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FIBERWISE RETRACTION AND SHAPE PROPERTIES OF THE ORBIT SPACE

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Abstract. From the point of view of retracts and shape theory, the category $G\text{-}TOP_B$ of G-spaces over a G-space B, where G is a compact group, is investigated. In particular, we prove that if B has only one orbit type and E is a metric $G\text{-}ANR$ over B, then the orbit space E/G is an ANR over B/G. As an application we construct a fiberwise G-orbit functor $\mu\colon G\text{-}SH_B \to SH_{B/G}$ on shape level.

1. Introduction

Many of the ideas of homotopy theory belong most naturally to the category $G\text{-}TOP_B$ of G-spaces over a given G-space B, where G is a topological group. An excellent demonstration of that provide two articles of I. M. James and G. B. Segal [16], [17] (see also [18, Ch. 8]), which have inspired this research. Here we study fiberwise retraction and shape properties of orbit spaces of G-spaces over B. The first main result we establish (Theorem 3.1) has no counterpart in the ordinary theory of retracts and provides a fiberwise version of the following result of S. A. Antonyan [4], [5]:

Theorem 1.1. *Let G be a compact group, $N \subseteq G$ be a closed normal subgroup, and X be a metric $G\text{-}A(N)R$-space. Then X/N is a $G/N\text{-}A(N)R$-space. In particular X/G is an $A(N)R$-space.*

This result is the crucial tool in what follows. It will be applied also in the form of the following equivalent assertion:

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Theorem 1.2. *Let G be a compact group, $N \subseteq G$ be a closed normal subgroup and A be a metric G-space such that A/N admits a G/N-equivariant closed embedding into a metric G/N-space X. Then there exists a metric G-space Y, which contains A as a G-invariant subspace and $Y/N = X$.*

The equivalence of Theorem 1.1 and Theorem 1.2 is proved in [6].

Our Theorem 3.1 generalizes the above Theorem 1.1. At the same time we show that it is not true for arbitrary base B. Namely, the condition "B is an ANE over B/G" is a necessary condition (Proposition 3.2), while the condition "B is a G-ANE over B/G" is sufficient in Theorem 3.1. Here B/G is regarded as a G-space with the trivial G-action.

We prove for G a Lie group (Theorem 3.3) that every G-space B with all orbits of the same orbit type, is a G-ANE over its orbit space B/G.

Conversely, if B is a connected G-ANE over B/G, then B has only one orbit type. Theorem 3.7 is just the finite-dimensional analogue of Theorem 3.1.

In §3 we make use the results of §2 to develop a fiberwise shape theory for arbitrary G-spaces over a given metric G-space B, where the acting group G is assumed to be compact. There are several approaches to the non-equivariant fiber shape theory [8], [10], [19], [26]. For equivariant shape theory we refer to [7], [11], [23] and [25]. The description of our fiber G-shape category $G\text{-}SH_B$ is based on the general construction of shape categories in [22].

Finally, applying Theorem 3.1, in particular, we establish that the G-orbit functor $\pi\colon G\text{-}TOP_B \to TOP_{B/G}$ naturally induces a corresponding functor $\mu\colon G\text{-}SH_B \to SH_{B/G}$ on shape level (Theorem 4.9).

2. Basic notions, facts and notations

Throughout the paper it is assumed, unless the contrary is stated, that G is a compact Hausdorff group which we keep fixed. All topological spaces are assumed to be Tychonov.

The basic definitions and results of the theory of G-spaces or topological transformation groups can be found in G. Bredon [9] and in R. Palais [24].

A survey of the equivariant theory of retracts was given in [1]. For equivariant fiberwise theory of retracts we refer to [17], [18, Ch. 8].

By G/H we always denote the left coset space of G by a closed subgroup H, endowed with the action of G by left translations.

If X is a G-space and $N \subseteq G$ is a closed normal subgroup, then the N-orbit space X/N admits a natural G/N-action defined by

$$(gN)(N(x)) = N(gx) \tag{2.1}$$

where $gN \in G/N$ and $N(x)$ denotes the N-orbit of $x \in X$.

Hereafter, we will always mean the action (1) on the G/N-space X/N.

If $H \subseteq G$ is a subgroup then the class of subgroups of G which are conjugate to H is denoted by (H), i.e., $(H) = \{gHg^{-1}|g \in G\}$. The class (H) is often called a G-orbit type or simply an orbit type. Let (H) and (K) be two orbit types. One says that $(H) \leq (K)$ if H is conjugate to some subgroup of K. If in addition $(H) \neq (K)$, we say that $(H) < (K)$. It is easy to see that the relation $\leq$ is a partial ordering on the set of all G-orbit types. Now suppose that X is a G-space and $x \in X$. The subgroup $G_x = \{g \in G \mid gx = x\}$ of G is called the stabilizer of the point x; since $G_{gx} = gG_xg^{-1}$ for any $g \in G$ we have $(G_x) = (G_{gx})$. If $H \subseteq G$ is a subgroup, we denote by $X[H]$ the subset of H-fixed points of X, i.e., $X[H] = \{x \in X| \ H \subseteq G_x\}$. It is well-known that $X[H]$ is a closed $N(H)$-invariant subspace of X, where $N(H)$ is the normalizer of H in G (see [9, Ch. I, §5]).

Let X be a G-space and $H \subseteq G$ be a closed subgroup. We denote by HS the subset $\{hs \mid h \in H, s \in S\}$ of X.

A subset $S \subseteq X$ is called [24, p. 27] an H-slice in X if

(1) GS is open in X and S is closed in GS,
(2) S is H-invariant, i.e., $HS = S$,
(3) for each $g \in G$ not in H, gS is disjoint from S.

If in addition $GS = X$ then S is called a *global* H-slice of X.

Clearly, if $f\colon Z \to X$ is a G-map and S is an H-slice in X, then $f^{-1}(S)$ is an H-slice in Z.

In the sequel we will need the following useful property of a slice: if Q is a global H-slice of a G-space X and R is a global H-slice of a G-space Y, then for each H-equivariant map $f\colon Q \to R$, the map $F\colon X \to Y$ defined by $F(gx) = gf(x)$ is a well-defined G-map (see for example [24, Proposition 1.7.10]).

