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A CLASS OF 2×2–MATRIX FUNCTIONS

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Abstract. For a special class 2×2–matrix functions Ω operator representations of $\Omega(z)$ and $\widehat{\Omega}(z) := -\Omega(z)^{-1}$ by means of self-adjoint linear relations A and $\widehat{A}$ in a Krein space $\mathcal{K}$ are given. Since $\widehat{A}$ is a 2-dimensional peturbation of A, results of [LMM] imply that "singularities of positive type" of Ω remain singularities of positive type of $\widehat{\Omega}$ with the possible exception of isolated points which have a "finite negative index".

1. Introduction

The 2×2–matrix functions considered in this note are of the form

$$\widehat{\Omega}(z) = -\begin{pmatrix} m_1(z) & 1 \\ 1 & -m_2(z) \end{pmatrix}^{-1}, \tag{1.1}$$

where $m_1(z)$ and $m_2(z)$ are Nevanlinna functions, that is, they are defined and holomorphic in the upper and lower half plane and

$$\frac{\Im\, m_j(z)}{\Im\, z} \geq 0 \text{ if } \Im\, z \neq 0, \;\; j = 1, 2.$$

If we denote by $\sigma_{ess}(m_j)$ the set of nonisolated singularities of $m_j(z)$, it is clear that outside of $\sigma_{ess}(m_1) \cup \sigma_{ess}(m_2)$ the singularities of $\widehat{\Omega}(z)$ are just poles. The main result of this note is that e.g. for each point $\lambda_0 \in \sigma_{ess}(m_1) \setminus \sigma_{ess}(m_2)$ there exists an open interval Δ, $\lambda_0 \in \Delta$, such that

$$\widehat{\Omega}(z) = \widehat{\Omega}_{\Delta}(z) + \widehat{\Omega}_{\Delta'}(z) \tag{1.2}$$

where $\widehat{\Omega}_{\Delta'}(z)$ is holomorphic in Δ, $\widehat{\Omega}_{\Delta}(z)$ is holomorphic outside of the closure of Δ and $\widehat{\Omega}_{\Delta}(z)$ belongs either to the Nevanlinna class or to a generalized

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Nevanlinna class $\mathcal{N}_\kappa^{2\times 2}$ for some positive integer κ (for the definition of this class see Section 2). In the latter case λ_0 is a generalized pole of nonpositive type of $\widehat{\Omega}_\Delta(z)$ which can also be characterized analytically, see [BL].

This result is proved by using a realization of the matrix function $\widehat{\Omega}(z)$. The main operator $\widehat{A}$ in this realization, which is selfadjoint in a Krein space, is a two-dimensional perturbation of the main operator A of the realization of the original function

$$\Omega(z) := \begin{pmatrix} m_1(z) & 1 \\ 1 & -m_2(z) \end{pmatrix}.$$

The second tool for the proof of (1.2) is the classification of the real spectral points of a self-adjoint operator in a Krein space as those of positive, negative etc. type, see [LMM]. From [LMM], Theorem 5.1, it is then easy to obtain the decomposition (1.2). The fact that the operator $\widehat{A}$ is a two–dimensional perturbation of A is proved in Section 3 using an operator identity and following the lines of [BGK]. In [KL], Section 1.6, a similar but (in the Krein space case) weaker result was proved.

Finally we mention, that matrix functions of the form (1.1) arise e.g. with the study of boundary value problems with eigenvalue–depending boundary conditions, see, e.g., [DLS].

2. Preliminaries

Let $(\mathcal{K}, [\,\cdot\,,\,\cdot\,])$ be a Krein space, $U\in\mathcal{L}(\mathcal{K})$ be a unitary operator in $\mathcal{K}$ and let $\Gamma:\mathbb{C}^n\to\mathcal{K}$ be a bounded linear operator. By Γ^+ we denote the adjoint of Γ, defined by the relation $[\Gamma x, f] = (x, \Gamma^+ f)_{\mathbb{C}^n}$ for $x\in\mathbb{C}^n$, $f\in\mathcal{K}$. We fix some point $z_0\in\mathbb{C}\setminus\mathbb{R}$ and put $\lambda(z) := \dfrac{z-\overline{z_0}}{z-z_0}$. With an arbitrary $n\times n$–matrix Q_0 we define the matrix function

$$Q(z) := Q_0^* + (z_0-\overline{z_0})\Gamma^+\left(I - \frac{1}{\lambda(z)}U\right)^{-1}\Gamma \tag{2.1}$$

