $A : D(A) \subseteq H \to H$ *satisfy the assumptions of theorem 3.5 and* $\nu \geq \nu_0 > 0$. *For* $f \in H_s(D_\nu - \mathrm{i}) \otimes H_m(\mathrm{i}\nu + A)$ *the solution* $u = (D_0 - A)^{-1} f$ *of*

$$(D_0 - A)u = f,$$

satisfies

$$u \in \bigcap_{h \in [-k-1, 1] \cap \mathbb{Z}} H_{s+h}(D_\nu - \mathrm{i}) \otimes H_{m-h-k}(\mathrm{i}\nu + A).$$

Proof. From

$$(D_0 - A)u = f = (D_\nu - (\mathrm{i}\nu + A))u = (\mathrm{i}\nu + A)(\mathrm{i}\nu + A)^{-1} f$$

we obtain

$$(D_\nu - (\mathrm{i}\nu + A))(u - (\mathrm{i}\nu + A)^{-1} f) = D_\nu (\mathrm{i}\nu + A)^{-1} f. \tag{4.5}$$

Let now $k \in \mathbb{N}$ be the regularity defect of $(D_0 - A)$ then we have

$$u - (\mathrm{i}\nu + A)^{-1} f \in H_{s-1-k}(D_\nu - \mathrm{i}) \otimes H_{m+1}(\mathrm{i}\nu + A).$$

Since

$$(\mathrm{i}\nu + A)^{-1} f \in H_s(D_\nu - \mathrm{i}) \otimes H_{m+1}(\mathrm{i}\nu + A),$$

we find

$$u \in H_{s-1-k}(D_\nu - \mathrm{i}) \otimes H_{m+1}(\mathrm{i}\nu + A). \tag{4.6}$$

From

$$(D_0 - A)u = f = ((D_\nu + \mathrm{i}) - (\mathrm{i}(\nu+1) + A))u = (D_\nu + \mathrm{i})(D_\nu + \mathrm{i})^{-1} f$$

we obtain similarly

$$((D_\nu + \mathrm{i}) - (\mathrm{i}(\nu+1) + A))(u - (D_\nu + \mathrm{i})^{-1} f) = (\mathrm{i}(\nu+1) + A)(D_\nu + \mathrm{i})^{-1} f \tag{4.7}$$

From (4.6) we obtain by induction for any $n \in \mathbb{N}$

$$((D_\nu + \mathrm{i}) - (\mathrm{i}(\nu+1) + A))(u - (D_\nu + \mathrm{i})^{-1} \textstyle\sum_{j=0}^{n-1} (\mathrm{i}(\nu+1) + A)^j (D_\nu + \mathrm{i})^{-j} f) =$$
$$= (\mathrm{i}(\nu+1) + A)^n (D_\nu + \mathrm{i})^{-n} f.$$

As above we conclude

$$u - (D_\nu + \mathrm{i})^{-1} \sum_{j=0}^{n-1} (\mathrm{i}(\nu+1) + A)^j (D_\nu + \mathrm{i})^{-j} f \in H_{s-k+n}(D_\nu - \mathrm{i}) \otimes H_{m-n}(\mathrm{i}\nu + A)$$

and with

$$(D_\nu + \mathrm{i})^{-1} \sum_{j=0}^{n-1} (\mathrm{i}(\nu+1) + A)^j (D_\nu + \mathrm{i})^{-j} f \in H_{s+1}(D_\nu - \mathrm{i}) \otimes H_{m-n+1}(\mathrm{i}\nu + A)$$

we establish

$$u \in H_{s-k+n}(D_\nu - \mathrm{i}) \otimes H_{m-n}(\mathrm{i}\nu + A) \text{ for } n = 0, 1, 2, \ldots, k+1.$$

Together with (4.6) we get indeed

$$u \in H_{s-k+n}(D_\nu - \mathrm{i}) \otimes H_{m-n}(\mathrm{i}\nu + A) \text{ for } n = -1, 0, 1, 2, ..., k+1. \tag{4.8}$$

□

REMARK 4.2. Note that the result of lemma 4.1 is also reflected in lemma 3.6.

