GLASNIK MATEMATIČKI
Vol. 35(55)(2000), 179 – 190

APPROXIMATION OF GREEN'S FUNCTION AND APPLICATION

Ibrahim Aganović, Zvonimir Tutek and Krešo Veselić
University of Zagreb, Croatia and Fernuniversität, Hagen, Germany

Abstract. We consider a non local boundary value problem for elliptic operator on a two dimensional domain with a small hole around origin. The precise asymptotics in terms of diameter of the hole of values of solution on boundary of the hole is described by appropriate values of the Green function associated with the origin. In case of elliptical hole it is proved that solutions converge uniformly toward the Green function associated with the origin as diameter of the ellipse tends to zero.

1. Introduction

Let $\mathcal{C} = \{x = (x_1, x_2) \in \mathbf{R}^2 \ : \ |x| < 1\}$ be the unit circle in $\mathbf{R}^2$. The Green function $G_{\mathcal{C}}$ on $\mathcal{C}$ (associated with the origin) of the Laplace operator is given by the formula:

$$G_{\mathcal{C}}(x) = \frac{1}{2\pi} \ln \frac{1}{|x|}.$$

For small $\varepsilon > 0$ let $\mathcal{C}_\varepsilon = \{x \in \mathbf{R}^2 \ : \ \varepsilon < |x| < 1\}$ be the ring around origin. It is easy to verify that the problem

$$\begin{cases} \Delta u_\varepsilon = 0 \quad \text{in} \quad \mathcal{C}_\varepsilon, \\ u_\varepsilon|_{|x|=1} = 0, \\ u_\varepsilon|_{|x|=\varepsilon} = U_\varepsilon \quad \text{(unknown constant)}, \\ \displaystyle\int_{|x|=\varepsilon} \frac{\partial u_\varepsilon}{\partial \nu}\, dS = 1, \end{cases}$$

has a unique solution $(u_\varepsilon, U_\varepsilon)$ given by

$$u_\varepsilon = G_{\mathcal{C}}|_{\mathcal{C}_\varepsilon} \quad \text{and} \quad U_\varepsilon = \frac{1}{2\pi} \ln \frac{1}{\varepsilon}. \tag{1.1}$$

2000 *Mathematics Subject Classification.* 35J25.

Equalities in (1.1) do not hold in case of an elliptic operator on arbitrary domain with a 'small' hole around origin; parameter $\varepsilon > 0$ describes the diameter of a hole. First conclusion in (1.1) holds only approximately in certain function space, see Remark 3.2, [1], [2] and [4]. Instead of second equality in (1.1) we show that the ratio between U_ε and certain constants defined by appropriate fundamental solution tends to 1 as ε tends to zero, see Theorem 5.1. The obtained result is then applied for decoupling of the nonlocal boundary value problem (6.2) posed on two domains with small holes.

2. Statement of the problem

Let Ω and B be domains in $\mathbf{R}^2$ containing the origin. We assume that the boundaries of Ω and B are of the class C^1. For $\varepsilon > 0$ we define the domain Ω_ε with a hole in the following way. Let $(r_\varepsilon,\, \varepsilon > 0)$ be a decreasing sequence of positive numbers such that

$$r_\varepsilon \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

The ε-hole B_ε we define by $B_\varepsilon = r_\varepsilon\, B,\ \varepsilon > 0$. For ε small enough so that B_ε is compactly contained in Ω we introduce the domain with a hole

$$\Omega_\varepsilon = \Omega \setminus \overline{B}_\varepsilon.$$

By Γ and Γ_ε we denote the boundary of Ω and B_ε, respectively. Then

$$\partial\Omega_\varepsilon = \Gamma \cup \Gamma_\varepsilon.$$

Let $A(x) = (a_{ij}(x))$ be a second order matrix-valued function defined on $\overline{\Omega}$ and satisfying

$$\begin{aligned} &a_{12}(x) = a_{21}(x), \quad x \in \overline{\Omega}, \quad \text{and} \quad a_{ij} \in C^1(\overline{\Omega}), \quad i,j = 1,2, \\ &A(x)y \cdot y \geq \kappa |y|^2 \quad \text{for some constant} \quad \kappa > 0, \quad x \in \overline{\Omega}, \quad y \in \mathbf{R}^2. \end{aligned} \tag{2.1}$$

With the matrix A we associate the second order elliptic operator $\mathcal{L}$ on Ω by

$$\mathcal{L} = -div(A\nabla). \tag{2.2}$$

The formal setting of the ε-problem is:
find a function $u_\varepsilon : \Omega_\varepsilon \to \mathbf{R}$ and a number U_ε such that

$$\begin{cases} \mathcal{L}u_\varepsilon = 0 \quad \text{in} \quad \Omega_\varepsilon, \\ u_\varepsilon = 0 \quad \text{on} \quad \Gamma, \\ u_\varepsilon = U_\varepsilon \quad \text{on} \quad \Gamma_\varepsilon, \\ \displaystyle\int_{\Gamma_\varepsilon} \frac{\partial u_\varepsilon}{\partial \nu_{\mathcal{L}}}\, dS = 1. \end{cases} \tag{2.3}$$

Here $\frac{\partial}{\partial \nu_{\mathcal{L}}}$ denotes the conormal derivative, i.e. $\frac{\partial u_\varepsilon}{\partial \nu_{\mathcal{L}}} = A\nabla u_\varepsilon \cdot \nu$, where ν is the unit outer normal on Γ_ε.

