

COMPACT EMBEDDINGS OF VECTOR-VALUED SOBOLEV AND BESOV SPACES

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ABSTRACT. The main result of this paper is a generalization and sharpening of the Aubin-Dubinskii lemma concerning compact subsets in vector-valued Lebesgue spaces. In addition, there are given some new embedding results for vector valued Besov spaces.

1. INTRODUCTION AND MAIN RESULTS

Let E , E_0 , and E_1 be Banach spaces such that

$$(1.1) \quad E_1 \hookrightarrow E \hookrightarrow E_0 ,$$

with \hookrightarrow and \hookrightarrow denoting continuous and compact embedding, respectively. Suppose that $p_0, p_1 \in [1, \infty]$ and $T > 0$, that

$$(1.2) \quad \mathcal{V} \text{ is a bounded subset of } L_{p_1}((0, T), E_1) ,$$

and that

$$(1.3) \quad \partial\mathcal{V} := \{ \partial v ; v \in \mathcal{V} \} \text{ is bounded in } L_{p_0}((0, T), E_0) ,$$

where ∂ denotes the distributional derivative. Then the well-known ‘Aubin lemma’, more precisely, the ‘Aubin-Dubinskii lemma’ guarantees that

$$(1.4) \quad \mathcal{V} \text{ is relatively compact in } L_{p_1}((0, T), E) .$$

This result is proven in [Aub63, Théorème 1] and also in [Lio69, Théorème I.5.1], provided E_0 and E_1 are reflexive and $p_0, p_1 \in (1, \infty)$. It has also been derived by Dubinskii [Dub65] (see [Lio69, Théorème I.12.1]) with the same

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restrictions for p_0 and p_1 , but without the reflexivity hypothesis. (In fact, Dubinskii proves a slightly more sophisticated theorem in which the L_{p_1} -norm in (1.2) is replaced by a more general functional.)

A proof of (1.4), given assumptions (1.2) and (1.3) only, is due to Simon (see [Sim87, Corollary 4]). In fact, this author observes that (1.3) can be replaced by

$$(1.5) \quad \lim_{h \rightarrow 0^+} \|v(\cdot + h) - v\|_{L_{p_1}((0, T-h), E_0)} = 0, \quad \text{uniformly for } v \in \mathcal{V},$$

(see [Sim87, Theorem 5]). Note that the integrability exponents in (1.2) and (1.5) are equal.

Compactness theorems of ‘Aubin-Dubinskii type’ are very useful in the theory of nonlinear evolution equations and are employed in numerous research papers. Typical situations are as follows: (u_k) is a sequence of approximate solutions to a given nonlinear evolution equation. If it is possible to bound this sequence in $L_{p_1}(X, E_1)$ and if one can bound the sequence (∂u_k) in $L_{p_0}(X, E_0)$, then the Aubin-Dubinskii lemma guarantees that one can extract a subsequence which converges in $L_{p_1}(X, E)$. If it is then possible to pass to the limit in the approximating problems, whose solutions are the u_k , and if the limiting equation coincides with the original evolution equation, then the existence of a solution to the original problem has been established (cf. [Lio69] for an exposition of this technique). In many concrete cases it is rather difficult, if not impossible, to pass to the limit in nonlinear equations if (∂u_k) is only known to converge in $L_{p_1}(X, E)$. Convergence in ‘better spaces’, whose elements are more regular (in space or in time), is needed. Even if convergence in $L_{p_1}(X, E)$ is sufficient, it is often important to know that the limiting element belongs to a space with more regularity.

It is the purpose of this paper to prove compact embedding theorems of ‘Aubin-Dubinskii type’ involving spaces of higher regularity. For this we observe that in most practical cases it is possible to squeeze an interpolation space between E and E_1 (see Remark 7.4). Thus we replace assumption (1.1) by the slightly more restrictive condition:

$$(1.6) \quad E_1 \hookrightarrow E_0 \quad \text{and} \quad (E_0, E_1)_{\theta, 1} \hookrightarrow E \hookrightarrow E_0 \quad \text{for some } \theta \in (0, 1),$$

where $(\cdot, \cdot)_{\theta, q}$ denote the real interpolation functors (cf. [BL76] or [Tri78] for the basic facts of interpolation theory; also see [Ama95, Section I.2] for a summary). Note that the compactness assumption in (1.6) is weaker than the one in (1.1). Moreover, it is well-known that $(E_0, E_1)_{\theta, 1} \hookrightarrow E \hookrightarrow E_0$ iff $E_1 \hookrightarrow E \hookrightarrow E_0$ and

$$\|x\|_E \leq c \|x\|_{E_0}^{1-\theta} \|x\|_{E_1}^{\theta}, \quad x \in E_1,$$

(e.g., [BL76, Theorem 3.5.2] or [Tri78, Lemma 1.10.1]). Here and below c denotes positive constants which may differ from formula to formula. Intuitively, the parameter $1 - \theta$ measures the ‘distance’ between E_1 and E .

In order to formulate our main result involving assumptions (1.2) and (1.6) we need some notation. Throughout this paper it is always assumed that $p, p_0, p_1 \in [1, \infty]$, unless explicit restrictions are given, and that $0 < \theta < 1$. Then

$$\frac{1}{p\theta} := \frac{1-\theta}{p_0} + \frac{\theta}{p_1} .$$

Given $s \in \mathbb{R}^+ := [0, \infty)$, we denote by $W_p^s((0, T), E)$ the Sobolev-Slobodeckii space of order s of E -valued distributions on $(0, T)$, which is defined in analogy to the scalar case (see Section 2). We also put $c^0([0, T], E) := C([0, T], E)$; and $c^s([0, T], E)$ is, for $0 < s < 1$, the Banach space of all s -Hölder-continuous E -valued functions on $[0, T]$ satisfying

$$\lim_{r \rightarrow 0} \sup_{\substack{0 < x, y < T \\ 0 < |x-y| < r}} \frac{\|u(x) - u(y)\|}{|x - y|^s} = 0 ,$$

the ‘little Hölder space’ of order s .