We are now ready to show the following equivalence result.

THEOREM 4.3. *Let $A : D(A) \subseteq H \to H$ satisfy the assumptions of theorem 3.5 and $f \in \chi_{\mathbb{R}^+}(m) H_0(D_\nu - \mathrm{i}) \otimes H_{-\infty}(\mathrm{i}\nu + A)$ be given (for $\nu \geq \nu_0$). Then the solution $u \in \chi_{\mathbb{R}^+}(m) H_1(D_\nu - \mathrm{i}) \otimes H_{-\infty}(\mathrm{i}\nu + A)$ of the initial value problem (4.3) and (4.4) is given by*

$$u = (D_0 - A)^{-1}(f + \frac{1}{2\pi \mathrm{i}} \delta \otimes u_0).$$

Conversely, if $u \in \chi_{\mathbb{R}^+}(m)H_1(D_\nu - \mathrm{i}) \otimes H_{-\infty}(\mathrm{i}\nu + A)$ *solves the initial value problem then*

$$(D_0 - A)u = f + \frac{1}{2\pi\mathrm{i}}\delta \otimes u_0.$$

Proof. Consider the solution $u = (D_0 - A)^{-1}(f + \frac{1}{2\pi\mathrm{i}}\delta \otimes u_0)$ of the equation

$$(D_0 - A)u = f + \frac{1}{2\pi\mathrm{i}}\delta \otimes u_0.$$

Then

$$\begin{aligned} \langle (D_0 - A)u - f | \varphi \otimes \psi \rangle_{\nu,0,0} &= \left\langle \tfrac{1}{2\pi\mathrm{i}}\delta \otimes u_0 | \varphi \otimes \psi \right\rangle_{\nu,0,0} \\ &= -\tfrac{1}{2\pi\mathrm{i}}\varphi(0)\left\langle u_0 | \psi \right\rangle_0 = 0 \end{aligned}$$

for all $\varphi \in \overset{\circ}{C}_\infty(\mathbb{R} - \{0\})$. This proves (4.3) formally. Next we want to show the initial condition (4.4) and that

$$u \in \chi_{\mathbb{R}^+}(m)H_1(D_\nu - \mathrm{i}) \otimes H_{-\infty}(\mathrm{i}\nu + A).$$

Since $D_0\chi_{\mathbb{R}^+} = \frac{1}{2\pi\mathrm{i}}\delta$ in $H_{-1}(D_\nu - \mathrm{i})$ for $\nu\rangle 0$ we have

$$(D_0 - A)(u - \chi_{\mathbb{R}^+} \otimes u_0) = f - \chi_{\mathbb{R}^+} \otimes Au_0 \in H_0(D_\nu - \mathrm{i}) \otimes H_{-\infty}(\mathrm{i}\nu + A)$$

According to lemma 4.1 we have now

$$u - \chi_{\mathbb{R}^+} \otimes u_0 \in H_1(D_\nu - \mathrm{i}) \otimes H_{-\infty}(\mathrm{i}\nu + A).$$

Because of causality

$$\operatorname{supp}_0 u \subseteq [0, \infty[,$$

and so with Sobolev's imbedding result $(u - \chi_{\mathbb{R}^+} \otimes u_0)$ is continuous and vanishes on $\mathbb{R}^-$. Consequently,

$$u(0+) - u_0 = V(0-) = 0.$$

Since $v = u$ on $\mathbb{R}^+$ we have as desired

$$u(0+) = u_0.$$

Moreover,

$$u - \chi_{\mathbb{R}^+} \otimes u_0 = u - \chi_{\mathbb{R}^+}(m)\beta \otimes u_0 \in H_1(D_\nu - \mathrm{i}) \otimes H_{-\infty}(\mathrm{i}\nu + A)$$