Problem (2.3) is a mathematical model of some physical problems, see [1], [2] and [4]. A simple example is the equilibrium in the gravity field of an anisotropic membrane clamped at its boundary and supporting (around origin) a thin rigid cylinder, where the cylinder is sealed to the membrane. The boundary condition $(2.3)_{3,4}$ on Γ_ε is not the classical one because the global behaviour instead of the local one is prescribed. Note that the problem (2.3) is an example of a linear non-local boundary value problem.

3. Weak formulation

The appropriate function space for the problem (2.3) is the space

$$\mathcal{H}_\varepsilon = \{v \in H_0^1(\Omega)\,;\, v = \mathit{const.} \text{ on } B_\varepsilon\}.$$

$\mathcal{H}_\varepsilon$ is a Hilbert space with the scalar product

$$(u, v) = \int_\Omega \nabla u \cdot \nabla v \, dx \quad \Big(= \int_{\Omega_\varepsilon} \nabla u \cdot \nabla v \, dx\Big), \quad u, v \in \mathcal{H}_\varepsilon.$$

For $v \in \mathcal{H}_\varepsilon$ we set

$$V_\varepsilon = v|_{B_\varepsilon} \quad \text{(a constant)}.$$

Note that $\mathcal{H}_\varepsilon$ is isomorphic to the space

$$\{v \in H^1(\Omega_\varepsilon)\,;\, v|_\Gamma = 0,\ v|_{\Gamma_\varepsilon} = \mathit{const.}\};$$

here the equalities $v|_\Gamma = 0$ and $v|_{\Gamma_\varepsilon} = \mathit{const.}$ are in the sense of traces of functions from the Sobolev space $H^1(\Omega_\varepsilon)$. We will use both definitions of the space $\mathcal{H}_\varepsilon$. The proof of the following result is simple and it can be found in [4].

Lemma 3.1. *Problem (2.3) has a unique solution*

$$(u_\varepsilon, U_\varepsilon) \in \mathcal{H}_\varepsilon \times \mathbf{R},$$

where u_ε is a unique solution of the variational equation

$$\int_{\Omega_\varepsilon} A \nabla u_\varepsilon \cdot \nabla v \, dx = V_\varepsilon, \quad v \in \mathcal{H}_\varepsilon, \tag{3.1}$$

while

$$U_\varepsilon = \int_{\Omega_\varepsilon} A\, \nabla u_\varepsilon \cdot \nabla u_\varepsilon \, dx. \tag{3.2}$$

Remark 3.2. The right-hand side of (3.1) is in fact a functional

$$\mathcal{H}_\varepsilon \ni v \to V_\varepsilon = v|_{\Gamma_\varepsilon} \in \mathbf{R}. \tag{3.3}$$

It is obviously linear and bounded for a fixed ε, but their norms are not bounded with respect to ε, see [4]. Thus the sequence $(u_\varepsilon,\, \varepsilon > 0)$ is not bounded in $H_0^1(\Omega)$. On the other hand, the sequence $(u_\varepsilon,\, \varepsilon > 0)$ is bounded in $W_0^{1,p}(\Omega)$ for all $p \in [1, 2)$ and it is converging toward the Green function associated with the origin as ε tends to zero in the space $W_0^{1,p}(\Omega)$, see [4]. Still this knowledge does not imply the asymptotics (5.2).

In the sequel we will use an equivalent formulation of the problem (3.1). Let

$$\mathcal{F}_\varepsilon(v) = \frac{1}{2}\int_{\Omega_\varepsilon} A\nabla v \cdot \nabla v \, dx - V_\varepsilon, \quad v \in \mathcal{H}_\varepsilon. \tag{3.4}$$

From the classical theory of calculus of variations it follows that $u_\varepsilon \in \mathcal{H}_\varepsilon$ is a solution of (3.1) if and only if $u_\varepsilon \in \mathcal{H}_\varepsilon$ is a minimizer of $\mathcal{F}_\varepsilon(v)$ over $\mathcal{H}_\varepsilon$, i.e.

$$\mathcal{F}_\varepsilon(u_\varepsilon) = \inf_{v\in\mathcal{H}_\varepsilon} \mathcal{F}_\varepsilon(v). \tag{3.5}$$

From (3.2) and (3.5) it follows that

$$\mathcal{F}_\varepsilon(u_\varepsilon) = -\frac{1}{2}U_\varepsilon. \tag{3.6}$$