The following device (due to G. Segal) called the colon construction by I. M. James [18, Ch. 4], is often useful in the study of G-spaces. For G-spaces X and Y let us denote by $(X : Y)$ the orbit space W/G, where $W \subset X \times Y$ is the invariant subspace consisting of pairs (x, y) such that $G_x \subseteq G_y$.

In this article we work in the category $G\text{-}TOP_B$ of G-spaces over a given G-space B, which is called the base. A G-space over B consists of a G-space E and a G-map $p\colon E \to B$ called the projection.

Usually E alone is a sufficient notation. Thus B is regarded as a G-space over itself with the projection the identity map. Moreover any product $X \times B$ of G-spaces is regarded as a G-spaces over B with the natural projection $X \times B \to B$. Let X, Y be G-spaces over B with the projections p, q, respectively. By a G-map $f\colon X \to Y$ over B we mean a G-map in the ordinary sense such that $qf = p$. With this definition of morphisms the category $G\text{-}TOP_B$ is defined.

Let us recall the definition of N-orbit functor $\pi : G\text{-}TOP_B \to G/N\text{-}TOP_{B/N}$ where $N \subseteq G$ is a closed normal subgroup. For any G-space E over

B with projection $p : E \to B$ we denote $\pi(E) = E/N$, $\pi(B) = B/N$ and $\pi(p) = p/N$, where $(p/N)(N(x)) = N(p(x))$ for every N-orbit $N(x) \in E/N$. If we consider E/N and B/N as G/N-spaces with G/N-action (1), the map $p/N : E/N \to B/N$ becomes a G/N-map. So E/N naturally is a G/N-space over B/N.

Let F be another G-space over B and $f : E \to F$ be a G-map over B. Then f induces a G/N-map $f/N : E/N \to F/N$ defined by $(f/N)(N(x)) = N(f(x))$, $N(x) \in E/N$. One easily verifies that f/N is a G/N-map over B/N. Putting $\pi(f) = f/N$ we obtain the desired functor π.

Remark 2.1. Sometimes we will need to regard $\pi(E)$ also as **a G-space** over $\pi(B)$, where G acts on $\pi(E)$ and $\pi(B)$ via the natural homomorphism $G \to G/N$ (see e.g., the proof of Theorem 3.1).

In this case $\pi(f)\colon \pi(E) \to \pi(F)$ is a G-map over $\pi(B)$ for any G-map $f\colon E \to F$ over B. So, one also can regard π as a functor from $G\text{-}TOP_B$ into $G\text{-}TOP_{B/N}$.

For future references, we will work with the diagram of G-spaces as follows:

$$\begin{array}{ccc} A & \xrightarrow{\;f\;} & E \\ {\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ X & \xrightarrow{\;\phi\;} & B \end{array} \qquad (2.2)$$

where (X, A) is a metric G-pair, i.e., a pair in which X is a metric G-space, A is a closed invariant subspace of X and i is the inclusion map.

Let E be a G-space over B. Then E is called a G-ANE over B (notation: $E \in G\text{-}ANE_B$), if the following equivariant fiberwise extension property holds for all metric G-pairs (X, A) over B:

For any G-maps f, ϕ which make the above diagram (2) commutative, there exist an invariant neighborhood U of A in X and a G-map $\psi\colon U \to E$ such that $\psi|A = f$ and $p\psi = \phi|U$. If in addition we can always take $U = X$, then we say that E is a G-AE over B (notation: $E \in G\text{-}AE_B$). The map ψ is called a G-extension of f over ϕ.

Let E be a metric G-space over B. Then E is called a G-ANR over B (notation: $E \in G\text{-}ANR_B$) provided for any metric G-space X over B and any closed G-embedding $E \hookrightarrow X$ over B there exist an invariant neighborhood U of E in X and a G-retraction $r\colon U \to E$ over B. If in addition we can always take $U = X$, then we say that E is a G-AR over B (notation: $E \in G\text{-}AR_B$).

Let $n \geq 0$ be an integer and E be a G-space over B. Then E is called a G-$A(N)E(n)$ over B if the above fiberwise extension property holds for all metric G-pairs (X, A) with $\dim X/G \leq n$.

If in the above definitions instead of a metric G-pair (X, A) we take a G-pair (X, A) from a given class $\mathcal{K}$ of G-spaces over B, then we obtain in a similar way the notions of G-$A(N)E(\mathcal{K})$, G-$A(N)E(n)(\mathcal{K})$ spaces over B. When the base B has only one point then the above definitions become the usual definitions of G-$A(N)E(\mathcal{K})$ and G-$A(N)E(n)(\mathcal{K})$. In [17], [18] the case $\mathcal{K} = \mathcal{P}-$ the class of all paracompact G-spaces was considered.

Let $f_0, f_1\colon E \to E'$ be G-maps over B. A G-homotopy over B of f_0 into f_1 is a homotopy in the ordinary sense which is a G-map over B at each stage of the deformation. The G-space E is called G-contractible over B if there is a G-section $s\colon B \to E$ (i.e., $ps = id_B$) such that sp and id_E are G-homotopic over B.

In an obvious way one obtains a G-equivariant fiber homotopy category over B denoted by $[G\text{-}TOP_B]$, whose objects are G-spaces over B and whose morphisms are classes of G-homotopic G-maps over B. There is a homotopy functor $G\text{-}TOP_B \to [G\text{-}TOP_B]$ which keeps the objects fixed and takes G-maps f over B into their G-homotopy classes $[f]$ over B.

3. Main results

THEOREM 3.1. *Let $N \subseteq G$ be a closed normal subgroup and E be a metric G-ANE (resp., a G-AE) over a (resp., metric) G-space B. Suppose also that B is a G-ANE over B/N. Then E/N is a G/N-ANE (resp., a G/N-AE) over B/N. In particular E/G is an ANE (resp., an AE) over B/G whenever B is a G-ANE over B/G.*

Before proceeding with the proof, let us show that the restriction on B in Theorem 3.1 is essential.

First we observe that for every topological space Z and each integer $n \geq 1$ the n-fold power Z^n possesses a natural action of the symmetric group on n letters S_n defined by the formula: $g*(z_1, \ldots, z_n) = (z_{g^{-1}(1)}, \ldots, z_{g^{-1}(n)})$ where $g \in S_n$ and $(z_1, \ldots, z_n) \in Z^n$.