on the set $\mathcal{D} := \{z\in\mathbb{C} : \lambda(z)\in\rho(U)\}$. Since $0\in\rho(U)$, the set $\mathcal{D}$ contains a neighbourhood of $\overline{z_0}$ and a neighbourhood of z_0 and $Q(z)$ is holomorphic on $\mathcal{D}$. Moreover $\mathcal{D}$ is symmetric with respect to the real axis and, evidently, $Q_0^* = Q(\overline{z_0})$. Additionally we suppose that also the function $Q(z)$ is symmetric, that is

$$Q(\overline{z}) = Q(z)^* \quad \text{for} \quad z\in\mathcal{D}. \tag{2.2}$$

This last relation is equivalent to

$$\frac{Q_0 - Q_0^*}{z_0-\overline{z_0}} = \Gamma^+\Gamma. \tag{2.3}$$

Conversely, suppose that an $n\times n$–matrix function $Q(z)$ is given, which is holomorphic on a symmetric nonempty open set $\mathcal{D}$ and which satisfies the relation (2.2). Then $Q(z)$ admits an (essentially unique) minimal representation of the form (2.1), at least on each open set $\mathcal{D}'$ with sufficiently smooth boundary, such that the closure of $\mathcal{D}'$ is contained in $\mathcal{D}$, see [DLS]. The representation (2.1) is called *minimal* if

$$\mathcal{K} = \text{c.l.s.}\,\{(U-\lambda)^{-1}\Gamma x:\ \lambda \in \rho(U), x \in \mathbb{C}^n\}.$$

With the self-adjoint linear relation

$$A := (\overline{z_0} - z_0 U)(I-U)^{-1}, \tag{2.4}$$

the representation (2.1) becomes

$$Q(z) = Q_0^* + (z-\overline{z_0})\Gamma^+(I + (z-z_0)(A-z)^{-1})\Gamma. \tag{2.5}$$

Here A is a self-adjoint, possibly unbounded operator if and only if $1 \notin \sigma_p(U)$.

The representation (2.5) simplifies if the condition

$$\text{ran}\Gamma \subset \mathcal{D}(A) \tag{2.6}$$

is satisfied. In fact, in this case with $\Gamma_0 := (A-\overline{z_0})\Gamma$ and $S := Q_0^* - \Gamma^+(A-z_0)\Gamma$ the representation reduces to

$$Q(z) = S + \Gamma_0^+(A-z)^{-1}\Gamma_0. \tag{2.7}$$

Recall, that an $n \times n$–matrix function $Q(z)$ which is meromorphic in the upper and the lower half plane belongs to the *generalized Nevanlinna class* $\mathcal{N}_\kappa^{n\times n}$ if the kernel

$$K_Q(z,\zeta) := \frac{Q(z)-Q(\zeta)^*}{z-\overline{\zeta}}$$

has κ negative squares. It was shown in [KL] that these are the functions which allow a minimal representation (2.1) with a π_κ-space $\mathcal{K}$.

As a more particular case, consider a function $m(z) \in \mathcal{N}_0 := \mathcal{N}_0^{1\times 1}$, that is, $m(z)$ is a complex function which is holomorphic in the upper and the lower half plane and has the property

$$\frac{\Im m(z)}{\Im z} \geq 0 \text{ if } \Im z \neq 0.$$

It is well known that $m(z)$ admits an integral representation

$$m(z) = \alpha + \beta z + \int_{-\infty}^{+\infty} \left(\frac{1}{t-z} - \frac{t}{1+t^2}\right) d\sigma(t), \tag{2.8}$$

here $\alpha \in \mathbb{R}$, $\beta \geq 0$, and σ is a positive measure such that $\displaystyle\int_{-\infty}^{+\infty} \frac{1}{1+t^2}\, d\sigma(t) < \infty$. The representation (2.8) of the function $m(z)$ leads in an easy way to an operator representation of $m(z)$. Indeed, the space $\mathcal{K}$ and the operators appearing in (2.1) and (2.5) can be chosen as follows:

$$\mathcal{K} := L^2_\sigma(\mathbb{R}) \oplus \mathbb{C}_\beta,$$

with $\mathbb{C}_\beta := \mathbb{C}$, equipped with the inner product $(\xi, \eta)_\beta := \beta\xi\,\overline{\eta}$. We fix some $z_0 \in \mathbb{C} \setminus \mathbb{R}$ and define the linear operator $\Gamma : \mathbb{C} \to \mathcal{K}$ by the relation

$$\Gamma 1 := \begin{pmatrix} \dfrac{1}{t - z_0} \\ 1 \end{pmatrix}.$$

Then the adjoint operator is

$$\Gamma^+ \begin{pmatrix} f(t) \\ \xi \end{pmatrix} = \beta\xi + \int\limits_{-\infty}^{\infty} \frac{f(t)}{t - \overline{z_0}}\, d\sigma(t).$$