THEOREM 1.1. *Let (1.2) and (1.6) be satisfied. Suppose that either*

$$(1.7) \quad s_0 := 1 \quad \text{and} \quad (1.3) \text{ is true,}$$

or

$$(1.8) \quad \left. \begin{array}{l} 0 < s_0 < 1, \quad p_0 \leq p_1, \text{ and} \\ \|v(\cdot + h) - v\|_{L_{p_0}((0, T-h), E_0)} \leq ch^{s_0}, \quad 0 < h < T, \quad v \in \mathcal{V} . \end{array} \right\}$$

Then \mathcal{V} is relatively compact in

$$(1.9) \quad W_p^s((0, T), E) \quad \text{if } 0 \leq s < (1-\theta)s_0 \text{ and } s-1/p < (1-\theta)s_0-1/p\theta ,$$

and in

$$(1.10) \quad c^s([0, T], E) \quad \text{if } 0 \leq s < (1-\theta)s_0 - 1/p\theta .$$

Let (1.2), (1.3), and (1.6) be satisfied. In [Sim87, Corollary 8] it is shown that \mathcal{V} is relatively compact in

$$(1.11) \quad L_p((0, T), E) \quad \text{if } 1-\theta \leq 1/p\theta < 1/p ,$$

and in

$$(1.12) \quad C([0, T], E) \quad \text{if } 1-\theta > 1/p\theta .$$

Note that (1.9) implies in this case that \mathcal{V} is relatively compact in $L_p((0, T), E)$ if

$$1/p\theta - (1-\theta) < 1/p .$$

Hence we can admit values $p > p_\theta$ if $1-\theta < 1/p\theta$, in contrast to (1.11) where $p < p_\theta$ is required. Furthermore, (1.9) implies in the present situation that

\mathcal{V} is relatively compact in

$$W_{p_\theta}^s((0, T), E) \quad \text{if } 0 \leq s < 1 - \theta .$$

Since (1.10) shows that \mathcal{V} is relatively compact in $c^s([0, T], E)$ if $0 \leq s < 1 - \theta - 1/p_\theta$, we see that Theorem 1.1 is a substantial improvement over Simon's extension of the Aubin-Dubinskii lemma, provided condition (1.6) is satisfied.

In [Sim87, Theorem 7] it is also shown that \mathcal{V} is relatively compact in $L_{p_\theta}((0, T), E)$ if (1.2), (1.5), and (1.6) are true. Theorem 1.1 gives a considerable sharpening of this result, provided (1.5) is replaced by its quantitative version (1.8).

Suppose that V and H are Hilbert spaces such that $V \xrightarrow{d} H$. Then, identifying H with its (anti-)dual H' , it follows that $V \xrightarrow{d} H \xrightarrow{d} V'$. It is known (e.g., [LM72]) that $H = (V', V)_{1/2, 2}$. Hence, letting $(E_0, E_1) := (V', V)$ and $E := H$, condition (1.6) is satisfied with $\theta := 1/2$. Setting $p_0 := p_1 := 2$, we infer from (1.9) that \mathcal{V} is relatively compact in $L_p((0, T), H)$ for $1 \leq p < \infty$. It is also known that \mathcal{V} is continuously — but not compactly — injected in $C([0, T], H)$ (see [Mig95]). This shows that Theorem 1.1 is sharp. It should be noted that Simon's result (1.11) guarantees only that \mathcal{V} is relatively compact in $L_p((0, T), H)$ for $1 \leq p < 2$.

Theorem 1.1 is a special case of much more general results which are also valid if $(0, T)$ is replaced by a sufficiently regular bounded open subset of \mathbb{R}^n . Its proof is given in Section 5.

In the next section we introduce vector-valued Besov spaces on \mathbb{R}^n and recall some of their basic properties. In particular, we prove an interpolation theorem extending an earlier result due to Grisvard. In Section 4 we discuss vector-valued Besov spaces on X and prove compact embedding theorems for them. In Section 5 we derive an analogue of the Rellich-Kondrachov theorem for vector-valued Sobolev spaces on X as well as a compact embedding theorem for intersections of Sobolev-Slobodeckii spaces. The last section contains a renorming result for Sobolev-Slobodeckii spaces. We close this paper by commenting on the regularity assumptions for X .

We are indebted to E. Maître for bringing [Mig95] to our attention.

2. SOME FUNCTION SPACES

Let X be an open subset of \mathbb{R}^n . Suppose that E is a Banach space, that $1 \leq p \leq \infty$, and $m \in \mathbb{N}$. Then the Sobolev space $W_p^m(X, E)$ is the Banach space of all $u \in L_p(X, E)$ such that the distributional derivatives $\partial^\alpha u$ belong to $L_p(X, E)$ for $|\alpha| \leq m$, endowed with the usual norm $\|\cdot\|_{m,p}$. Furthermore, $BUC^m(X, E)$ is the closed linear subspace of $W_\infty^m(X, E)$ consisting

of all u such that $\partial^\alpha u$ is bounded and uniformly continuous on X , that is, $\partial^\alpha u \in BUC(X, E)$, for $|\alpha| \leq m$.

If $0 < \theta < 1$, put

$$[u]_{\theta,p} := \begin{cases} \left[\int_{X \times X} \left(\frac{\|u(x) - u(y)\|_E}{|x - y|^\theta} \right)^p \frac{d(x, y)}{|x - y|^n} \right]^{1/p}, & p < \infty, \\ \sup_{\substack{x, y \in X \\ x \neq y}} \frac{\|u(x) - u(y)\|_E}{|x - y|^\theta}, & p = \infty. \end{cases}$$

Then we set

$$W_p^{m+\theta}(X, E) := \left(\{ u \in W_p^m(X, E) ; \|u\|_{m+\theta,p} < \infty \}, \|\cdot\|_{m+\theta,p} \right),$$

where

$$\|u\|_{m+\theta,p} := \|u\|_{m,p} + \max_{|\alpha|=m} [\partial^\alpha u]_{\theta,p}.$$

If $p < \infty$ then $W_p^{m+\theta}(X, E)$ is a vector-valued Slobodeckii space, and

$$W_\infty^{m+\theta}(X, E) = BUC^{m+\theta}(X, E),$$

the subspace of $BUC^m(X, E)$ consisting of all u such that $\partial^\alpha u$ is uniformly θ -Hölder continuous for $|\alpha| = m$.