Our first result concerning the asymptotic behaviour of the sequence $(U_\varepsilon,\, \varepsilon > 0)$ is the following

THEOREM 3.3. *The sequence $(U_\varepsilon,\, \varepsilon > 0)$ is decreasing, i.e.*

$$U_{\varepsilon_1} \geq U_{\varepsilon_2} \quad \textit{for} \quad \varepsilon_1 < \varepsilon_2.$$

PROOF. Let $\varepsilon_1 < \varepsilon_2$. Then $B_{\varepsilon_1} \subset B_{\varepsilon_2}$ because the sequence $(r_\varepsilon,\, \varepsilon > 0)$ is decreasing, hence $\mathcal{H}_{\varepsilon_2} \subset \mathcal{H}_{\varepsilon_1}$. So the functional $\mathcal{F}_{\varepsilon_1}$ is well defined on the space $\mathcal{H}_{\varepsilon_2}$; moreover, it holds

$$\mathcal{F}_{\varepsilon_1}(v) = \mathcal{F}_{\varepsilon_2}(v), \quad v \in \mathcal{H}_{\varepsilon_2}.$$

This equality, (3.5) and (3.6) imply

$$-\frac{U_{\varepsilon_2}}{2} = \mathcal{F}_{\varepsilon_2}(u_{\varepsilon_2}) = \inf_{v\in\mathcal{H}_{\varepsilon_2}} \mathcal{F}_{\varepsilon_2}(v) = \mathcal{F}_{\varepsilon_1}(u_{\varepsilon_2}) \geq \inf_{v\in\mathcal{H}_{\varepsilon_1}} \mathcal{F}_{\varepsilon_1}(v) = \mathcal{F}_{\varepsilon_1}(u_{\varepsilon_1}) = -\frac{U_{\varepsilon_1}}{2}.$$

□

REMARK 3.4. In fact we proved the following monotonicity. Let H_1 and H_2 be two holes in Ω with boundaries of the class C^1 and such that $0 \in H_1 \subset H_2$. Let (u_1, U_1) and (u_2, U_2) be solutions of the problem (2.3) in the domains $\Omega \setminus \overline{H}_1$ and $\Omega \setminus \overline{H}_2$, respectively. Then $U_1 \geq U_2$.

4. ASYMPTOTICS IN CASE OF SPECIAL HOLES

A variant of the following result is proved in [1]. The proof we give now is valid for a domain with a hole in $\mathbf{R}^n$ and an elliptic operator of the form (2.2). The result is essential for the proof of Theorem 4.2.

LEMMA 4.1. *Let $\varepsilon > 0$ be fixed. If a function $\varphi \in C^2(\Omega_\varepsilon) \cap C^1(\overline{\Omega}_\varepsilon)$ satisfies*

$$\begin{cases} \mathcal{L}\varphi = 0 \quad \textit{in} \quad \Omega_\varepsilon, \\ M = \max\limits_{\Gamma} \varphi < \min\limits_{\Gamma_\varepsilon} \varphi = m, \end{cases} \tag{4.1}$$

then

$$\int_{\Gamma_\varepsilon} \frac{\partial \varphi}{\partial \nu_{\mathcal{L}}}\, dS = -\int_{\Gamma} \frac{\partial \varphi}{\partial \nu_{\mathcal{L}}}\, dS > 0. \tag{4.2}$$

PROOF. Let us consider the auxiliary problem

$$\begin{cases} \mathcal{L}v = 0 \quad \text{in} \quad \Omega_\varepsilon, \\ v = m \quad \text{on} \quad \Gamma_\varepsilon, \\ v = \varphi \quad \text{on} \quad \Gamma. \end{cases} \tag{4.3}$$

The problem (4.3) has a unique solution $v \in C^2(\Omega_\varepsilon) \cap C^1(\overline{\Omega}_\varepsilon)$, see [3]. Moreover, the function v attains its global maximum on $\overline{\Omega}_\varepsilon$ at every point from Γ_ε, so the maximum principle implies

$$\frac{\partial v}{\partial \nu} > 0 \quad \text{on} \quad \Gamma_\varepsilon.$$

But v is a constant function on Γ_ε and this constant is a maximal value of v on $\overline{\Omega}_\varepsilon$, so it holds on Γ_ε:

$$\nabla v(x) = \alpha(x)\, \nu(x) \quad \text{with} \quad \alpha > 0.$$

This equality and positive definiteness (2.1) of the matrix A imply:

$$A(x)\, \nabla v(x) \cdot \nu(x) = \alpha(x)\, A(x)\, \nu(x) \cdot \nu(x) > 0, \quad x \in \Gamma_\varepsilon.$$

Now it follows from (4.3)

$$\int_{\Gamma} \frac{\partial v}{\partial \nu_{\mathcal{L}}}\, dS = -\int_{\Gamma_\varepsilon} \frac{\partial v}{\partial \nu_{\mathcal{L}}}\, dS < 0. \tag{4.4}$$