With the operator $U : \mathcal{K} \to \mathcal{K}$:

$$U \begin{pmatrix} f(t) \\ \xi \end{pmatrix} := \begin{pmatrix} \dfrac{t - \overline{z_0}}{t - z_0} f(t) \\ \xi \end{pmatrix}$$

the representation (2.1) for the function m becomes

$$m(z) = \overline{m(z_0)} + (z - \overline{z_0})\beta + \int\limits_{-\infty}^{\infty} \left(\frac{1}{t - z} - \frac{1}{t - \overline{z_0}} \right) d\sigma(t).$$

Inserting the point z_0 yields

$$2i\Im m(z_0) = (z_0 - \overline{z_0})\beta + \int\limits_{-\infty}^{\infty} \left(\frac{1}{t - z_0} - \frac{1}{t - \overline{z_0}} \right) d\sigma(t),$$

and therefore

$$m(z) = a - (\Re z_0)\beta + \beta z + \int\limits_{-\infty}^{\infty} \left(\frac{1}{t - z} - \frac{t - 2\Re z_0}{|\, t - \overline{z_0}\,|^2} \right) d\sigma(t)$$

with $a := \Re m(z_0)$ is an operator representation of $m(z)$. If, in particular, $z_0 = i$ this representation coincides with the integral representation (2.8). Clearly,

$$1 \notin \sigma_p(U) \quad \Longleftrightarrow \quad \beta = 0,$$

and in this case $\mathcal{K} = L^2_\sigma(\mathbb{R})$ and the self-adjoint operator A is the operator of multiplication by the independent variable. For this example the assumption (2.6) is satisfied if and only if the measure σ is finite, and then

$$\Gamma_0 1 = \frac{t - \overline{z_0}}{t - z_0}, \quad \Gamma_0^+ f = \int\limits_{-\infty}^{\infty} f(t)\, \frac{t - z_0}{t - \overline{z_0}}\, d\sigma(t),$$

and the representation (2.7) becomes

$$m(z) = s + \int_{-\infty}^{+\infty} \frac{d\sigma(t)}{t - z}.$$

3. Realization of the inverse function

In this section, starting from a representation of the function $Q(z)$ in the form (2.1), (2.5) or (2.7), we find a corresponding representation of the inverse function

$$\widehat{Q}(z) := -Q(z)^{-1}, \quad z \in \mathcal{D}. \tag{3.1}$$

In the following, a closed linear operator is said to be *boundedly invertible*, if its inverse exists and is defined on the whole space (and hence bounded). The following lemma (in a slightly different form) can be found e.g. in [BGK] and [CZ].

Lemma 3.1. *Let $\mathcal{X}_1$ and $\mathcal{X}_2$ be Banach spaces, and let A and D be densely defined and boundedly invertible operators in $\mathcal{X}_1$ and $\mathcal{X}_2$, respectively, and let B and C be bounded linear operators from $\mathcal{X}_2$ into $\mathcal{X}_1$ and from $\mathcal{X}_1$ into $\mathcal{X}_2$, respectively. Then the relation*

$$(D + CA^{-1}B)^{-1} = D^{-1} - D^{-1}C(A + BD^{-1}C)^{-1}BD^{-1}$$

holds whenever $A + BD^{-1}C$ is boundedly invertible.

Proof. We have for $x \in \mathcal{D}(D)$

$$\begin{aligned}
&(D^{-1} - D^{-1}C(A + BD^{-1}C)^{-1}BD^{-1})(D + CA^{-1}B)x \\
&= x - D^{-1}C(A + BD^{-1}C)^{-1}Bx \\
&\quad + D^{-1}CA^{-1}Bx - D^{-1}C(A + BD^{-1}C)^{-1}BD^{-1}CA^{-1}Bx \\
&= x - D^{-1}C(A + BD^{-1}C)^{-1}[A - (A + BD^{-1}C) + BD^{-1}C]A^{-1}Bx = x,
\end{aligned}$$

since the expression in the square brackets is zero. Also, if $x \in \mathcal{X}_2$ we obtain

$$\begin{aligned}
&(D + CA^{-1}B)(D^{-1} - D^{-1}C(A + BD^{-1}C)^{-1}BD^{-1})x \\
&= x + CA^{-1}BD^{-1}x - C(A + BD^{-1}C)^{-1}BD^{-1}x \\
&\quad - CA^{-1}BD^{-1}C(A + BD^{-1}C)^{-1}BD^{-1}x \\
&= x + CA^{-1}[(A + BD^{-1}C) - A - BD^{-1}C](A + BD^{-1}C)^{-1}BD^{-1}x = x,
\end{aligned}$$

where again the expression in the square brackets vanishes. □

In the following we assume that the domain of holomorphy $\mathcal{D}$ of the function $Q(z)$ consists of at most two components and that for some $w_0 \in \mathcal{D}$ the matrix $Q(w_0)$ is invertible. Then, if $w_0 \neq \overline{w_0}$, also $Q(\overline{w_0})$ is invertible, hence $Q(z)$ is invertible on $\mathcal{D}$ with the possible exception of a set of isolated points. Without loss of generality we can suppose that $Q(z)$ is invertible at the point z_0 from Section 2. Using Lemma 3.1 we can easily prove the following theorem.