If $m > 0$ and $0 \leq \theta < 1$ then $W_p^{-m+\theta}(X, E)$ [resp. $BUC^{-m}(X, E)$] is the Banach space of all E -valued distributions u on X having a representation

$$u = \sum_{|\alpha| \leq m} \partial^\alpha u_\alpha$$

with $u_\alpha \in W_p^\theta(X, E)$ [resp. $u_\alpha \in BUC^\theta(X, E)$], equipped with the norm

$$u \mapsto \|u\|_{-m+\theta,p} := \inf \left(\sum_{|\alpha| \leq m} \|u_\alpha\|_{\theta,p} \right),$$

the infimum being taken over all such representations, and p being equal to ∞ if $u_\alpha \in BUC^\theta(X, E)$. Thus the ‘Sobolev-Slobodeckii scale’ $W_p^s(X, E)$, $s \in \mathbb{R}$, is well-defined for each $p \in [1, \infty]$, as is the ‘Hölder scale’ $BUC^s(X, E)$, $s \in \mathbb{R}$. Moreover,

$$\mathcal{D}(X, E) \hookrightarrow W_p^s(X, E) \cap BUC^s(X, E) \hookrightarrow W_p^s(X, E) + BUC^s(X, E) \hookrightarrow \mathcal{D}'(X, E)$$

for $s \in \mathbb{R}$. Here $\mathcal{D}(X, E)$ is the space of all E -valued test functions on X endowed with the usual inductive limit topology, and $\mathcal{D}'(X, E) = \mathcal{L}(\mathcal{D}(X), E)$ is the space of E -valued distributions on X , with \mathcal{L} denoting the space of continuous linear maps, equipped with the topology of uniform convergence on bounded sets.

We also define the scale of ‘little Hölder spaces’ $buc^s(X, E)$, $s \in \mathbb{R}$, by setting

$$buc^m(X, E) := BUC^m(X, E)$$

and by denoting by

$$buc^{m+\theta}(X, E) \text{ the closure of } BUC^{m+1}(X, E) \text{ in } BUC^{m+\theta}(X, E)$$

for $m \in \mathbb{Z}$ and $\theta \in (0, 1)$. Then $u \in BUC^{m+\theta}(X, E)$ belongs to $buc^{m+\theta}(X, E)$ iff

$$\lim_{r \rightarrow 0} \sup_{\substack{x, y \in X \\ 0 < |x-y| < r}} \frac{\|\partial^\alpha u(x) - \partial^\alpha u(y)\|_E}{|x-y|^\theta} = 0, \quad |\alpha| = m,$$

(cf. [Lun95, Proposition 0.2.1], for example).

Throughout the remainder of this paper we suppose that

$$X \text{ is a smoothly bounded open subset of } \mathbb{R}^n,$$

which means that \overline{X} is a compact n -dimensional C^∞ -submanifold of \mathbb{R}^n with boundary. This assumption is imposed for convenience and can be considerably relaxed (see the last paragraph of Section 7).

It follows that $BUC^s(X, E) = C^s(\overline{X}, E)$ for $s \in \mathbb{R}^+$ by identifying $u \in BUC^s(X, E)$ with its unique continuous extension $\overline{u} \in C^s(\overline{X}, E)$. For this reason we put

$$C^s(\overline{X}, E) := BUC^s(X, E), \quad c^s(\overline{X}, E) := buc^s(X, E)$$

for all $s \in \mathbb{R}$.

Henceforth, we always suppose that E , E_0 , and E_1 are complex Banach spaces. The real case can be covered by complexification. We also suppose that $s, s_0, s_1 \in \mathbb{R}$ and put $s_\theta := (1 - \theta)s_0 + \theta s_1$.

3. BESOV SPACES ON \mathbb{R}^n

Fix a radial $\psi := \psi_0 \in \mathcal{D}(\mathbb{R}^n) := \mathcal{D}(\mathbb{R}^n, \mathbb{C})$ with $\psi(\xi) = 1$ for $|\xi| < 1$ and $\psi(\xi) = 0$ for $|\xi| \geq 2$. Put

$$\psi_k(\xi) := \psi(2^{-k}\xi) - \psi(2^{-k+1}\xi), \quad \xi \in \mathbb{R}^n, \quad k \in \mathbb{N} \setminus \{0\},$$

and $\psi_k(D) := \mathcal{F}^{-1}\psi_k\mathcal{F}$, where \mathcal{F} is the Fourier transform on $\mathcal{S}'(\mathbb{R}^n, E) := \mathcal{L}(\mathcal{S}(\mathbb{R}^n), E)$ and $\mathcal{S}(\mathbb{R}^n)$ is the Schwartz space of rapidly decreasing smooth functions on \mathbb{R}^n . Then the Besov space $B_{p,q}^s(\mathbb{R}^n, E)$ of E -valued distributions on \mathbb{R}^n is defined to be the vector subspace of $\mathcal{S}'(\mathbb{R}^n, E)$ consisting of all u satisfying

$$\|u\|_{s,p,q} := \left\| \left(2^{sk} \|\psi_k(D)\|_{L_p(\mathbb{R}^n, E)} \right)_{k \in \mathbb{N}} \right\|_{\ell_q} < \infty.$$

It is a Banach space with this norm, and different choices of ψ lead to equivalent norms.

In this section we simply write \mathfrak{F} for $\mathfrak{F}(\mathbb{R}^n, E)$ if the latter is a locally convex space of E -valued distributions on \mathbb{R}^n , that is, $\mathfrak{F}(\mathbb{R}^n, E) \hookrightarrow \mathcal{D}'(\mathbb{R}^n, E)$, and no confusion seems likely.