Let us consider the second auxiliary problem:

$$\begin{cases} \mathcal{L}w = 0 \quad \text{in} \quad \Omega_\varepsilon, \\ w = \varphi - m \quad \text{on} \quad \Gamma_\varepsilon, \\ w = 0 \quad \text{on} \quad \Gamma. \end{cases} \tag{4.5}$$

The function w belongs to the space $C^2(\Omega_\varepsilon) \cap C^1(\overline{\Omega}_\varepsilon)$ and it attains its mimimal value on $\overline{\Omega}_\varepsilon$ at every point from Γ (and at some points from Γ_ε). In a similar way as in the derivation of the inequality (4.4) we conclude that

$$\int_{\Gamma} \frac{\partial w}{\partial \nu_{\mathcal{L}}}\, dS = -\int_{\Gamma_\varepsilon} \frac{\partial w}{\partial \nu_{\mathcal{L}}}\, dS \le 0. \tag{4.6}$$

Because $\varphi = v + w$, inequality (4.2) follows from (4.4) i (4.6). $\square$

Throughout the remaining part of this section let B be the ellipse

$$B = \{x \in \mathbf{R}^2 \,;\ A^{-1}(0)\, x \cdot x < 1\}. \tag{4.7}$$

In this special case we find the precise asymptotics of the sequence $(U_\varepsilon,\, \varepsilon > 0)$. The obtained result will be applied in the next section to the analysis of asymptotics of $(U_\varepsilon,\, \varepsilon > 0)$ in the case of arbitrary canonical hole B.

Parametrix (fundamental solution), singular solution and the Green function, see below, are functions of two variables. In our analysis one of their variables will be equal to zero, so we consider them as functions of the single variable. According to Mikhlin's book [5], the function

$$F(x) = \frac{1}{2\pi\sqrt{det A(0)}} \ln \frac{1}{\sqrt{A^{-1}(0)x \cdot x}}, \quad x \in \overline{\Omega}, \quad x \neq 0, \tag{4.8}$$

is called the parametrix of the equation $\mathcal{L}u = 0$ in Ω associated with $0 \in \Omega$. A singular solution of the equation $\mathcal{L}u = 0$ in Ω associated with $0 \in \Omega$ is any function S which satisfies, see [5],

$$\begin{cases} \mathcal{L}S = 0 \quad \text{in} \quad \Omega \setminus \{0\}, \\ S(x) = F(x) + \psi(x), \quad x \in \Omega \setminus \{0\}, \end{cases} \tag{4.9}$$

where the function $\psi \in C^2(\Omega \backslash \{0\}) \cap C(\overline{\Omega})$ has the following behaviour at $x = 0$:

$$\psi(x) = O\Big(\frac{1}{\ln \frac{1}{|x|}}\Big). \tag{4.10}$$

Green's function G of the operator $\mathcal{L}$ in the domain Ω associated with $0 \in \Omega$ can be defined as follows:

$$G(x) = S(x) + \omega(x), \quad x \in \Omega \setminus \{0\}, \tag{4.11}$$

where ω is a solution of the boundary value problem

$$\begin{cases} \mathcal{L}\omega = 0 \quad \text{in} \quad \Omega, \\ \omega = -S \quad \text{on} \quad \Gamma. \end{cases} \tag{4.12}$$

Existence of parametrix and Green's function is proved in [6]. Note that the behaviour of Green's function at $0 \in \Omega$ is described by the behaviour of the parametrix F at $0 \in \Omega$.

The following result is intuitively acceptable because the restriction of the parametrix F to Γ_ε is a constant in the present setting and the behaviour of the Green function G near zero is determined by F.

THEOREM 4.2. *Let B be the ellipse (4.7) and let $((u_\varepsilon, U_\varepsilon),\, \varepsilon > 0)$ be the sequence of solutions of the problem (2.3). Then*

$$\max_{x \in \Gamma_\varepsilon} |U_\varepsilon - G(x)| \to 0 \quad as \quad \varepsilon \to 0. \tag{4.13}$$

PROOF. Let us assume that (4.13) does not hold. Then there exists a subsequence $(\varepsilon_k,\ k \in \mathbf{N})$ and $\delta > 0$ such that:

$$\begin{cases} \varepsilon_k \to 0 \quad \text{as} \quad k \to \infty, \\ \max\limits_{x \in \Gamma_{\varepsilon_k}} |U_{\varepsilon_k} - G(x)| \geq \delta > 0, \quad k \in \mathbf{N}. \end{cases}$$

Without loss of generality we assume that

$$\max_{x \in \Gamma_{\varepsilon_k}} [U_{\varepsilon_k} - G(x)] \geq \delta > 0 \quad k \in \mathbf{N}. \tag{4.14}$$