with $\beta \in C_\infty(\mathbb{R})$ such that $\operatorname{supp}(\beta)$ is bounded below and $\beta \equiv 1$ on $\mathbb{R}^+$. Thus, finally

$$u \in \chi_{\mathbb{R}^+}(m)\beta \otimes u_0 + \chi_{\mathbb{R}^+}(m)H_1(D_\nu - \mathrm{i}) \otimes H_{-\infty}(\mathrm{i}\nu + A)$$

and so

$$u \in \chi_{\mathbb{R}^+}(m)H_1(D_\nu - \mathrm{i}) \otimes H_{-\infty}(\mathrm{i}\nu + A).$$

Let now such a u solve the initial value problem. We calculate with $\varphi \in \overset{\circ}{C}_\infty(\mathbb{R}) \otimes_a H_j(\mathrm{i}\nu + A)$, j sufficiently large,

$$\begin{aligned}
&\langle (D_0 - A)u|\varphi\rangle_{\nu,0,0} \\
&\quad = \langle (D_\nu - \mathrm{i}\nu - A)u|\varphi\rangle_{\nu,0,0} \\
&\quad = \langle u|(D_\nu + \mathrm{i}\nu - A^*)\varphi\rangle_{\nu,0,0} \\
&\quad = \textstyle\int_0^\infty \langle u(t)|(D_\nu\varphi)(t) + (\mathrm{i}\nu - A^*)\varphi(t)\rangle_0 \exp(-4\pi\nu t)dt \\
&\quad = \textstyle\int_0^\infty \{\langle u(t)|(D_\nu\varphi)(t)\rangle_0 - \langle (\mathrm{i}\nu + A)u(t)|\varphi(t)\rangle_0\} \exp(-4\pi\nu t)dt.
\end{aligned}$$

Together with (4.3) this yields

$$\begin{aligned}
&\langle (D_0 - A)u|\varphi\rangle_{\nu,0,0} \\
&\quad = \textstyle\int_0^\infty \{\langle u(t)|(D_\nu\varphi)(t)\rangle_0 - \langle D_\nu u(t) - f(t)|\varphi(t)\rangle_0\} \exp(-4\pi\nu t)dt \\
&\quad = \langle f|\varphi\rangle_{\nu,0,0} + \textstyle\int_0^\infty \{\langle u(t)|(D_\nu\varphi)(t)\rangle_0 - \langle D_\nu u(t)|\varphi(t)\rangle_0\} \exp(-4\pi\nu t)dt \\
&\quad = \langle f|\varphi\rangle_{\nu,0,0} + \textstyle\int_0^\infty (D_0 F)(t)dt,
\end{aligned}$$

with

$$F(t) := \langle \exp(-2\pi\nu t)u(t)|\exp(-2\pi\nu t)\varphi(t)\rangle_0$$

for $t \in \mathbb{R}^+$. Thus,

$$\langle (D_0 - A)u|\varphi\rangle_{\nu,0,0} = \langle f|\varphi\rangle_{\nu,0,0} - \frac{1}{2\pi\mathrm{i}}F(0+),$$

or with $F(0+) = \langle u(0+)|\varphi(0)\rangle_0$,

$$\langle (D_0 - A)u|\varphi\rangle_{\nu,0,0} = \langle f|\varphi\rangle_{\nu,0,0} - \frac{1}{2\pi\mathrm{i}}\langle u_0|\varphi(0)\rangle_0 = \left\langle f + \frac{1}{2\pi\mathrm{i}}\delta \otimes u_0|\varphi\right\rangle_{\nu,0,0}.$$

This confirms our claim. In particular, the solution of the initial value problem is uniquely determined and satisfies the continuous dependence estimate induced by (3.7). ∎

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Technische Universität Dresden
FR Mathematik, Institut für Analysis
D-01062 DRESDEN, Germany
E-mail: picard@math.tu-dresden.de

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