Denote $E = (Q \times Q)^n$ and $B = Q^n$ where Q is the Hilbert cube. So, E and B can be regarded as G-spaces with $G = S_n$. Define the projection $p\colon E \to B$ as follows: $p(x_1, \cdots, x_n) = (h(x_1), \cdots, h(x_n))$ for any $(x_1, \cdots, x_n) \in E$, where $h\colon Q \times Q \to Q$ is the first projection. Clearly p is an S_n-map. We claim that E is an S_n-AE over B. Indeed, let f and ϕ be S_n-maps making commutative the diagram (2). Denote by T the discrete S_n-space $\{1, 2, ..., n\}$. Consider maps $f'\colon A \times T \to Q \times Q$ and $\phi' : X \times T \to Q$ defined by $f'(a,t) = f(a)_t$ and $\phi'(x,t) = \phi(x)_t$ respectively, where $a \in A$, $x \in X$ and $t \in T$. Consider $X \times T$ as an S_n-space with the diagonal action. One easily sees that f' is a continuous map over ϕ' and that both of them are S_n-invariant maps. Therefore these maps induce canonically continuous maps $f^* : (A \times T)/S_n \to Q \times Q$ and $\phi^* : (X \times T)/S_n \to Q$ such that f^* is a map over ϕ^*. Clearly $(X \times T)/S_n$ is metric, $(A \times T)/S_n$ is closed in $(X \times T)/S_n$ and $Q \times Q$ is an AE over Q with h the projection (because the Hilbert cube is an AE). Hence there is a continuous extension $\psi^*\colon (X \times T)/S_n \to Q \times Q$ of f^* over ϕ^*. Let

$p\colon X\times T\to (X\times T)/S_n$ be the S_n-orbit map and $F'=\psi^*p\colon X\times T\to Q\times Q$. Then F' is continuous and invariant. Now the map $F\colon X\to E$ defined by the formula $F(x)_t=F'(x,t),\ \ x\in X,\ t\in T$, is the required S_n-equivariant extension of f over ϕ and the claim is proved.

However E/S_n is not an AE over B/S_n as it is shown by V. V. Fedorchuk [13], [14, p. 242]. In fact E/S_n is not even an ANE over B/S_n. Indeed, since $E\in S_n\text{-}AE_B$, and B is metric, E is S_n-contractible over B. This easily implies the contractibility of E/S_n over B/S_n and since a fiberwise contractible ANE is a fiberwise AE, we conclude that E/S_n is not an ANE over B/S_n.

Proof of Theorem 3.1. Let (X,A) be a metric G/N-pair and let $f\colon A\to E/N$, $\phi\colon X\to B/N$ be G/N-maps such that $(p/N)f=\phi|A$, where $p/N\colon E/N\to B/N$ is the canonical G/N-map induced by the projection $p\colon E\to B$. We must show that f admits a G/N-neighborhood extension $\Phi: W\to E/N$ over ϕ. Define $A'\subseteq A\times E$ to be the pull-back of the G-space E with respect to f. Then A' is a G-invariant subspace of $A\times E$ endowed with the diagonal action of G (by Remark 2.1 we can consider A as a G-space). Since in this case N acts trivially on A we have $A'/N=A$ (see [15, §4.1]). Let $\lambda\colon A'\to A$, $f'\colon A'\to E$ be the corresponding projections. Since A' is metric, we are in position to apply the above Theorem 1.2, according to which there is a metric G-space X' which contains A' as a G-invariant closed subspace and $X'/N=X$. Let denote by $\mu\colon X'\to X$ the orbit map and by $j\colon A'\to X'$ the inclusion map. Consider the commutative diagram

$$\begin{array}{ccc} A' & \xrightarrow{pf'} & B \\ {\scriptstyle j}\big\downarrow & & {\scriptstyle \rho}\big\downarrow \\ X' & \xrightarrow{\phi\mu} & B/N \end{array}$$

where ρ is the N-orbit map. As B is a $G\text{-}ANE$ over B/N there exists a G-extension $F'\colon U\to B$ of pf' over $\phi\mu$ defined on some G-neighborhood U of A' in X'.

So, we have $(\phi\mu)|U=\rho F'$ and $F'|A'=pf'$. Since $E\in G\text{-}ANE_B$ it follows that there exist a G-neighborhood V of A' in U and a G-extension $F\colon V\to E$ of f' over F'. Denote shortly by Φ the G/N-map F/N. We claim that $\Phi\colon V/N\to E/N$ is the desired G/N-extension of f over ϕ. Indeed, first we note that $W{=}V/N$ is a G/N-neighborhood of A in X. Now let $a\in A$ and let $\nu: E\to E/N$ be the N-orbit map. Then $a=\lambda(a')=\mu(a')$ for some $a'\in A'$, and hence, $\Phi(a)=\nu F(a')=\nu f'(a')=f\lambda(a')=f(a)$, i.e., $\Phi i=f$. For every $x\in V/N$ there is $x'\in V$ such that $x=\mu(x')$ and then $\Phi(x)=\nu F(x')$. Consequently, $(p/N)\Phi(x)=(p/N)\nu F(x')=\rho pF(x')=\rho F'(x')=\phi\mu(x')=\phi(x)$, i.e., Φ is a map over ϕ and the proof in the "$G\text{-}ANE$" case is completed.

If in addition E is $G\text{-}AE$ over B and B is metric, then E is G-contractible over B by [16]. This implies that E/N is G/N-contractible over B/N. Since

E/N is a G/N-ANE over B/N by the former case, it then follows from Proposition 2.3 of [16] that E/N is a G/N-AE over B/N.

Concerning the restriction on base B in Theorem 3.1 we have the following result (for simplicity we consider only the case $N = G$):

PROPOSITION 3.2. *Let G be a compact Lie group and B be a metric G-space. Suppose that for any G-space E over B which is a G-ANE_B, the orbit space E/G is an ANE over B/G. Then B is an ANE over B/G.*

PROOF. Take $E = G \times B$ with the G-action $g\star(h,b) = (hg^{-1}, gb)$. Since G is a G-ANE [24, Corollary 1.6.7] we conclude that E is a G-ANE over B. By the hypothesis it then follows that E/G is an ANE over B/G. Now observe that the map $\phi\colon (G \times B)/G \to B$ defined by $\phi[g,b] = gb$, $[g,b] \in (G \times B)/G$ is an homeomorphism over B/G (see, for example [9, p. 113]). Thus B is an ANE over B/G.