Using (4.11) and continuity of ω it is easy to show that

$$\max_{x\in\Gamma_{\varepsilon_k}}[U_{\varepsilon_k}-G(x)]-\min_{x\in\Gamma_{\varepsilon_k}}[U_{\varepsilon_k}-G(x)]\to 0 \quad \text{as} \quad k\to\infty,$$

thus, because of (4.14), there exists $m>0$ such that for k large enough it holds:

$$\min_{x\in\Gamma_{\varepsilon_k}}[U_{\varepsilon_k}-G(x)]\geq m>0. \tag{4.15}$$

For $k\in\mathbf{N}$ we introduce the auxiliary function v_k by

$$v_k(x)=u_{\varepsilon_k}(x)-G(x), \quad x\in\Omega\setminus\{0\}.$$

For k large enough the function v_k satisfies the assumptions of Lemma 4.1 in the domain Ω_{ε_k}, hence

$$\int_{\Gamma_{\varepsilon_k}}\frac{\partial v_k}{\partial\nu_{\mathcal{L}}}\,dS>0. \tag{4.16}$$

On the other hand, $(2.3)_4$ and the following simple properties

$$\int_{\Gamma_{\varepsilon_k}}\frac{\partial S}{\partial\nu_{\mathcal{L}}}\,dS=1 \quad \text{and} \quad \int_{\Gamma_{\varepsilon_k}}\frac{\partial\omega}{\partial\nu_{\mathcal{L}}}\,dS=0,$$

imply that for k large enough it holds :

$$\int_{\Gamma_{\varepsilon_k}}\frac{\partial v_k}{\partial\nu_{\mathcal{L}}}\,dS=0. \tag{4.17}$$

Inequality (4.16) and equality (4.17) are in contradiction. $\square$

Convergence stated in (4.13) describes the sense in which the first equality in (1.1) should be replaced in case of general elliptic operator. The following result, which is a simple consequence of (4.8), (4.10) and (4.13), replaces the second equality in (1.1) in case of elliptic operator.

COROLLARY 4.3. *Let B be the ellipse (4.7) and let $((u_\varepsilon,U_\varepsilon),\,\varepsilon>0)$ be the sequence of solutions of the problem (2.3). Then there exists a constant $C>0$ such that*

$$\Bigg|\frac{U_\varepsilon}{\frac{1}{2\pi\sqrt{detA(0)}}\ln\frac{1}{r_\varepsilon}}-1\Bigg|\leq\frac{C}{\ln\frac{1}{r_\varepsilon}}. \tag{4.18}$$

Theorem 4.2 and the maximum principle have another simple consequence; analogous result has been proved in [1] in case of the Laplace operator.

COROLLARY 4.4.

$$u_\varepsilon\to G \quad \textit{as} \quad \varepsilon\to 0$$

uniformly on compact sets from $\overline{\Omega}\setminus\{0\}$.

5. Asymptotics in case of general hole

In this section the canonical hole B is again an arbitrary domain in $\mathbf{R}^2$ with the boundary of the class C^1 such that $0 \in B$. Although the statements of Theorem 4.2 and Corollaries 4.3 and 4.4 do not hold, we will find the precise asymptotics of the sequence $(U_\varepsilon,\, \varepsilon > 0)$. This asymptotics is limited to the two-dimensional domains with a hole and it is improvement of the corresponding result from [4]. Our proof is based on the monotonicity result from Remark 3.4 and on asymptotics (4.18) for ellipses.

Let $\overline{R} > 0$ ($\underline{R} > 0$) be such that the ellipse

$$E_c = \{x \in \mathbf{R}^2 \,;\, A^{-1}(0)\, x \cdot x < \overline{R}^2\} \quad (E_i = \{x \in \mathbf{R}^2 \,;\, A^{-1}(0)\, x \cdot x < \underline{R}^2\})$$

is the circumscribed (inscribed) ellipse to B. The ellipses $E_c^\varepsilon = r_\varepsilon\, E_c$ and $E_i^\varepsilon = r_\varepsilon\, E_i$ are then circumscribed and inscibed, respectively, to B_ε, i.e.

$$E_i^\varepsilon \subset B_\varepsilon \subset E_c^\varepsilon, \quad \varepsilon > 0. \tag{5.1}$$