THEOREM 3.2. *Suppose that the function $Q(z)$ admits the representation* (2.1),

$$Q(z) := Q_0^* + (z_0 - \overline{z_0})\Gamma^+ \left(I - \frac{1}{\lambda(z)}U\right)^{-1} \Gamma,$$

with a unitary operator U in a Krein space $\mathcal{K}$, and that the matrix Q_0 is invertible. Then the inverse function $\widehat{Q}(z)$ admits the representation

$$\widehat{Q}(z) = -Q_0^{-*} + (z_0 - \overline{z_0})\widehat{\Gamma}^+ \left(I - \frac{1}{\lambda(z)}\widehat{U}\right)^{-1} \widehat{\Gamma},$$

with $\widehat{\Gamma} := \Gamma Q_0^{-1} \in \mathcal{L}(\mathbb{C}^n, \mathcal{K})$ and the unitary operator $\widehat{U} := U - (z_0 - \overline{z_0})\Gamma Q_0^{-1}\Gamma^+ U$ in $\mathcal{K}$. If the representation of $Q(z)$ is minimal, then the representation of $\widehat{Q}(z)$ is also minimal.

PROOF. Lemma 3.1 implies

$$\widehat{Q}(z) = -Q_0^{-*} + (z_0 - \overline{z_0})Q_0^{-*}\Gamma^+ \left(I - \frac{1}{\lambda(z)}U + (z_0 - \overline{z_0})\Gamma Q_0^{-*}\Gamma^+\right)^{-1} \Gamma Q_0^{-*}.$$

The operator $B := I + (z_0 - \overline{z_0})\Gamma Q_0^{-*}\Gamma^+$ is boundedly invertible, in fact, taking into account relation (2.3) we easily get

$$B^{-1} = (I + (z_0 - \overline{z_0})\Gamma Q_0^{-*}\Gamma^+)^{-1} = I - (z_0 - \overline{z_0})\Gamma Q_0^{-1}\Gamma^+ \tag{3.2}$$

and hence

$$\widehat{Q}(z)^{-1} = -Q_0^{-*} + (z_0 - \overline{z_0})Q_0^{-*}\Gamma^+ \left(I - \frac{1}{\lambda}B^{-1}U\right)^{-1} B^{-1}\Gamma Q_0^{-*}.$$

Further, (2.3) also implies $B^{-1}\Gamma Q_0^{-*} = \Gamma Q_0^{-1}$ and, finally,

$$\widehat{U} := (I + (z_0 - \overline{z_0})\Gamma Q_0^{-*}\Gamma^+)^{-1}U = U - (z_0 - \overline{z_0})\Gamma Q_0^{-1}\Gamma^+ U. \tag{3.3}$$

The relation (3.2) shows that B and hence also $\widehat{U}$ are unitary. In order to show the minimality of the representation, for every $\varepsilon > 0$, $x \in \mathbb{C}^n$ and $\lambda \in \rho(U)$ we have to find $y \in \mathbb{C}^n$ and $\eta \in \rho(\widehat{U})$ with

$$\|\, (U - \lambda)^{-1}\Gamma x - (\widehat{U} - \eta)^{-1}\widehat{\Gamma} y \,\| < \varepsilon. \tag{3.4}$$

With the particular choice $y := Q_0 x - (z_0 - \overline{z_0})\Gamma^+ U(U - \lambda)^{-1}\Gamma x$ it is easy to see that $\eta \in \rho(\widehat{U})$ can be chosen such that the inequality (3.4) holds. □

According to relation (3.3), the unitary operator $\widehat{U}$ in the representation of $\widehat{Q}(z)$ is an n–dimensional perturbation of U. The analogous result for a function $Q(z)$ with a representation (2.5) is the following theorem.