THEOREM 3.3. *Let G be a compact Lie group and suppose B is an arbitrary Tychonov G-space with all orbits of type G/H. Then B is a G-ANE over B/N for every closed normal subgroup $N \subseteq G$. Conversely, if B is connected and B is a G-ANE over B/G, then B has only one orbit type (even in the case of G an arbitrary compact group).*

For the proof of Theorem 3.3 we need the following three lemmas:

LEMMA 3.4. *Let G be a compact Lie group, $H \subseteq G$ be a closed subgroup and $N \subseteq G$ be a closed normal subgroup. Then G/H is a G-ANE over G/HN.*

PROOF. Let (X, A) be a metric G-pair and let f, ϕ be G-maps such that the diagram (2) with $E = G/H$, $B = G/HN$ and $p\colon G/H \to G/HN$ the natural projection, is commutative. Put $Q = \phi^{-1}(eHN)$ where e is the unity of G and denote $S = Q \cap A$. Then Q is a global HN-slice of X, S is a closed HN-invariant subspace of Q and f maps S into $(HN)/H = p^{-1}(eHN) \subseteq G/H$. Since $(HN)/H$ is an HN-ANE [24, Corollary 1.6.7], $f|S$ admits an HN-equivariant extension F defined on some HN-invariant neighborhood V of S in Q. Then setting $\tilde{F}(gv) = gF(v)$ for each $g \in G$, $v \in V$, we obtain a G-extension $\tilde{F}$ of f over ϕ defined on the G-neighborhood GV of A in X.

LEMMA 3.5. *Let E be a G-space over B and suppose that there exists an open invariant covering $\{V_j | j \in J\}$ of B such that $E_j = p^{-1}(V_j)$ is a G-AE_{V_j} (resp., a G-ANE_{Vj}) for each index j. Then E is a G-AE_B (resp., a G-ANE_B).*

PROOF. Because of [18, Proposition 8.48] it is sufficient to show that $(Z\colon E)$ is an AE (resp., an ANE) over $(Z\colon B)$ for all metric G-spaces Z. Clearly $\{(Z\colon V_j) | j \in J\}$ is an open covering of $(Z\colon B)$. Since E_j is a G-ANE

over V_j [18, Proposition 8.48] using again [18, Proposition 8.48], we have that $(Z\colon E_j)$ is an AE (resp., an ANE) over $(Z\colon V_j)$ for each index j. So, by [18, Proposition 8.25], $(Z\colon E)$ is an AE (resp., an ANE) over $(Z\colon B)$.

Lemma 3.6. *Let $H \subseteq G$ be a closed subgroup and E be a G-ANE (resp., a G-AE) over B. Then E is an H-ANE (resp., an H-AE) over B.*

Proof. We consider only the "H-ANE" case. The "H-AE" case is similar. Let (X, A) be a metric H-pair, f, ϕ be H-maps such that the diagram (2) is commutative. Consider the G-space Z for which X is a global H-slice (for Z one can take the twisted product $G\times_H X$ [9, Ch. II, §3]). According to [24, Theorem 1.7.10] the maps f, ϕ uniquely determine G-equivariant maps $f' : GA \to E$ and $\phi' : Z \to B$ such that $f'|A = f$, $\phi'|X = \phi$. Clearly $pf' = \phi'|GA$. Since $E \in G$-ANE_B there exist a G-neighborhood U of GA in Z and a G-extension $\psi' : U \to E$ of f' over ϕ'. Putting $V = U \cap X$ and $\psi = \psi'|V$ we obtain the desired H-equivariant extension $\psi : V \to E$ of f over ϕ.

Proof of Theorem 3.3. By Theorem 5.8 of [9, Ch. II] the G-space B constitutes a fibre bundle over B/G. Therefore we can cover B by open sets V_α of the form $V_\alpha = G/H\times U_\alpha$ where G acts trivially on U_α. For B/N we have the open invariant covering $\{V_\alpha/N\}$ and according to Lemma 3.5, it suffices to show that V_α is a G-ANE over V_α/N. Since $V_\alpha/N = (G/H \times U_\alpha)/N = G/HN \times U_\alpha$ and since G/H is a G-ANE over G/HN (Lemma 3.4), it then follows that $G/H \times U_\alpha$ is a G-ANE over $G/HN \times U_\alpha$. This proves the first part of Theorem 3.3

To prove the second part, suppose that B is a connected G-space, which is a G-ANE over B/G. Suppose also that B has more than one orbit type. Then one can find two different orbit types in B, say (H_1) and (H_2), such that either $(H_1) < (H_2)$ or (H_1) and (H_2) are not comparable. Let W denotes the normalizer of H_2 in G. Consider $B[H_2]$ the set of H_2-fixed points of B. Clearly $B[H_2]$ is a W-invariant subspace of B, so it can be regarded as a W-space. Put $X = B/H_2$ the H_2-orbit space of B. Since H_2 is a closed normal subgroup of W, the group W acts naturally on X. Set $A = B[H_2]$. Evidently A can be regarded as a closed W-invariant subspace of the W-space X. Consider the diagram

$$\begin{array}{ccc} A & \xrightarrow{\ f\ } & B \\ {\scriptstyle j}\big\downarrow & & {\scriptstyle p}\big\downarrow \\ X & \xrightarrow{\ \phi\ } & B/G \end{array}$$

where f is the natural inclusion and $\phi(H_2(x)) = G(x)$ for all $H_2(x) \in B/H_2 = X$. Clearly f and ϕ are W-maps and the diagram commutes. Since B is a W-ANE over B/G (Lemma 3.6), there exist a W-invariant neighborhood U of A

in X and a W-extension $\psi : U \to B$ of f over ϕ. As H_2 acts trivially on X it follows that the image $\psi(U)$ must lie in $B[H_2]$. Let $r : B \to X$ denotes the H_2-orbit map. Putting $\psi' = \psi r$, $V = r^{-1}(U)$ we get the following commutative diagram

$$\begin{array}{ccc} A = B[H_2] & \xrightarrow{f} & B[H_2] \subset B \\ {\scriptstyle j}\downarrow & & {\scriptstyle p}\downarrow \\ V & \xrightarrow{p'} & V/G \subset B/G \end{array}$$

where $p' = p|V$.