Let $((\overline{u}_\varepsilon, \overline{U}_\varepsilon),\, \varepsilon > 0)$ and $((\underline{u}_\varepsilon, \underline{U}_\varepsilon),\, \varepsilon > 0)$ be the solution of the following problem, respectively:

$$\begin{cases} \mathcal{L}\overline{u}_\varepsilon = 0 \quad \text{in} \quad \Omega \setminus \overline{E_c^\varepsilon}, \\ \overline{u}_\varepsilon = 0 \quad \text{on} \quad \Gamma, \\ \overline{u}_\varepsilon = \overline{U}_\varepsilon \quad \text{on} \quad \partial E_c^\varepsilon, \\ \displaystyle\int_{\partial E_c^\varepsilon} \frac{\partial \overline{u}_\varepsilon}{\partial \nu_{\mathcal{L}}}\, dS = 1, \end{cases} \qquad \begin{cases} \mathcal{L}\underline{u}_\varepsilon = 0 \quad \text{in} \quad \Omega \setminus \overline{E_i^\varepsilon}, \\ \underline{u}_\varepsilon = 0 \quad \text{on} \quad \Gamma, \\ \underline{u}_\varepsilon = \underline{U}_\varepsilon \quad \text{on} \quad \partial E_i^\varepsilon, \\ \displaystyle\int_{\partial E_i^\varepsilon} \frac{\partial \underline{u}_\varepsilon}{\partial \nu_{\mathcal{L}}}\, dS = 1, \end{cases}$$

The above problems are the same as the problem (2.3), but posed on different domains; recall that $\underline{U}_\varepsilon$ and $\overline{U}_\varepsilon$ are numbers.

Theorem 5.1.

$$\frac{U_\varepsilon}{\frac{1}{2\pi\sqrt{detA(0)}} \ln \frac{1}{r_\varepsilon}} \to 1 \quad as \quad \varepsilon \to 0. \tag{5.2}$$

Proof. Remark 3.4 and (5.1) imply:

$$\overline{U_\varepsilon} \le U_\varepsilon \le \underline{U_\varepsilon}, \quad \varepsilon > 0.$$

For $\varepsilon > 0$ small enough is $r_\varepsilon < 1$, thus

$$\frac{\overline{U_\varepsilon}}{\frac{1}{2\pi\sqrt{detA(0)}} \ln \frac{1}{r_\varepsilon}} \le \frac{U_\varepsilon}{\frac{1}{2\pi\sqrt{detA(0)}} \ln \frac{1}{r_\varepsilon}} \le \frac{\underline{U_\varepsilon}}{\frac{1}{2\pi\sqrt{detA(0)}} \ln \frac{1}{r_\varepsilon}}, \quad \varepsilon > 0. \tag{5.3}$$

Let us analyse the asymptotic behaviour of the first member in the inequality (5.3). From the obvious equality

$$\frac{\overline{U_\varepsilon}}{\frac{1}{2\pi\sqrt{detA(0)}} \ln \frac{1}{r_\varepsilon}} = \frac{\overline{U_\varepsilon}}{\frac{1}{2\pi\sqrt{detA(0)}} \ln \frac{1}{r_\varepsilon\, \overline{R}}} (1 + \frac{\ln \overline{R}}{\ln r_\varepsilon})$$

we conclude that the first member in the inequality (5.3) converges to 1 as ε tends to zero because of (4.18).

In a similar way it can be shown that the third member in the inequality (5.3) converges to 1 as ε tends to zero, so (5.2) follows from (5.3). $\square$

6. Nonlocal problem on two domains with holes

First we define the problem considered. Let Ω_i and B_i be domains in $\mathbf{R}^2$ with boundaries of the class C^1 and such that $y^i \in \Omega_i$ and $0 \in B_i$, $i = 1, 2$. The sets $y^i + B_i$ are the canonical holes around y^i, $i = 1, 2$. As before, let $(r_\varepsilon,\ \varepsilon > 0)$ be a decreasing sequence which converges to zero as ε tends to zero. The ε-hole in Ω_i around the point $y^i \in \Omega_i$, $i = 1, 2$, we define by

$$B_i^\varepsilon = r_\varepsilon\,(y^i + B_i), \quad i = 1, 2.$$

Let us introduce the notations:

$$\Omega_i^\varepsilon = \Omega_i \setminus \overline{B}_i^\varepsilon, \quad \Gamma_i = \partial\Omega_i, \quad \Gamma_i^\varepsilon = \partial B_i^\varepsilon, \quad i = 1, 2.$$

Let $A_k(x) = (a_{ij}^{(k)}(x))$ be a second order matrix-valued function defined on $\overline{\Omega}_k$, $k = 1, 2$, satisfying

$$\begin{aligned} &a_{12}^{(k)}(x) = a_{21}^{(k)}(x),\ x \in \overline{\Omega}_k, \quad \text{and} \quad a_{ij}^{(k)} \in C^1(\overline{\Omega}_k),\ i, j, k = 1, 2, \\ &A^{(k)}(x) y \cdot y \geq \kappa_k |y|^2,\ x \in \overline{\Omega}_k,\ y \in \mathbf{R}^2,\ k = 1, 2, \end{aligned} \tag{6.1}$$

where κ_k is a positive constant. With the matrix A_i we associate the elliptic operator $\mathcal{L}_i$ on Ω_i by

$$\mathcal{L}_i = -div(A_i\,\nabla), \quad i = 1, 2.$$

Throughout this section let $\varepsilon > 0$ be small enough. For such ε we consider the following problem:
find functions $u_i^\varepsilon : \Omega_i^\varepsilon \to \mathbf{R}$ $(i = 1, 2)$ and a number U^ε such that