It follows that

$$(3.1) \quad \mathcal{S} \hookrightarrow B_{p,q_1}^{s_1} \hookrightarrow B_{p,q_0}^{s_0} \hookrightarrow \mathcal{S}' , \quad s_1 > s_0 ,$$

and

$$(3.2) \quad B_{p,q_0}^s \hookrightarrow B_{p,q_1}^s , \quad q_0 < q_1 .$$

Moreover,

$$(3.3) \quad B_{p_1,q}^{s_1} \hookrightarrow B_{p_0,q}^{s_0} , \quad s_1 > s_0 , \quad s_1 - n/p_1 = s_0 - n/p_0 .$$

Besov spaces are stable under real interpolation, that is, if $0 < \theta < 1$ then

$$(3.4) \quad (B_{p,q_0}^{s_0}, B_{p,q_1}^{s_1})_{\theta,q} \doteq B_{p,q}^{s_\theta} , \quad s_0 \neq s_1 .$$

They are related to Slobodeckii and Hölder spaces by

$$(3.5) \quad B_{p,p}^s \doteq W_p^s , \quad s \in \mathbb{R} \setminus \mathbb{Z} ,$$

and

$$(3.6) \quad B_{p,1}^m \hookrightarrow W_p^m \hookrightarrow B_{p,\infty}^m , \quad m \in \mathbb{Z} , \quad p < \infty .$$

Moreover, $B_{p,p}^m \neq W_p^m$ for $m \in \mathbb{Z}$ unless $p = 2$ and E is a Hilbert space. Note that (3.4)–(3.6) imply

$$(3.7) \quad (W_p^{s_0}, W_p^{s_1})_{\theta,q} \doteq B_{p,q}^{s_\theta} , \quad s_0 \neq s_1 , \quad p < \infty .$$

It is also true that

$$(3.8) \quad B_{\infty,1}^m \hookrightarrow BUC^m \hookrightarrow B_{\infty,\infty}^m , \quad m \in \mathbb{Z} ,$$

and $B_{\infty,\infty}^m$ is the Zygmund space \mathcal{C}^m for $m \in \mathbb{N} \setminus \{0\}$ (e.g., [Tri83] for the scalar case). Hence we infer from (3.4) and (3.5) that

$$(3.9) \quad (BUC^{s_0}, BUC^{s_1})_{\theta,q} \doteq B_{\infty,q}^{s_\theta} .$$

The definition and the above properties of vector-valued Besov spaces are literally the same as in the scalar case (for which we refer to [Tri78], [Tri83], [Tri92], and [BL76]). The proofs carry over from the scalar to the vector-valued setting by employing the Fourier multiplier theorem of Proposition 4.5 of [Ama97]. A detailed and coherent treatment containing many additional results will be given in [Ama99]. For earlier (partial) results and different approaches we refer to [Gri66], [Sch86], and [Tri97, Section 15], as well as to the other references cited in [Ama97]. Embedding theorems for vector-valued Besov and Slobodeckii spaces on an interval are also derived in [Sim90], but with s , s_0 , and s_1 restricted to the interval $[0, 1]$.

We define the little Besov space $b_{p,q}^s$ to be the closure of $B_{p,q}^{s+1}$ in $B_{p,q}^s$. Then

$$(3.10) \quad b_{p,q}^s := \begin{cases} B_{p,q}^s, & p \vee q < \infty, \quad s \in \mathbb{R}, \\ \text{buc}^s, & p = q = \infty, \quad s \in \mathbb{R} \setminus \mathbb{Z}, \end{cases}$$

and

$$(3.11) \quad b_{p,q}^s \text{ is the closure of } B_{p,q}^t \text{ in } B_{p,q}^s \text{ for } t > s$$

(see [Ama97, Propositions 5.3 and 5.4 and Remark 5.5(b)] and [Ama99]).

Denoting by \xrightarrow{d} dense embedding, it follows that

$$(3.12) \quad \mathcal{S} \xrightarrow{d} B_{p,q_1}^{s_1} \xrightarrow{d} B_{p,q_0}^{s_0} \xrightarrow{d} b_{p,\infty}^{s_0} \xrightarrow{d} \mathcal{S}', \quad p < \infty,$$

if either $s_1 = s_0$ and $1 \leq q_1 \leq q_0 < \infty$, or $s_1 > s_0$ and $q_0 \vee q_1 < \infty$ (see [Ama97, Remark 5.5(a)]).

The following interpolation theorem for vector-valued Besov spaces will be of particular importance for us.

THEOREM 3.1. *Let (E_0, E_1) be an interpolation couple and suppose that $s_0 \neq s_1$ and $p_0, p_1, q_0, q_1 \in [1, \infty)$. Then*

$$(B_{p_0, q_0}^{s_0}(\mathbb{R}^n, E_0), B_{p_1, q_1}^{s_1}(\mathbb{R}^n, E_1))_{\theta, q_\theta} \doteq B_{p_\theta, q_\theta}^{s_\theta}(\mathbb{R}^n, (E_0, E_1)_{\theta, q_\theta}),$$

provided $p_\theta = q_\theta$.

PROOF. We denote by $\ell_q^s(E)$ the subspace of $E^{\mathbb{N}}$ consisting of all $u = (u_k)$ satisfying

$$\|u\|_{\ell_q^s(E)} := \|(2^{sk} u_k)_{k \in \mathbb{N}}\|_{\ell_q} < \infty.$$

It is a Banach space with this norm. If (F_0, F_1) is an interpolation couple then

$$(3.13) \quad (\ell_{q_0}^{s_0}(F_0), \ell_{q_1}^{s_1}(F_1))_{\theta, q_\theta} \doteq \ell_{q_\theta}^{s_\theta}((F_0, F_1)_{\theta, q_\theta})$$

(e.g., [BL76, Theorem 5.6.2] or [Tri78, Theorem 1.18.1]). Furthermore ([Tri78, Theorem 1.18.4]),

$$(3.14) \quad (L_{p_0}(\mathbb{R}^n, E_0), L_{p_1}(\mathbb{R}^n, E_1))_{\theta, p_\theta} \doteq L_{p_\theta}(\mathbb{R}^n, (E_0, E_1)_{\theta, p_\theta}).$$

From [Ama97, Lemma 5.1] we know that $B_{p,q}^s$ is a retract of $\ell_q^s(L_p)$. Hence the assertion follows from (3.13), (3.14), and [Tri78, Theorem 1.2.4] or [Ama95, Proposition I.2.3.2]. \square

Theorem 3.1 generalizes a result of Grisvard [Gri66, formula (6.9) on p. 179] who considers the case $p_j = q_j$ and $n = 1$. It should be noted that Grisvard's proof does not extend to $n > 1$ since, in general, $W_p^m(\mathbb{R}^n, E)$ is not isomorphic to $L_p(\mathbb{R}^n, E)$.