We claim that there is a point $y \in V$ such that $G(y) \cap B[H_2] = \emptyset$. Indeed, if the contrary is true, then $GV = GB[H_2]$. Since G is compact, $GB[H_2]$ is closed (see [9, Ch. I, §1, Corollary 1.3] or [24, Proposition 1.1.1]) and since V is open, GV is open as well. So $GB[H_2]$ being a closed-open subset of B must coincide with B as B is connected. But this is impossible, because in B there is a point $z \in B$ with $G_z = H_1$ and we have $(G_z) = (H_1) \not\geq (H_2)$, while $(G_x) \geq (H_2)$ for all $x \in GB[H_2]$. This completes the proof of the claim. Now by the commutativity of the above diagram we have $p\psi'(y)$=$p'(y) = p(y)$ i.e., $\psi'(y)$ and y have the same G-orbit. This is a contradiction since $\psi'(y) \in B[H_2]$ and $G(y) \cap B[H_2] = \emptyset$.

Theorem 3.7. *Let $N \subseteq G$ be a closed normal subgroup and E be a metric G-$ANE(k)$ over a G-space B, where $k \geq 0$ is an integer. Suppose also that B is a G-$ANE(k)$ over B/N. Then E/N is a G/N-$ANE(k)$ over B/N.*

Furthermore, if B is metric, $dimB/G \leq k$, $dimE/G \leq k-1$ and E is a G-$AE(k)$ over B, then E/N is a G/N-$AE(k)$ over B/N.

Proof. The proof of the first part of this theorem is analogous to that of Theorem 3.1, the only additional condition on the dimension of the orbit space also holds because $X'/N = X$ implies $X'/G = X/G$.

Consider the "*G-AE*"case. Since B is metric and $dimB/G \leq k$, the condition $E \in G$-$AE_B(k)$ easily implies that the projection $p : E \to B$ admits an equivariant section $s : B \to E$. Consider the following commutative diagram of G-maps:

$$\begin{array}{ccc} E \times \{0\} \cup E \times \{1\} & \xrightarrow{f} & E \\ \downarrow & & {\scriptstyle p}\downarrow \\ E \times [0,1] & \xrightarrow{\phi} & B \end{array}$$

where $f(x,0) = x$, $f(x,1) = sp(x)$ and $\phi(x,t) = p(x)$ for $x \in E$, $t \in [0,1]$. Since $dimE/G \leq k-1$ we have $dim(E \times [0,1])/G \leq k$. Therefore, using the condition $E \in G$-$AE_B(k)$, we obtain a G-contraction of E over B. This implies that E/N is G/N-contractible over B/N. Since E/N is a G/N-$ANE(k)$ over B/N by the previous case, it then follows that E/N is a G/N-$AE(k)$ over B/N [17, Proposition 2.3].

4. Equivariant fiberwise shape

Throughout of this section we assume that B is a given metric G-space. Here we define a shape category for arbitrary G-spaces over B. In our development we follow the method of resolutions introduced in the case of ordinary shape by S. Mardešić [20], [21] and extended to the equivariant case in [7]. A general procedure is described in [22, Ch. I, §2], which associates a shape category $SH_{\mathcal{T},\mathcal{P}}$ with every pair consisting of a category $\mathcal{T}$ and of a dense subcategory $\mathcal{P}$. The equivariant shape category over a base B is the shape category associated in this way with the pair $\mathcal{T} = [G\text{-}TOP_B]$, $\mathcal{P} = [G\text{-}ANR_B]$ where $[G\text{-}ANR_B]$ denotes the full subcategory of $[G\text{-}TOP_B]$ consisting of all G-spaces over B, which have the fiberwise G-homotopy type of some G-ANR_B space.

In the realization of the outlined program the crucial tool is the notion of fiberwise G-resolution defined below.

Consider inverse systems $\underline{E} = (E_\lambda, p_{\lambda\lambda'}, \Lambda)$ in the category $G\text{-}TOP_B$. This means that every E_λ is a G-space over B and every $p_{\lambda\lambda'}\colon E_{\lambda'} \to E_\lambda$, $\lambda \le \lambda'$ is a G-map over B. If every E_λ is a G-ANR over B we will say that $\underline{E}$ is a G-ANR_B-system.

We refer to [22, Ch. I, §§1,2] for definitions of basic terms (pro-category, expansion, dense subcategory, etc.).

If E is a G-space over B, a morphism in $pro - G\text{-}TOP_B$, $\underline{p}\colon E \to \underline{E}$ consists of G-maps over B, $p_\lambda\colon E \to E_\lambda$ such that $p_\lambda = p_{\lambda\lambda'} p_{\lambda'}$ for $\lambda \le \lambda'$.

Definition 4.1. *A morphism $\underline{p}\colon E \to \underline{E}$ of $pro - G\text{-}TOP_B$ is called a G-resolution of the G-space E over B, provided for every $P \in G\text{-}ANR_B$ and every open covering ω of P the following conditions are satisfied:*

- (R_1) *If $f\colon E \to P$ is a G-map over B, then there exist a $\lambda \in \Lambda$ and a G-map $h\colon E_\lambda \to P$ over B such that hp_λ and f are ω-near.*
- (R_2) *There is an open covering ω' of P with the following property: whenever $\lambda \in \Lambda$ and $h_0, h_1\colon E_\lambda \to P$ are G-maps over B such that $h_0 p_\lambda$ and $h_1 p_\lambda$ are ω'-near, then there is a $\lambda' \ge \lambda$ such that $h_0 p_{\lambda\lambda'}$ and $h_1 p_{\lambda\lambda'}$ are ω-near.*

If each E_λ is a G-ANR_B then we say that $\underline{p}$ is a G-ANR_B-resolution of E.

Theorem 4.2. *Every G-space E over B admits a G-ANR_B-resolution $\underline{q}\colon E \to \underline{E}$.*

PROOF. It follows from the proof of [7, Theorem 1] that E admits an ordinary (not over B) G-ANR resolution $\underline{r}\colon E \to \underline{X} = (X_\lambda, r_{\lambda\lambda'}, \Lambda)$ which satisfies the following strongest condition instead of (R_1) with B the singleton:

(R_1') If $f\colon E \to L$ is a G-map in some G-ANR space L then there are an index $\lambda \in \Lambda$ and a G-map $h\colon X_\lambda \to L$ such that $hr_\lambda = f$.