$$\begin{cases} \mathcal{L}_i\, u_i^\varepsilon = 0 \quad \text{in} \quad \Omega_i^\varepsilon,\ i = 1, 2, \\ u_i^\varepsilon = 0 \quad \text{on} \quad \Gamma_i,\ i = 1, 2, \\ u_i^\varepsilon = U^\varepsilon \quad \text{on} \quad \Gamma_i^\varepsilon,\ i = 1, 2, \\ \displaystyle\sum_{i=1}^{2} \int_{\Gamma_i^\varepsilon} \frac{\partial u_i^\varepsilon}{\partial \nu_{\mathcal{L}_i}}\, dS = 1. \end{cases} \tag{6.2}$$

For example, problem (6.2) is the mathematical model of oil exploitation process from layered oil fields by a single well; for more details we refer to [1].

The appropriate function space $\mathbf{H}^\varepsilon$ for the problem (6.2) is a product of spaces which are similar to the space $\mathcal{H}_\varepsilon$ introduced in Section 2:

$$\mathcal{H}_i^\varepsilon = \{v \in H_0^1(\Omega_i)\ ;\ v|_{B_i^\varepsilon} = const.\}, \quad i = 1, 2,$$

$$\mathbf{H}^\varepsilon = \{(v_1, v_2) \in \mathcal{H}_1^\varepsilon \times \mathcal{H}_2^\varepsilon\ ;\ v_1|_{B_1^\varepsilon} = v_2|_{B_2^\varepsilon}\}.$$

Function spaces $\mathcal{H}_i^\varepsilon$ are Hilbert spaces with the scalar products

$$(u,v)_i = \int_{\Omega_i^\varepsilon} \nabla u \cdot \nabla v \, dx \quad (= \int_{\Omega_i} \nabla u \cdot \nabla v \, dx \ ), \quad u,v \in \mathcal{H}_i^\varepsilon, \quad i=1,2.$$

The space $\mathbf{H}^\varepsilon$ is a Hilbert space with the natural scalar product. Note that the space $\mathcal{H}_i^\varepsilon$ is isomorphic to the space

$$\{v \in H^1(\Omega_i^\varepsilon)\, ;\, v|_{\Gamma_i^\varepsilon} = const., \quad v|_{\Gamma_i} = 0\}, \quad i=1,2.$$

For a function v_i from $\mathcal{H}_i^\varepsilon$ we introduce the notation

$$V_i^\varepsilon = v_i|_{B_i^\varepsilon} \quad (= v_i|_{\Gamma_i^\varepsilon} \ ), \quad i=1,2.$$

If $(v_1,v_2) \in \mathbf{H}^\varepsilon$, then by definition $V_1^\varepsilon = V_2^\varepsilon$, thus for $(v_1,v_2) \in \mathbf{H}^\varepsilon$ we set

$$V^\varepsilon = v_1|_{\Gamma_1^\varepsilon} \ (= v_1|_{B_1^\varepsilon}) = v_2|_{\Gamma_2^\varepsilon} \ (= v_2|_{B_2^\varepsilon}).$$

Lemma 6.1. *The problem (6.2) has a unique solution*

$$((u_1^\varepsilon, u_2^\varepsilon), U^\varepsilon) \in \mathbf{H}^\varepsilon \times \mathbf{R},$$

where $(u_1^\varepsilon, u_2^\varepsilon) \in \mathbf{H}^\varepsilon$ *is a unique solution of the variational equation*

$$\sum_{i=1}^{2} \int_{\Omega_i^\varepsilon} A_i \, \nabla u_i^\varepsilon \cdot \nabla v_i \, dx = V^\varepsilon, \quad (v_1,v_2) \in \mathbf{H}^\varepsilon, \tag{6.3}$$

while U^ε *is given by the formula*

$$U^\varepsilon = \sum_{i=1}^{2} \int_{\Omega_i^\varepsilon} A_i \, \nabla u_i^\varepsilon \cdot \nabla u_i^\varepsilon \, dx. \tag{6.4}$$

Proof. In a similar way as in the proof of Lemma 3.1 we obtain from (6.2):

$$\int_{\Omega_i^\varepsilon} A_i \, \nabla u_i^\varepsilon \cdot \nabla v_i \, dx = V_i^\varepsilon \int_{\Gamma_i^\varepsilon} \frac{\partial u_\alpha^\varepsilon}{\partial \nu_{\mathcal{L}_i}} \, dS, \quad v_i \in \mathcal{H}_i^\varepsilon, \quad i=1,2. \tag{6.5}$$

Taking the test-functions v_1 and v_2 such that $v = (v_1,v_2) \in \mathbf{H}^\varepsilon$ and summing the two equations from (6.5) we obtain (6.3) because of $(6.2)_4$. The existence and uniqueness of solution $(u_1^\varepsilon, u_2^\varepsilon) \in \mathbf{H}^\varepsilon$ of the problem (6.3) follows from (6.1), continuity of linear functional of the form (3.3) (for ε fixed) and the Lax–Milgram lemma. Equality (6.4) is a simple consequence of (6.3). ◻

In the remaining part of this section we decouple the system (6.3) and prove the convergence of the sequence $(u_i^\varepsilon, \, \varepsilon > 0)$ toward the multiple of the corresponding Green function as ε tends to zero, $i=1,2$.