4. BESOV SPACES ON X

We denote by $r_{\overline{X}} \in \mathcal{L}(C(\mathbb{R}^n, E), C(\overline{X}, E))$ the operator of point-wise restriction, $u \mapsto u|_{\overline{X}}$, and recall that $r_X \in \mathcal{L}(\mathcal{D}'(\mathbb{R}^n, E), \mathcal{D}'(X, E))$ is the restriction operator in the sense of distribution, that is,

$$r_X u(\varphi) := u(\varphi), \quad u \in \mathcal{D}'(\mathbb{R}^n, E), \quad \varphi \in \mathcal{D}(X).$$

Observe that coretractions for $r_{\overline{X}}$ and r_X are extension operators.

The following extension theorem is of basic importance for the study of spaces of distributions on X . Here and below we set

$$\mathcal{W}_p^s(Y, E) := \begin{cases} W_p^s(Y, E), & p < \infty, \\ BUC^s(Y, E), & p = \infty, \end{cases}$$

for $s \in \mathbb{R}$ and $Y \in \{\mathbb{R}^n, X\}$.

THEOREM 4.1. *r_X is a retraction from $\mathcal{S}'(\mathbb{R}^n, E)$ onto $\mathcal{D}'(X, E)$ and there exists a coretraction e_X for r_X which is independent of E . Moreover, $r_X \supset r_{\overline{X}}$, and r_X belongs to*

$$\begin{aligned} & \mathcal{L}(\mathcal{S}(\mathbb{R}^n, E), C^\infty(\overline{X}, E)) \cap \mathcal{L}(\mathcal{W}_p^s(\mathbb{R}^n, E), \mathcal{W}_p^s(X, E)) \\ & \quad \cap \mathcal{L}(buc^s(\mathbb{R}^n, E), c^s(\overline{X}, E)). \end{aligned}$$

Furthermore, e_X is an element of

$$\begin{aligned} & \mathcal{L}(C^\infty(\overline{X}, E), \mathcal{S}(\mathbb{R}^n, E)) \cap \mathcal{L}(\mathcal{W}_p^s(X, E), \mathcal{W}_p^s(\mathbb{R}^n, E)) \\ & \quad \cap \mathcal{L}(c^s(\overline{X}, E), buc^s(\mathbb{R}^n, E)), \end{aligned}$$

and it is a coretraction for r_X in each case.

PROOF. By a standard partition of unity argument the proof is reduced to establishing a corresponding statement if X is replaced by a half-space of \mathbb{R}^n . In this case the theorem is deduced by constructing an extension operator along the lines of [Ham75, Part II]. For details and generalizations we refer to [Ama99]. \square

Now we define the Besov spaces of E -valued distributions on X by

$$B_{p,q}^s(X, E) := r_X B_{p,q}^s(\mathbb{R}^n, E),$$

equipped with the obvious quotient space topology.

PROPOSITION 4.2. *r_X is a retraction from $B_{p,q}^s(\mathbb{R}^n, E)$ onto $B_{p,q}^s(X, E)$ and e_X is a corresponding coretraction.*

PROOF. Fix $s_0 < s < s_1$ and put $\theta := (s - s_0)/(s_1 - s_0)$. Then

$$(\mathcal{W}_p^{s_0}(\mathbb{R}^n, E), \mathcal{W}_p^{s_1}(\mathbb{R}^n, E))_{\theta, q} \doteq B_{p, q}^s(\mathbb{R}^n, E)$$

thanks to (3.7) and (3.9). By Theorem 4.1 the diagrams of continuous linear maps

$$\begin{array}{ccc} \mathcal{W}_p^{s_j}(\mathbb{R}^n, E) & \xrightarrow{r_X} & \mathcal{W}_p^{s_j}(X, E) \\ & \searrow e_X & \swarrow \text{id} \\ & & \mathcal{W}_p^{s_j}(X, E) \end{array}$$

are commutative. Hence the assertion follows by interpolation. \square

COROLLARY 4.3. *Assertions (3.1)–(3.12) as well as Theorem 3.1 remain valid if \mathbb{R}^n is replaced by X , provided we substitute $C^\infty(\overline{X}, E)$ and $\mathcal{D}'(X, E)$ for \mathcal{S} and \mathcal{S}' , respectively.*

PROOF. This is deduced from Proposition 4.2 by standard arguments. \square

In the following (4.x), where $x \in \{1, \dots, 12\}$, denotes the analogue of formula (3.x) with \mathbb{R}^n replaced by X , as well as \mathcal{S} and \mathcal{S}' replaced by $C^\infty(\overline{X}, E)$ and $\mathcal{D}'(X, E)$, respectively.

Now it is easy to prove the following compact embedding theorem.

THEOREM 4.4. *Suppose that $E_1 \hookrightarrow E_0$. Then*

$$B_{p, q}^{s_1}(X, E_1) \hookrightarrow B_{p, q}^{s_0}(X, E_0), \quad s_1 > s_0.$$

PROOF. Fix $\sigma_0 < s_0 < s_1 < \sigma_1$ and $\sigma \in (0, 1)$ such that $\sigma_0 < 0$ and $\sigma < \sigma_1 - n/p$. Then we infer from (4.1)–(4.3) and (4.5), (4.6) that

$$B_{p, q}^{\sigma_1}(X, E_1) \hookrightarrow B_{\infty, \infty}^{\sigma_1 - n/p}(X, E_1) \hookrightarrow C^\sigma(\overline{X}, E_1)$$

and

$$C(\overline{X}, E_0) \hookrightarrow L_p(X, E_0) \hookrightarrow B_{p, q}^{\sigma_0}(X, E_0).$$

Since, by the Arzéla-Ascoli theorem, $C^\sigma(\overline{X}, E_1)$ is compactly embedded in $C(\overline{X}, E_0)$, it follows that $B_{p, q}^{\sigma_1}(X, E_1) \hookrightarrow B_{p, q}^{\sigma_0}(X, E_0)$. Now the assertion is a consequence of (4.4) and the Lions-Peetre compactness theorem for the real interpolation method. \square

COROLLARY 4.5. (i) *Suppose that $E_1 \hookrightarrow E_0$. If $s_1 > s_0$ and $s_1 - n/p_1 > s_0 - n/p_0$ then*

$$B_{p_1, q_1}^{s_1}(X, E_1) \hookrightarrow b_{p_0, q_0}^{s_0}(X, E_0).$$

(ii) Suppose that

$$E_1 \hookrightarrow E_0 \quad \text{and} \quad (E_0, E_1)_{\theta, p_\theta} \hookrightarrow E .$$