THEOREM 3.3. *Suppose that the function $Q(z)$ admits a representation (2.5),*

$$Q(z) = Q_0^* + (z - \overline{z_0})\Gamma^+(A - z_0)(A - z)^{-1}\Gamma,$$

with a self-adjoint operator A in the Krein space $\mathcal{K}$. If the matrix Q_0 is invertible and

$$\ker\left(I - \Gamma Q_0^{-1}\Gamma^+(A - \overline{z_0})\right) = \{0\}, \tag{3.5}$$

then the inverse function $\widehat{Q}(z)$ admits the representation

$$\widehat{Q}(z) = -Q_0^{-*} + (z - \overline{z_0})\widehat{\Gamma}^+(\widetilde{A} - z_0)(\widetilde{A} - z)^{-1}\widehat{\Gamma},$$

where $\widehat{\Gamma} = \Gamma Q_0^{-1}$ and for the self-adjoint operator $\widetilde{A}$ it holds

$$(\widetilde{A} - z_0)^{-1} - (A - z_0)^{-1} = -\Gamma Q_0^{-1}\Gamma^+(A - \overline{z_0})(A - z_0)^{-1}. \tag{3.6}$$

The relation (3.6) implies that the difference of the resolvents of A and $\widetilde{A}$ is an n–dimensional operator. The condition (3.5) is needed in order to assure that the inverse function admits a representation with a self-adjoint operator, that is that $1 \notin \sigma_p(\widehat{U})$ for the operator $\widehat{U}$ which exists according to Theorem 3.2. The next theorem is the corresponding result for a function $Q(z)$ with a representation (2.7).

THEOREM 3.4. *Suppose that the function $Q(z)$ admits a representation (2.7),*

$$Q(z) = S + \Gamma_0^+(A - z)^{-1}\Gamma_0,$$

with a self-adjoint operator A in the Krein space $\mathcal{K}$. If the matrix S is invertible, then the inverse function $\widehat{Q}(z)$ admits the representation

$$\widehat{Q}(z) = -S^{-1} + \widehat{\Gamma}_0^+(\widehat{A} - z)^{-1}\widehat{\Gamma}_0$$

with $\widehat{\Gamma}_0 := \Gamma_0 S^{-1}$ and the self-adjoint operator $\widehat{A} := A + \Gamma_0 S^{-1}\Gamma_0^+$.

In the situation of Theorem 3.4 the difference $\widehat{A} - A$ is an n–dimensional operator. The proof of Theorem 3.3 is similar to the proof of Theorem 3.2, Theorem 3.4 follows immediately from Lemma 3.1. Both proofs are left to the reader.

4. A matrix function and its inverse

In this section we consider the matrix function

$$\Omega(z) := \begin{pmatrix} m_1(z) & 1 \\ 1 & -m_2(z) \end{pmatrix} \tag{4.1}$$

with functions $m_j \in \mathcal{N}_0,\ j = 1, 2$. Let

$$m_j(z) = \alpha_j + \beta_j z + \int_{-\infty}^{+\infty} \left(\frac{1}{t - z} - \frac{t}{1 + t^2}\right) d\sigma_j(t), \quad j = 1, 2, \tag{4.2}$$

be their integral representations (2.8). As was explained in the paragraph following formula (2.8), they imply operator representations (2.1) with the Hilbert spaces $\mathcal{K}_j := L^2_{\sigma_j} \oplus \mathbb{C}_{\beta_j}$ and operators Γ_j, U_j for $j = 1, 2$:

$$m_j(z) = \overline{m_j(z_0)} + (z_0 - \overline{z_0})\Gamma_j^+ \left(I - \frac{1}{\lambda(z)} U_j\right)^{-1} \Gamma_j; \tag{4.3}$$

and, if the conditions

$$\beta_j = 0, \quad \sigma_j(\mathbb{R}) < \infty, \quad j = 1, 2, \tag{4.4}$$

are satisfied, operator representations (2.5) with operators Γ_{j0} and A_j:

$$m_j(z) = s_j + \Gamma_{j0}^+ (A_j - z)^{-1} \Gamma_{j0}. \tag{4.5}$$

Now we introduce the Krein space $\mathcal{K} := \mathcal{K}_1 [\dot{+}] \mathcal{K}_2$ with the inner product

$$[x, y] := (x_1, y_1)_{\mathcal{K}_1} - (x_2, y_2)_{\mathcal{K}_2},$$
$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \ y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \ x_j, y_j \in \mathcal{K}_j, \ j = 1, 2,$$

and the following operators $\Gamma : \mathbb{C}^2 \to \mathcal{K}$ and $U \in \mathcal{L}(\mathcal{K})$:

$$\Gamma := \begin{pmatrix} \Gamma_1 & 0 \\ 0 & \Gamma_2 \end{pmatrix}, \ U := \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix}. \tag{4.6}$$