For $i \in \{1,2\}$ we consider the following problem:
find $(\varphi_i^\varepsilon, \Phi_i^\varepsilon) \in \mathcal{H}_i^\varepsilon \times \mathbf{R}$ such that

$$\begin{cases} \mathcal{L}_i^\varepsilon \, \varphi_i^\varepsilon = 0 \quad \text{in} \quad \Omega_i^\varepsilon, \\ \varphi_i^\varepsilon = 0 \quad \text{on} \quad \Gamma_i, \\ \varphi_i^\varepsilon = \Phi_i^\varepsilon \quad \text{on} \quad \Gamma_i^\varepsilon, \\ \displaystyle\int_{\Gamma_i^\varepsilon} \frac{\partial \varphi_i^\varepsilon}{\partial \nu_{\mathcal{L}_i}} \, dS = 1. \end{cases} \tag{6.6}$$

According to Lemma 3.1, problem (6.6) has a unique solution $(\varphi_i^\varepsilon, \Phi_i^\varepsilon) \in \mathcal{H}_i^\varepsilon \times \mathbf{R}$, $i = 1, 2$. It is easy to verify that the solution $(u_1^\varepsilon, u_2^\varepsilon) \in \mathbf{H}^\varepsilon$ of the problem (6.3) satisfies:

$$u_1^\varepsilon = \frac{\Phi_2^\varepsilon}{\Phi_1^\varepsilon + \Phi_2^\varepsilon} \, \varphi_1^\varepsilon, \quad u_2^\varepsilon = \frac{\Phi_1^\varepsilon}{\Phi_1^\varepsilon + \Phi_2^\varepsilon} \, \varphi_2^\varepsilon. \tag{6.7}$$

Let G_i be the Green function of the operator $\mathcal{L}_i$ on Ω_i corresponding to the point $y^i \in \Omega_i$, $i = 1, 2$. Then the following convergence holds, see [4]:

$$\forall p \in [1,2), \quad \varphi_i^\varepsilon \to G_i \quad \text{in} \quad W_0^{1,p}(\Omega_i) \quad \text{as} \quad \varepsilon \to 0, \quad i = 1, 2. \tag{6.8}$$

Convergences stated in (5.2) and (6.8) and equalities (6.7) have as a consequence the following decoupling result.

THEOREM 6.2. *Let*

$$c_1 = \frac{\sqrt{detA_1(y^1)}}{\sqrt{detA_1(y^1)} + \sqrt{detA_2(y^2)}}, \quad c_2 = \frac{\sqrt{detA_2(y^2)}}{\sqrt{detA_1(y^1)} + \sqrt{detA_2(y^2)}}.$$

Then

$$\forall p \in [1,2), \quad u_i^\varepsilon \to c_i \, G_i \quad \textit{in} \quad W_0^{1,p}(\Omega_i) \quad \textit{as} \quad \varepsilon \to 0, \quad i = 1, 2.$$

REFERENCES

[1] I. Aganović, K. Veselić, *On a singular contact problem for two–dimensional Laplace equation*, Rendiconti di Matematica (4), Vol. **11** (1978), 521–532.

[2] A. Damlamian, Li Ta–Tsien, *Comportement limite des solutions de certain problemes mixtes pour les equations paraboliques*, J. Math. pures et appl. **61** (1982), 113–130.

[3] D. Gilbarg, N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer, Berlin, 1983.

[4] Li Ta–Tsien, J. F. Rodrigues, Zhu Chen, *Limit behaviour of solutions to a class of non–linear and non–local boundary value problems*, Asymptotic Analysis **9** (1994), 297–310.

[5] S. G. Mikhlin, *Course in Mathematical Physics (in Russian)*, Nauka, Moskva, 1968.

[6] C. Miranda, *Partial Differential Equations of Elliptic Type (in Russian)*, Moskva, 1957.

Department of Mathematics
University of Zagreb
Bijenička 30
10000 Zagreb, Croatia
E-mail: aganovic@math.hr

Department of Mathematics
University of Zagreb
Bijenička 30
10000 Zagreb, Croatia
E-mail: tutek@math.hr

Fernuniversität
Gesamthochschule
Fachbereich Mathematik
Postfach 940
58000 Hagen 1, Germany
E-mail: Kresimir.Veselic@FernUni-Hagen.de

Received: 20.07.98
Revised: 4.06.99