If $s_\theta > s$ and $s_\theta - n/p_\theta > s - n/p$ then

$$B_{p_0, q_0}^{s_0}(X, E_0) \cap B_{p_1, q_1}^{s_1}(X, E_1) \hookrightarrow b_{p, q}^s(X, E) .$$

PROOF. (i) Since X is bounded, it is obvious that

$$C^m(\overline{X}, E) \hookrightarrow W_p^m(X, E) \hookrightarrow W_{\overline{p}}^m(X, E) , \quad 1 \leq \overline{p} < p , \quad m \in \mathbb{Z} .$$

Thus it is an easy consequence of (4.1), (4.5), (4.7), and (4.9) that

$$B_{p, q}^s(X, E) \hookrightarrow B_{\overline{p}, q}^s(X, E) , \quad 1 \leq \overline{p} < p .$$

Fix $p \in [1, p_1]$ and $s \in (s_0, s_1)$ such that $t := s - n(1/p - 1/p_0) < s$ and suppose that $s_0 < \sigma < \tau < t$. Then we infer from (4.1)–(4.3), Theorem 4.4, and the above embedding that

$$\begin{aligned} B_{p_1, q_1}^{s_1}(X, E_1) &\hookrightarrow B_{p, q_1}^s(X, E_1) \hookrightarrow B_{p_0, q_1}^t(X, E_1) \hookrightarrow B_{p_0, q_0}^\tau(X, E_1) \\ &\hookrightarrow B_{p_0, q_0}^\sigma(X, E_0) \hookrightarrow b_{p_0, q_0}^{s_0}(X, E_0) , \end{aligned}$$

where the last embedding follows from (4.11).

(ii) Fix $\sigma_j < s_j$ such that $s - n/p < \sigma_\theta - n/p_\theta$. Then

$$B_{p_0, q_0}^{s_0}(X, E_0) \cap B_{p_1, q_1}^{s_1}(X, E_1) \hookrightarrow B_{p_0, p_0}^{\sigma_0}(X, E_0) \cap B_{p_1, p_1}^{\sigma_1}(X, E_1) .$$

Since

$$B_{p_0, p_0}^{\sigma_0}(X, E_0) \cap B_{p_1, p_1}^{\sigma_1}(X, E_1) \hookrightarrow B_{p_j, p_j}^{\sigma_j}(X, E_j) , \quad j = 0, 1 ,$$

interpolation gives

$$\begin{aligned} B_{p_0, p_0}^{\sigma_0}(X, E_0) \cap B_{p_1, p_1}^{\sigma_1}(X, E_1) &\hookrightarrow (B_{p_0, p_0}^{\sigma_0}(X, E_0), B_{p_1, p_1}^{\sigma_1}(X, E_1))_{\theta, p_\theta} \\ &= B_{p_\theta, p_\theta}^{\sigma_\theta}(X, (E_0, E_1)_{\theta, p_\theta}) , \end{aligned}$$

where the last equality follows from Theorem 3.1 and Corollary 4.3. Now it suffices to apply (i). \square

5. SOBOLEV-SLOBODECKII SPACES ON X

As an easy consequence of the preceding results we obtain the following vector-valued version of the Rellich-Kondrachov theorem.

THEOREM 5.1. *Suppose that $E_1 \hookrightarrow E_0$. If $s_1 > s_0$ and $s_1 - n/p_1 > s_0 - n/p_0$ then*

$$W_{p_1}^{s_1}(X, E_1) \hookrightarrow W_{p_0}^{s_0}(X, E_0) .$$

If $0 \leq s < s_1 - n/p_1$ then

$$W_{p_1}^{s_1}(X, E_1) \hookrightarrow c^s(\overline{X}, E_0) .$$

PROOF. Fix $\sigma_0, \sigma_1 \in (s_0, s_1)$ with $\sigma_1 > \sigma_0$ such that $\sigma_1 - n/p_1 > \sigma_0 - n/p_0$. Then (4.5), (4.6), and Corollary 4.5(i) imply

$$W_{p_1}^{s_1}(X, E_1) \hookrightarrow B_{p_1, p_1}^{\sigma_1}(X, E_1) \hookrightarrow b_{p_0, p_0}^{\sigma_0}(X, E_0) .$$

Now the assertion follows from (4.10) and (4.5). \square

It is also easy to prove a compact embedding theorem involving intersections of Sobolev-Slobodeckii spaces as well as interpolation spaces E_θ .

THEOREM 5.2. *Suppose that*

$$(5.1) \quad E_1 \hookrightarrow E_0 \quad \text{and} \quad (E_0, E_1)_{\theta, p_\theta} \hookrightarrow E \hookrightarrow E_0 .$$

Then

$$(5.2) \quad W_{p_0}^{s_0}(X, E_0) \cap W_{p_1}^{s_1}(X, E_1) \hookrightarrow W_p^s(X, E) ,$$

provided

$$(5.3) \quad s < s_\theta \quad \text{and} \quad s - n/p < s_\theta - n/p_\theta .$$

If $0 \leq s < s_\theta - n/p_\theta$ then

$$(5.4) \quad W_{p_0}^{s_0}(X, E_0) \cap W_{p_1}^{s_1}(X, E_1) \hookrightarrow c^s(\overline{X}, E) .$$

PROOF. Since $E_1 \hookrightarrow E_0$, interpolation theory guarantees that

$$E_1 \hookrightarrow (E_0, E_1)_{\vartheta, p_\vartheta} \hookrightarrow (E_0, E_1)_{\theta, 1} , \quad \theta < \vartheta < 1 .$$

Hence (4.2) and the second part of (5.1) show that $(E_0, E_1)_{\vartheta, p_\vartheta} \hookrightarrow E$. Fix $\vartheta \in (\theta, 1)$ sufficiently close to θ such that $s - n/p < s_\vartheta - n/p_\vartheta$ if (5.3) holds, and such that $s < p_\vartheta - n/p_\vartheta$ if $s_\theta - n/p_\theta > 0$. Now the assertion is an easy consequence of Corollary 4.5(ii) and (4.1), (4.5), and (4.6). \square

REMARKS 5.3.