Then $\Omega(z)$ admits the operator representation

$$\Omega(z) = \Omega(z_0)^* + (z_0 - \overline{z_0})\Gamma^+ \left(I - \frac{1}{\lambda(z)} U\right)^{-1} \Gamma. \tag{4.7}$$

If in the representations (4.2) the assumptions (4.4) are satisfied, then with the operators $\Gamma_0 : \mathbb{C}^2 \to \mathcal{K}$, A in $\mathcal{K}$ and the 2×2–matrix S:

$$\Gamma_0 := \begin{pmatrix} \Gamma_{10} & 0 \\ 0 & \Gamma_{20} \end{pmatrix}, \ A := \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, \ S := \begin{pmatrix} s_1 & 1 \\ 1 & -s_2 \end{pmatrix} \tag{4.8}$$

the operator representation

$$\Omega(z) = S + \Gamma_0^+ (A - z)^{-1} \Gamma_0. \tag{4.9}$$

holds.

In the following we suppose that $\det \Omega(z) = -1 - m_1(z) m_2(z) \not\equiv 0$. We are interested in the structure of the singularities of the inverse matrix function

$$\widehat{\Omega}(z) := -\Omega(z)^{-1} = \frac{1}{1 + m_1(z) m_2(z)} \begin{pmatrix} -m_2(z) & -1 \\ -1 & m_1(z) \end{pmatrix}.$$

Evidently, this function $\widehat{\Omega}(z)$ exists and is analytic at least on the comlement of the set $\operatorname{supp}\sigma_1 \cup \operatorname{supp}\sigma_2$ with possible exception of a sequence of isolated points which are zeros of the function $1 + m_1(z) m_2(z)$ and hence poles of $\widehat{\Omega}(z)$.

An operator representation of the function $\widehat{\Omega}(z)$ is easily obtained from the operator representations (4.3) or (4.5) of the functions $m_1(z)$ and $m_2(z)$.

To this end we fix $z_0 \in \mathbb{C} \setminus \mathbb{R}$ such that $1 + m_1(z_0)m_2(z_0) \neq 0$. Then from Theorem 3.2 we get

$$\widehat{\Omega}(z) = \widehat{\Omega}(z_0)^* + (z_0 - \overline{z_0})\widehat{\Gamma}^+ \left(I - \frac{1}{\lambda(z)}\widehat{U} \right)^{-1} \widehat{\Gamma} \tag{4.10}$$

with $\widehat{\Gamma} := \Gamma\Omega(z_0)^{-1}$ and $\widehat{U} := U - (z_0 - \overline{z_0})\Gamma\Omega(z_0)^{-1}\Gamma^+ U$, where Γ and U are defined in (4.6). If the assumptions (4.4) are satisfied, it follows that also S is invertible and hence we obtain from Theorem 3.4 the representation

$$\widehat{\Omega}(z) = -S^{-1} + \widehat{\Gamma}_0^+ \left(\widehat{A} - z \right)^{-1} \widehat{\Gamma}_0 \tag{4.11}$$

with $\widehat{\Gamma}_0 := \Gamma_0 S^{-1}$ and

$$\begin{aligned} \widehat{A} :&= A + \Gamma_0 S^{-1} \Gamma_0^+ \\ &= \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} + \frac{1}{1 + s_1 s_2} \begin{pmatrix} s_2 \Gamma_{10}\Gamma_{10}^* & -\Gamma_{10}\Gamma_{20}^* \\ \Gamma_{20}\Gamma_{10}^* & s_1 \Gamma_{20}\Gamma_{20}^* \end{pmatrix}, \end{aligned} \tag{4.12}$$

where A, Γ_0 and S are given in (4.8).

Lemma 4.1. *The spectrum of the operator $\widehat{A}$ coincides with the set of singularities of the function $\widehat{\Omega}(z)$.*

Proof. Obviously, the function $\widehat{\Omega}(z)$ is holomorphic at every point $\lambda_0 \in \rho(\widehat{A})$. Conversely, let $\widehat{\Omega}(z)$ be holomorphic at $\lambda_0 \in \mathbb{C}$. Then for the inverse function $\Omega(z)$ the point λ_0 cannot be a non-isolated singularity. So we know, by construction, that $\lambda_0 \notin \sigma_{ess}(A)$, hence λ_0 is not an accumulation point of $\sigma(\widehat{A})$. If we consider the Riesz projection of $\widehat{A}$ at λ_0 and observe the minimality of the representation of $\widehat{\Omega}(z)$ it follows that $\lambda_0 \in \rho(\widehat{A})$. ☐