(see also [20, the proof of Theorem 13, formula (8)]). For each $\lambda, \lambda' \in \Lambda$ with $\lambda \leq \lambda'$ define the G-maps $t_\lambda\colon E \to X_\lambda \times B$ and $s_{\lambda\lambda'}\colon X_{\lambda'} \times B \to X_\lambda \times B$ by putting $t_\lambda(x) = (r_\lambda(x), p(x))$ and $s_{\lambda\lambda'} = r_{\lambda\lambda'} \times id_B$, where $p\colon E \to B$ is the projection. Then t_λ and $s_{\lambda\lambda'}$ become G-maps over B if we consider each $X_\lambda \times B$ as a G-space over B with the usual projection $X_\lambda \times B \to B$.

Let M be the set of all pairs $\mu = (\lambda, U)$ where $\lambda \in \Lambda$ and U is an invariant open neighborhood of $t_\lambda(E)$ in $X_\lambda \times B$. We order M by putting $\mu \leq \mu' = (\lambda', U')$ whenever $\lambda \leq \lambda'$ and $s_{\lambda\lambda'}(U') \subseteq U$. For every $\mu = (\lambda, U) \in M$ let $E_\mu = U$, $q_\mu = t_\lambda\colon E \to U$ and $q_{\mu\mu'} = s_{\lambda\lambda'}|U'\colon U' \to U$ if $\mu \leq \mu'$.

Clearly $\underline{E} = (E_\mu, q_{\mu\mu'}, M)$ is an inverse system of G-ANR spaces over B and $\underline{q} = (q_\mu)\colon E \to \underline{E}$ is a morphism of the category $pro-G\text{-}TOP_B$. We claim that $\underline{q}$ satisfies both conditions (R_1) and (R_2).

For (R_1) let P be a G-ANR over B and $f\colon E \to P$ be a G-map over B. Then there are normed linear G-space L such that $L \in G$-AE and L contains P as a closed invariant subspace [2, Corollary 5 and Corollary 8]. This implies a closed equivariant embedding over B of P into $L \times B$. Since $P \in G\text{-}ANR_B$ there is an invariant neighborhood V of P in $L \times B$ and a G-retraction $\eta\colon V \to P$ over B. Let $\alpha\colon L \times B \to L$ denotes the first projection. As $L \in G$-ANR, according to the condition (R_1') there exist an index $\lambda \in \Lambda$ and a G-map $\phi\colon X_\lambda \to L$ such that

$$\phi r_\lambda = \alpha f. \tag{4.3}$$

Define the map $\Phi\colon X_\lambda \times B \to L \times B$ as the product $\phi \times id_B$ and put $U = \Phi^{-1}(V)$, $\mu = (\lambda, U)$. Then (3) implies $t_\lambda(E) \subseteq U$ and therefore $\mu \in M$. Now setting $h = \eta(\Phi|U)$ we obtain a G-map $h : E_\mu \to P$ such that $hq_\mu = f$ and (R_1) is satisfied.

For (R_2) consider an open covering ω of P. We claim that $\omega' = \omega$ has the desired property. Indeed, let $\mu = (\lambda, U) \in M$ and $h_0, h_1\colon U \to P$ be G-maps over B such that $h_0 q_\mu$ and $h_1 q_\mu$ are ω-near G-maps. For each $x \in E$ there is a $W_x \in \omega$ such that $h_0 q_\mu(x),\ h_1 q_\mu(x) \in W_x$. By continuity of h_0 and h_1 there is an open neighborhood O_x of $q_\mu(x)$ in U such that $h_0(y)$, $h_1(y) \in W_x$ for all $y \in O_x$. Then $O = \bigcup_{x\in E} O_x$ is an open neighborhood of $q_\mu(E)$ in $E_\mu = U$ and the maps $h_0|O$ and $h_1|O$ are ω-near. Since G is compact and $q_\mu(E)$ is an invariant subset of $U \cap O$ one can find an open invariant neighborhood U' of $q_\mu(E)$ in $U \cap O$ [24, Proposition 1.1.14]. We now put $\mu' = (\lambda, U') \in M$. Note that $\mu \leq \mu'$ because $s_{\lambda\lambda}(U') = U' \subseteq U$. The maps $h_0 q_{\mu\mu'} = h_0|U'$ and $h_1 q_{\mu\mu'} = h_1|U'$ are indeed ω-near, because $U' \subseteq O$. This verifies condition (R_2).

The notion of a G-ANR_B-expansion is obtained by specializing the general categorical notion of expansion with respect to a category $\mathcal{T}$ and its subcategory $\mathcal{P}$ [22, Ch. I, §2]. In our case $\mathcal{T} = [G\text{-}TOP_B]$ and $\mathcal{P} = [G\text{-}ANR_B]$. So we have the following

DEFINITION 4.3. *A G-expansion over B or a fiberwise G-expansion of a G-space E over B consists of an inverse system* $[\underline{E}] = (E_\lambda, [p_{\lambda\lambda'}], \Lambda)$ *in* $[G\text{-}TOP_B]$ *and a morphism* $[\underline{p}]\colon E \to [\underline{E}]$ *in* $pro-[G\text{-}TOP_B]$*, i.e., a collection of fiberwise G-homotopy classes* $[p_\lambda]$ *of G-maps* $p_\lambda\colon E \to E_\lambda$, $\lambda \in \Lambda$ *over B such that* $p_{\lambda\lambda'}p_{\lambda'} \simeq_G p_\lambda$ *over B for* $\lambda \leq \lambda'$*, satisfying the following two conditions:*

(E_1) *If P is a G-ANR_B and* $f\colon E \to P$ *is a G-map over B then there exist a* $\lambda \in \Lambda$ *and a G-map* $h\colon E_\lambda \to P$ *over B such that* $hp_\lambda \simeq_G f$ *over B.*

(E_2) *If P is a G-ANR_B,* $\lambda \in \Lambda$ *and* $h_0, h_1\colon E_\lambda \to P$ *are G-maps over B satisfying* $h_0p_\lambda \simeq_G h_1p_\lambda$ *over B, then there is a* $\lambda' \geq \lambda$ *such that* $h_0p_{\lambda\lambda'} \simeq_G h_1p_{\lambda\lambda'}$ *over B.*

A G-ANR_B-expansion $[\underline{p}]$ *is a G-expansion such that each* E_λ *has the fiberwise G-homotopy type of some G-ANR_B.*

Clearly every inverse system $\underline{E} = (E_\lambda, p_{\lambda\lambda'}, \Lambda)$ in the category $G\text{-}TOP_B$ induces an inverse system $[\underline{E}] = (E_\lambda, [p_{\lambda\lambda'}], \Lambda)$ in the category $[G\text{-}TOP_B]$. Moreover, every morphism $\underline{p} = (p_\lambda, \Lambda)\colon E \to \underline{E}$ in $pro-G\text{-}TOP_B$ induces a morphism $[\underline{p}] = ([p_\lambda], \Lambda)\colon E \to [\underline{E}]$ in $pro-[G\text{-}TOP_B]$.