(a) Suppose that H is a Hilbert space. Then u belongs to $W_2^s(\mathbb{R}^n, H)$, where $s \in \mathbb{R}^+$, iff $u \in L_2(\mathbb{R}^n, H)$ and

$$(\xi \mapsto |\xi|^{2s} \widehat{u}(\xi)) \in L_2(\mathbb{R}^n, H) ,$$

with \widehat{u} denoting the Fourier transform of u . Thus assumption (5.1), modulo Theorem 5.2, generalizes a result of J.-L. Lions (cf. [Lio61, Théorème IV.2.2] and [Lio69, Théorème I.5.2]), who considers the case $n = 1$, $p = 2$, and $s_1 = 0$ with E, E_0 , and E_1 being Hilbert spaces satisfying $E_1 \hookrightarrow E \hookrightarrow E_0$.

(b) Theorem 1.1 also improves Corollary 9 of [Sim87] which, for $n = 1$, guarantees the validity of (5.2)–(5.4) for $s = 0$.

(c) Observe that there are no sign restrictions for s, s_0 , and s_1 in (5.3). Hence the first part of Theorem 5.2 is also valid if $s_0 < 0$, for example. In this connection it is important to know that, similarly as in the scalar case, Sobolev-Slobodeckii spaces of negative order can be characterized by duality.

More precisely: Denote by $\mathring{W}_p^s(X, E)$ the closure of $\mathcal{D}(X, E)$ in $W_p^s(X, E)$. Then, given a reflexive Banach space F ,

$$W_p^{-s}(X, F) \doteq [\mathring{W}_{p'}^s(X, F')]', \quad 1 < p < \infty ,$$

and

$$W_1^{-s}(X, F) \doteq [c^s(\overline{X}, F')]', \quad s \in \mathbb{R}^+ \setminus \mathbb{N} ,$$

with respect to the duality pairing induced by

$$(5.5) \quad \langle u', u \rangle := \int_X \langle u'(x), u(x) \rangle_{F'} dx , \quad u, u' \in \mathcal{D}(X, E) ,$$

where $\langle \cdot, \cdot \rangle_{F'} : F \times F' \rightarrow \mathbb{K}$ is the duality pairing between F and F' .

Consequently, if $1 < p < \infty$ then a subset \mathcal{V} of $W_p^{-s}(X, F)$ is bounded iff there exists a constant c such that

$$(5.6) \quad |\langle v, \varphi \rangle| \leq c \|\varphi\|_{s, p'} , \quad \varphi \in \mathcal{D}(X, F') , \quad v \in \mathcal{V} .$$

Similarly, a subset \mathcal{V} of $W_1^{-s}(X, F)$ is bounded iff (5.6) holds for all $\varphi \in C^\infty(\overline{X}, F')$. In concrete situations, estimates of this type are often rather easy to establish.

PROOF. Note that (5.5) extends by continuity from $\mathcal{D}(X, F) \times \mathcal{D}(X, F')$ to a bilinear form on $W_p^{-s}(X, F) \times W_{p'}^s(X, F')$ and from $\mathcal{D}(X, F) \times C^\infty(\overline{X}, F')$ to such a form on $W_1^{-s}(X, F) \times c^s(\overline{X}, F')$. For a proof of the duality assertions we refer to [Ama99, Chapter VII]. \square

(d) Suppose that (5.1) is satisfied and $\alpha \in \mathbb{N}^n$. Then

$$\partial^\alpha : W_{p_0}^{s_0}(X, E_0) \cap W_{p_1}^{s_1}(X, E_1) \rightarrow W_p^s(X, E) \text{ compactly ,}$$

provided

$$s < s_\theta \quad \text{and} \quad s - n/p < s_\theta - |\alpha| - n/p_\theta .$$

If $0 \leq s < s_\theta - |\alpha| - n/p_\theta$ then

$$\partial^\alpha : W_{p_0}^{s_0}(X, E_0) \cap W_{p_1}^{s_1}(X, E_1) \rightarrow c^s(\overline{X}, E) \text{ compactly .}$$

This generalizes Théorème 2 of [Aub63] as well as Simon's extension of it [Sim87, Corollary 10].

PROOF. Since

$$\partial^\alpha \in \mathcal{L}(W_{p_0}^{s_0}(X, E_0) \cap W_{p_1}^{s_1}(X, E_1), W_{p_0}^{s_0 - |\alpha|}(X, E_0) \cap W_{p_1}^{s_1 - |\alpha|}(X, E_1)) ,$$

the assertion follows from Theorem 5.2. \square

6. PROOF OF THEOREM 1.1

In order to derive Theorem 1.1 from the preceding results we need some preparation.

LEMMA 6.1. *Set*

$$V := V_{p_0, p_1}(E_0, E_1) := \{ v \in L_{p_1}((0, T), E_1) ; \partial v \in L_{p_0}((0, T), E_0) \} .$$

Then $V \doteq W_{p_0}^1((0, T), E_0) \cap L_{p_1}((0, T), E_1)$.

PROOF. It is clear that V is a Banach space and that

$$W_{p_0}^1((0, T), E_0) \cap L_{p_1}((0, T), E_1) \hookrightarrow V .$$

Moreover,

$$V \hookrightarrow C([0, T], E_0) \hookrightarrow L_{p_0}((0, T), E_0) ,$$

where we refer to [Tri78, Lemma 1.8.1], for example, for a proof of the first embedding. Now the assertion is obvious. \square

Put $X_h := X \cap (X - h)$ for $h \in \mathbb{R}^n$ and suppose that $p < \infty$. Also set

$$[u]_{\theta, p, \infty} := \sup_{\substack{h \in \mathbb{R}^n \\ h \neq 0}} \frac{\|u(\cdot + h) - u\|_{L_p(X_h, E)}}{|h|^\theta}$$

and, given $m \in \mathbb{N}$,

$$N_p^{m+\theta}(X, E) := \left(\{ u \in L_p(X, E) ; [\partial^\alpha u]_{\theta, p, \infty} < \infty, |\alpha| = m \}, \|\cdot\|_{m+\theta, p, \infty} \right) ,$$

where

$$\|u\|_{m+\theta, p, \infty} := \|u\|_p + \max_{|\alpha|=m} [\partial^\alpha u]_{\theta, p, \infty} .$$

Then $N_p^s(X, E)$, $s \in \mathbb{R}^+ \setminus \mathbb{N}$, are the Nikol'skii spaces of E -valued distributions on X . The proof for the scalar case (e.g., [Tri78, Section 2.5.1]) carries over to the vector-valued case to show that

$$(6.1) \quad N_p^s(X, E) \doteq B_{p, \infty}^s(X, E) , \quad s \in \mathbb{R}^+ \setminus \mathbb{N} ,$$

(cf. [Ama99, Section VII.3].)