Recall that for a bounded self-adjoint operator B in the Krein space $(\mathcal{K}, [\,\cdot\,,\,\cdot\,])$ the point $\lambda_0 \in \mathbb{R} \cap (\sigma_c(B) \cup \sigma_p(B))$ is called a *spectral point of positive type*, if for each sequence $(x_n) \subset \mathcal{K}$ with the properties $\|x_n\| = 1$, $\|(B - \lambda_0)x_n\| \to 0$ it follows that $\liminf_{n\to\infty} [x_n, x_n] > 0$. The set of all spectral points of positive type of B is denoted by $\sigma_+(B)$. The set $\sigma_-(B)$ of all spectral points of negative type of B is defined similarly. Further, the point $\lambda_0 \in \mathbb{R}$ belongs by definition to the set $\sigma_{-,f}(B)$, if there exists an interval (a, b), such that $\lambda_0 \in (a, b)$,

$$\{\lambda : a < Re\lambda < b,\ 0 < |\Im\lambda| < \eta\} \subset \rho(B) \text{ for some } \eta > 0, \tag{4.13}$$

$$(a, b)\backslash\{\lambda_0\} \subset \sigma_+(B) \cup \rho(B), \tag{4.14}$$

and for each interval $[\alpha, \beta]$ with $a < \alpha < \lambda_0 < \beta < b$ on the maximal spectral subspace $\mathcal{L}_{[\alpha,\beta]}(B)$ of the operator B the inner product $[\,\cdot\,,\,\cdot\,]$ has only finitely many negative squares (see [LMM], Section 5). In this case λ_0 is an eigenvalue of B with a nonpositive eigenvector. The set $\sigma_{+,f}(B)$ is defined analogously.

Now we return to the operators A and $\widehat{A}$ from (4.8) and (4.12), which correspond to the functions $\Omega(z)$ and $\widehat{\Omega}(z)$ according to (4.9) and (4.11), respectively. We assume in the sequel, that the conditions (4.4) are satisfied.

LEMMA 4.2. *For the operator A the relations* $\sigma(A) = \operatorname{supp}\sigma_1 \cup \operatorname{supp}\sigma_2$,

$$\sigma_+(A) = \operatorname{supp}\sigma_1 \setminus \operatorname{supp}\sigma_2, \quad \sigma_-(A) = \operatorname{supp}\sigma_2 \setminus \operatorname{supp}\sigma_1 \tag{4.15}$$

and $\sigma_{ess}(A) = \sigma_{ess}(A_1) \cup \sigma_{ess}(A_2)$ hold; $\sigma_{ess}(A_1)$ is the set of non-isolated points of $\operatorname{supp}\sigma_1$, *$\sigma_{ess}(A_2)$ is the set of non-isolated points of* $\operatorname{supp}\sigma_2$. *Further, $\lambda \in \sigma_{-,f}(A)$ if and only if λ is an isolated point of* $\operatorname{supp}\sigma_2$, *$\lambda \in \sigma_{+,f}(A)$ if and only if λ is an isolated point of* $\operatorname{supp}\sigma_1$.

PROOF. The first and the last claims are clear, only the relations in (4.15) need to be proved. We prove the first one, the proof of the second one is analogous. Consider $\lambda_0 \in \operatorname{supp}\sigma_1 \setminus \operatorname{supp}\sigma_2$. Then $\lambda_0 \in \sigma(A_1) \setminus \sigma(A_2)$. If (x^n) is a sequence of elements of $\mathcal{K}$ such that $\|x^n\|^2 = 1$ and $(A - \lambda_0)x^n \to 0$ if $n \to \infty$, then $(A_2 - \lambda_0)x_2^n \to 0$ and, since $\lambda_0 \in \rho(A_2)$, $x_2^n \to 0$. It follows that $\lim\,[x^n, x^n] = \lim \|x_1^n\|^2 = 1$ and hence $\lambda_0 \in \sigma_+(A)$. Conversely, if $\lambda_0 \in \operatorname{supp}\sigma_2$, then there exists a sequence $(x_2^n) \in \mathcal{K}_2$ such that $\|x_2^n\| = 1$ and $(A_2 - \lambda_0)x_2^n \to 0$. Then $(A - \lambda_0)(0\ x_2^n)^t \to 0$ and $\lim\,[x_2^n, x_2^n] = -1$, hence $\lambda_0 \notin \sigma_+(A)$. ∎

Since the difference $\widehat{A} - A$ is finite–dimensional, the results of Section 5 of [LMM] yield the following theorem.