In our development of equivariant fiberwise shape the next result is important.

THEOREM 4.4. *Let E be a G-space over B. If* $\underline{p}\colon E \to \underline{E}$ *is a G-resolution over B, then the induced morphism* $[\underline{p}]\colon E \to [\underline{E}]$ *is a G-expansion over B.*

The proof of this theorem proceeds in the same way as the proof of the analogous result in the case of ordinary shape (see the proof of [22, Ch. I §6.1, Theorem 2]) and of the equivariant one (see [7, Theorem 2]). However in the equivariant fiberwise case some specific difficulties arise. Below we show how to come over this difficulties. To be more rigorous and more complete, we repeat in our proof some of the arguments stated in the proof of [22, Ch. I, §6.1, Theorem 2].

We need the following three lemmas which are fiberwise analogues of [7, Proposition 3, Proposition 4 and Lemma 5].

First we recall that if ω is a covering of a space Y, then two maps $f, f'\colon X \to Y$ are said to be ω-near provided every $x \in X$ admits a $V \in \omega$ such that $f(x), f'(x) \in V$. For a homotopy $F : X \times I \to Y$ we say that it is an ω-homotopy provided every $x \in X$ admits a $V \in \omega$ such that $F(x \times I) \subseteq V$.

LEMMA 4.5. *Let Y be a G-ANR_B. Then every open covering* $\mathcal{U}$ *of Y admits an open covering* $\mathcal{U}'$ *of Y such that whenever* $f_0, f_1 : X \to Y$ *are* $\mathcal{U}'$*-near G-maps over B from an arbitrary* $X \in G\text{-}TOP_B$*, then there exists* *an equivariant $\mathcal{U}$-homotopy F over B from f_0 to f_1. Moreover, if for a given $x \in X$, $f_0(x) = f_1(x)$ then $F|x \times I$ is constant.*

Proof. By [2, Corollary 5] one can assume that Y is a closed invariant subspace of a normed linear G-space L. If $p : Y \to B$ is the projection then the map $h : Y \to L \times B$ defined by $h(y) = (y, p(y))$ is a closed G-embedding over B. As $Y \in G\text{-}ANR_B$ and $L \times B$ is metric, there exists an open invariant neighborhood D of $h(Y)$ in $L \times B$ and an equivariant retraction $r : D \to h(Y)$ over B. Let $\mathcal{V}$ be an open covering of D which refines $r^{-1}(h(\mathcal{U}))$ and consists of sets of the form $V_i \times W_j$ where V_i is an open ball from L and W_j is an open set from B. Put $\mathcal{V}'=\{(V_i \times W_j) \cap h(Y) :\ V_i \times W_j \in \mathcal{V}\}$. We claim that $\mathcal{U}' = h^{-1}(\mathcal{V}')$ has the desired property.

Let $f_0, f_1 : X \to Y$ be $\mathcal{U}'$-near G-maps over B from an arbitrary $X \in G\text{-}TOP_B$ with projection $l : X \to B$. We define a G-homotopy $\Phi : X \times I \to L$ from f_0 to f_1 by putting

$$\Phi(x,t) = (1-t)f_0(x) + tf_1(x), \qquad (x,t) \in X \times I. \tag{4.4}$$

As for every $x \in X$ there is an element $h^{-1}\big((V_i \times W_j) \times h(Y)\big) \in \mathcal{U}'$ which contains both $f_0(x)$ and $f_1(x)$, we see that V_i contains both $f_0(x)$ and $f_1(x)$. Since V_i is convex, we conclude that $\Phi(x \times I) \subseteq V_i$. As $l(x) = p(f_0(x))$ is the second coordinate of $hf_0(x)$, it then follows that W_j contains $l(x)$, which implies that $(\Phi(x,t), l(x)) \in V_i \times W_j \subseteq D$ for all $t \in I$. However, $V_i \times W_j$ is contained in a set $r^{-1}(h(U))$ where $U \in \mathcal{U}$. Therefore the map $F : X \times I \to Y$ defined by $F(x,t) = h^{-1}r\big(\Phi(x,t), l(x)\big)$ is a well-defined equivariant $\mathcal{U}$-homotopy over B from f_0 to f_1. Moreover, if $f_0(x) = f_1(x)$ then $\Phi|x \times I$ is constant and so is $F|x \times I$. F is equivariant because f_0, f_1 and r are equivariant and G acts linearly on L.

Lemma 4.6. *Let X be a metric G-space over B, let $A \subseteq X$ be a closed invariant subset and let Y be a $G\text{-}ANR_B$. Moreover, let f_0, $f_1 : X \to Y$ be equivariant maps over B and let $F : A \times I \to Y$ be an equivariant homotopy over B from $f_0|A$ to $f_1|A$. Then there exist an invariant neighborhood V of A in X and an equivariant homotopy $\tilde{F} : V \times I \to Y$ over B from $f_0|V$ to $f_1|V$, which extends F.*

Proof. Straightforward.