PROOF OF THEOREM 1.1. Clearly, we can assume that $p_0 \vee p_1 < \infty$.

Let (1.7) be satisfied. Then (1.2), (1.3), and Lemma 6.1 imply that \mathcal{V} is bounded in $W_{p_0}^1((0, T), E_0) \cap L_{p_1}((0, T), E_1)$. Hence the assertion is entailed by Theorem 5.2.

Suppose that assumption (1.8) is fulfilled. Then (6.1) shows that \mathcal{V} is bounded in $B_{p_0, \infty}^{s_0}((0, T), E_0)$. Hence it is bounded in $B_{p_0, \infty}^{s_0}((0, T), E_0) \cap L_{p_1}((0, T), E_1)$ by (1.6). Thus (4.1) and (4.6) imply that \mathcal{V} is bounded in $B_{p_0, \infty}^{s_0}((0, T), E_0) \cap B_{p_1, p_1}^{s_1}((0, T), E_1)$ for each $s_1 < 0$. Now the assertion follows from Corollary 4.5(ii) by means of the arguments used in the proof of Theorem 5.2. \square

7. FINAL REMARKS

So far we have not put any restriction, like reflexivity for example, on the Banach spaces under consideration. However, in order to prove an n -dimensional analogue to Lemma 6.1 we need such an additional assumption. For this we recall that a Banach space F is a UMD space if the Hilbert transform is a continuous self-map of $L_2(\mathbb{R}^n, F)$. Every UMD space is reflexive (but not conversely), and every Hilbert space is a UMD space. The class of UMD spaces enjoys many useful permanence properties. For example, each closed subspace of a UMD space is again a UMD space. For details and more information we refer to [Ama95, Subsection III.4.5].

EXAMPLE 7.1. Suppose that Ω is an open subset of some euclidean space. Then $W_p^s(\Omega)$ and every closed linear subspace thereof are UMD spaces, provided $1 < p < \infty$.

PROOF. If $m \in \mathbb{N}$ then $W_p^m(\Omega)$ is well-known to be isomorphic to a closed linear subspace of the M -fold product of $L_p(\Omega)$, where $M := \sum_{|\alpha| \leq m} 1$. Hence $W_p^m(\Omega)$ is a UMD space by Theorem III.4.5.2 in [Ama95]. Consequently, $\dot{W}_p^m(\Omega)$ is a UMD space as well. Thus $W_p^{-m}(\Omega) = [\dot{W}_p^m(\Omega)]'$ is also a UMD space, as follows from part (v) of Theorem III.4.5.2 in [Ama95]. Finally, part (vii) of that theorem, together with (3.5) and (3.7), implies the assertion. \square

If F is a UMD space then the Sobolev-Slobodeckii spaces $W_p^s(X, F)$ possess essentially the same properties as their scalar ancestors, provided $1 < p < \infty$. This is seen, for example, by the following proposition.

PROPOSITION 7.2. *Suppose that F is a UMD space and $1 < p < \infty$. Then, given $s \in \mathbb{R}$ and $m \in \mathbb{N}$,*

$$u \mapsto \|u\|_{s,p} + \sum_{|\alpha|=m} \|\partial^\alpha u\|_{s,p}$$

is an equivalent norm for $W_p^{s+m}(X, F)$.

PROOF. If F is a UMD space then Mihlin's multiplier theorem is valid in $L_p(\mathbb{R}^n, F)$ for $1 < p < \infty$ (and scalar symbols) (e.g., [Ama95, Theorem III.4.4.3]). Thus the well-known proof for scalar Sobolev spaces extends to the vector-valued setting in this case. \square

COROLLARY 7.3. *Suppose that E_0 is a UMD space and $1 < p_0 < \infty$. Then $W_{p_0}^m(X, E_0) \cap L_{p_1}(X, E_1) = \{ u \in L_{p_1}(X, E_1) ; \partial^\alpha u \in L_{p_0}(X, E_0), |\alpha| = m \}$ for $m \in \mathbb{N}$ and $1 \leq p_1 \leq \infty$.*

Lastly, we show that, in practice, the assumption that we can squeeze an interpolation space between E and E_1 is no serious restriction. In other words: in most applications assumption (1.6) is satisfied.

REMARK 7.4. In concrete applications it is most often the case that $E_j := W_{r_j}^{\sigma_j}(\Omega)$ for $j = 0, 1$ and $E := W_r^\sigma(\Omega)$, where Ω is a bounded smooth open subset of \mathbb{R}^d , σ_0 and σ_1 are real numbers with $\sigma_0 < \sigma < \sigma_1$, and $r, r_0, r_1 \in [1, \infty)$. Thanks to the classical Rellich-Kondrachov theorem $E_1 \hookrightarrow E_0$. Suppose that $\sigma_0 - d/r_0 < \sigma - d/r < \sigma_1 - d/r_1$. Fix $\vartheta \in (0, 1)$ such that

$$\sigma - d/r < \sigma_\vartheta - d/r_\vartheta < \sigma_1 - d/r_1, \quad \sigma < \sigma_\vartheta < \sigma_1,$$

and $\sigma_\vartheta \notin \mathbb{Z}$. Then we infer from (4.1) and (4.7) that

$$E_1 \hookrightarrow (E_0, E_1)_{\vartheta, 1} \hookrightarrow (E_0, E_1)_{\vartheta, r_\vartheta} \doteq W_{r_\vartheta}^{\sigma_\vartheta}(\Omega) \hookrightarrow E,$$

since, by making σ_1 slightly smaller and σ_0 slightly bigger, if necessary, we can suppose that $W_{r_j}^{\sigma_j}(\Omega) = B_{r_j, r_j}^{\sigma_j}(\Omega)$ for $j = 0, 1$. \square

For simplicity, we presupposed throughout that X be smooth. However, everything remains valid if we drop this hypothesis and assume instead that r_X possesses a coretraction with the properties stated in Theorem 4.1. This is known to be the case for a much wider class of subdomains of \mathbb{R}^n . We do not go into detail but refer to [Ama99]. The same observation applies to Ω , of course.

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