THEOREM 4.3. *Suppose that the assumptions* (4.4) *are satisfied. Then the following inclusions hold:*

$$\mathbb{R} \setminus \sigma_{ess}(A_2) \quad \subset \quad \sigma_+(\widehat{A}) \cup \sigma_{-,f}(\widehat{A}) \cup \rho(\widehat{A}),$$

$$\mathbb{R} \setminus \sigma_{ess}(A_1) \quad \subset \quad \sigma_-(\widehat{A}) \cup \sigma_{+,f}(\widehat{A}) \cup \rho(\widehat{A}),$$

in particular

$$\sigma_+(A) \cup (\rho(A) \cap \mathbb{R}) \quad \subset \quad \sigma_+(\widehat{A}) \cup \sigma_{-,f}(\widehat{A}) \cup \rho(\widehat{A}),$$

$$\sigma_-(A) \cup (\rho(A) \cap \mathbb{R}) \quad \subset \quad \sigma_-(\widehat{A}) \cup \sigma_{+,f}(\widehat{A}) \cup \rho(\widehat{A}).$$

The non-real spectrum of $\widehat{A}$ can accumulate only at points of $\sigma_{ess}(A_1) \cap \sigma_{ess}(A_2)$.

If a point λ_0 belongs to $\sigma_\pm(\widehat{A})$ or to $\sigma_{\mp,f}(\widehat{A})$, the singularities of the function $\widehat{\Omega}(z)$ can be described more precisely. We formulate this result for points of $\sigma_+(\widehat{A})$ and of $\sigma_{-,f}(\widehat{A})$.

THEOREM 4.4. *Suppose that the assumptions* (4.4) *are satisfied. If $\lambda \in \sigma_+(\widehat{A})$, then there exist an open interval Δ around λ_0 and functions*

$\widehat{\Omega}_\Delta(z)$ and $\widehat{\Omega}_{\Delta'}(z)$, such that $\widehat{\Omega}_\Delta(z)$ belongs to the Nevanlinna class $\mathcal{N}_0$ and is holomorphic outside of the closure of Δ, $\widehat{\Omega}_{\Delta'}(z)$ is holomorphic in Δ and

$$\widehat{\Omega}(z) = \widehat{\Omega}_\Delta(z) + \widehat{\Omega}_{\Delta'}(z). \tag{4.16}$$

If $\lambda \in \sigma_{-,f}(\widehat{A})$, then there exist an open interval Δ around λ_0 and functions $\widehat{\Omega}_\Delta(z)$ and $\widehat{\Omega}_{\Delta'}(z)$, such that $\widehat{\Omega}_\Delta(z)$ belongs to a generalized Nevanlinna class $\mathcal{N}_\kappa^{2\times 2}$ for some $\kappa > 0$ and is holomorphic outside of the closure of Δ, $\widehat{\Omega}_{\Delta'}(z)$ is holomorphic in Δ and the relation (4.16) *holds.*

Proof. If $\lambda_0 \in \sigma_+(\widehat{A})$, there exists an open interval Δ with $\lambda_0 \in \Delta$, $\Delta \subset \sigma_+(\widehat{A}) \cup \rho(\widehat{A})$ and

$$\{\lambda \in \mathbb{C} : \Re\lambda \in \Delta,\ 0 <| \Im\lambda |< \eta\} \subseteq \rho(\widehat{A}) \text{ for some } \eta > 0.$$

The corresponding maximal spectral subspace $(\mathcal{L}_\Delta(\widehat{A}), [\,\cdot\,,\,\cdot\,])$ is a Hilbert space. Therefore in the Krein space $\mathcal{K}$ there exists an orthogonal projection P_Δ onto $\mathcal{L}_\Delta(\widehat{A})$, and with the representation (4.11) the function $\widehat{\Omega}(z)$ can be decomposed as

$$\begin{aligned}\widehat{\Omega}(z) \;&=\; -S^{-1} + \left(P_\Delta\widehat{\Gamma}_0\right)^+ \left(\widehat{A} - z\right)^{-1} \left(P_\Delta\widehat{\Gamma}_0\right)\\ &\quad + \left((I - P_\Delta)\widehat{\Gamma}_0\right)^+ \left(\widehat{A} - z\right)^{-1} \left((I - P_\Delta)\widehat{\Gamma}_0\right).\end{aligned}$$

Since $\mathcal{L}_\Delta(\widehat{A})$ is a Hilbert space, $\sigma\left(\widehat{A}|\mathcal{L}_\Delta(\widehat{A})\right)$ is contained in the closure of Δ and $\sigma\left(\widehat{A}|\,(I - P_\Delta)\,\mathcal{K}\right) \cap \Delta = \emptyset$, the decomposition (4.16) follows. If $\lambda_0 \in \sigma_{-,f}(\widehat{A})$ the proof is analogous, we have only to observe that now the subspace $\left(\mathcal{L}_\Delta(\widehat{A}), [\,\cdot\,,\,\cdot\,]\right)$ is a π_κ-space. ∎

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