Lemma 4.7. *Let $X \in G\text{-}TOP_B$, let $P, P' \in G\text{-}ANR_B$ and let $f : X \to P'$, h_0, $h_1 : P' \to P$ be G-maps over B such that*

$$h_0 f \simeq_G h_1 f \quad \textit{over} \quad B. \tag{4.5}$$

Then there exist a $P'' \in G\text{-}ANR_B$ and G-maps $f' : X \to P''$, $h : P'' \to P'$ over B such that

$$hf' = f, \tag{4.6}$$

$$h_0 h \simeq_G h_1 h \quad \textit{over} \quad B. \tag{4.7}$$

Proof. By (4.5) there exists an equivariant homotopy $Q : X \times I \to P$ over B from $h_0 f$ to $h_1 f$. Let $C(I, P)$ be the G-space of all continuous maps $\varphi : I \to P$ endowed with the compact-open topology and the G-action defined by $(g\varphi)(t) = g(\varphi(t))$, $g \in G$, $\varphi \in C(I, P)$, $t \in I$. Consider the invariant subspace $F(I, P)$ of $C(I, P)$ consisting of all those maps $\varphi : I \to P$ for which the composition $p\varphi$ is a constant map, where $p : P \to B$ is the projection. In what follows we will consider $F(I, P)$ as a G-space over B equipped with the projection $F_p : F(I, P) \to B$ defined by the formula $F_p(\varphi) = p\varphi(0)$ (observe that $p\varphi(0) = p\varphi(t)$ for all $t \in I$).

Let $q : X \to F(I, P)$ be the G-map over B defined by $q(x)(t) = Q(x, t)$, $x \in X$, $t \in I$. We now define $P'' \subseteq P' \times F(I, P)$ by

$$P'' = \{(y, \varphi) \in P' \times F(I, P) |\ \varphi(0) = h_0(y),\ \varphi(1) = h_1(y)\}.$$

As in the proof of [7, Lemma 5], it can be checked that P'' is an invariant subset of $P' \times F(I, P)$ and that $f'(x) = (f(x), q(x))$ defines a G-map $f' : X \to P''$. Moreover, f' also becomes a G-map over B if we define the projection $l : P'' \to B$ by $l(y, \varphi) = p'(y)$, where $p' : P' \to B$ is the projection of P' (this follows easily from the fact that f is a G-map over B). Let $h : P'' \to P'$ be the first cartesian projection. Then h is a G-map over B and (6) holds.

In order to verify (7) consider the G-homotopy $H : P'' \times I \to P$ given by

$$H\big((y, \varphi), t\big) = \varphi(t), \quad (y, \varphi) \in P'' \times I,\ t \in I.$$

In the proof of [7, Lemma 5] it was shown that H is a G-homotopy from $h_0 h$ to $h_1 h$. Let us show that H is also a homotopy over B.

Indeed, $pH\big((y, \varphi), t\big) = p\varphi(t)$. By definition of $F(I, P)$ we have $p\varphi(t) = p\varphi(0)$ for all $t \in I$ and $\varphi(0) = h_0(y)$, implying $pH\big((y, \varphi), t\big) = ph_0(y)$. As h_0 is a map over B, we conclude that $ph_0(y) = l(y)$, and hence $pH\big((y, \varphi), t\big) = l(y)$. This verifies (7).

The proof of Lemma 4.7 will be completed if we show that P'' is a G-ANR_B or equivalently a G-ANE_B.

Let Z be a metric G-space over B, let $A \subseteq Z$ be a closed invariant subset and let $k : A \to P''$ be an equivariant map over B. We shall find an invariant neighborhood V of A in Z and an equivariant extension $\tilde{k} : V \to P''$ of k over B. Denote by $h' : P'' \to F(I, P)$ the second cartesian projection, which clearly is equivariant. Verify that h' is also a map over B. Indeed, $F_p h'(y, \varphi) = F_p(\varphi) = p\varphi(0) = ph_0(y) = l(y)$. Therefore $h'k : A \to F(I, P)$ is a G-map over B and it induces an equivariant homotopy $K : A \times I \to P$ over B defined by

$$K(a, t) = \big(h'k(a)\big)(t), \quad (a, t) \in A \times I.$$

In [7, p. 221] it is shown that K is a G-homotopy from $h_0 hk$ to $h_1 hk$.

Since P' is a G-ANE_B and $hk : A \to P'$ is a G-map over B, there exist an invariant neighborhood U of A in Z and an equivariant map $\tilde{k}' : U \to P'$ over B, which extends hk. One can now apply Lemma 4.6 to $h_0\tilde{k}'$, $h_1\tilde{k}'$ and K and conclude that there exist an equivariant neighborhood V of A in U and a G-homotopy $\tilde{K} : V \times I \to P$ over B from $h_0\tilde{k}'|V$ to $h_1\tilde{k}'|V$.

Consider the G-map $\tilde{k}'' : V \to F(I, P)$ given by $\tilde{k}''(z)(t) = \tilde{K}(z, t)$. As $\tilde{K}$ is a homotopy over B, we see that $\tilde{k}''(z)$ really belongs to $F(I, P)$ and $\tilde{k}''$ is a map over B. It is also continuous, equivariant and extends $h'k$ [7, the proof of Lemma 5]. Now we define

$$\tilde{k} : V \to P' \times F(I, P) \qquad \text{by} \qquad \tilde{k}(z) = (\tilde{k}'(z), \tilde{k}''(z)), \quad z \in V,$$

which clearly is equivariant and extends k. As $\tilde{k}'$ and $\tilde{k}''$ both are maps over B, we conclude that $\tilde{k}$ is a map over B too. Finally $\tilde{k}(z) \in P''$ for every $z \in V$ because

$$\tilde{k}''(z)(0) = \tilde{K}(z, 0) = h_0\tilde{k}'(z)$$

and

$$\tilde{k}''(z)(1) = \tilde{K}(z, 1) = h_1\tilde{k}'(z).$$

This completes the proof of Lemma 4.7.

Proof of Theorem 4.4. We must verify conditions (E_1) and (E_2) of Definition 4.3

(E_1). Let $P \in G$-ANR_B and let $f : E \to P$ be a G-map over B. By Lemma 4.5 we can choose an open covering ω of P such that any two ω-near G-maps over B into P are G-omotopic over B. By property (R_1) there are a $\lambda \in \Lambda$ and a G-map $h : E_\lambda \to P$ over B, such that hp_λ and f are ω-near maps and therefore $hp_\lambda \simeq_G f$ over B.

(E_2). Let $P \in G$-ANR_B, $\lambda \in \Lambda$ and let h_0, $h_1 : E_\lambda \to P$ be G-maps over B satisfying

$$h_0 p_\lambda \simeq_G h_1 p_\lambda \quad \text{over} \quad B. \tag{4.8}$$

We must find a $\lambda' \geq \lambda$ such that

$$h_0 p_{\lambda\lambda'} \simeq_G h_1 p_{\lambda\lambda'} \quad \text{over} \quad B. \tag{4.9}$$

Again by Lemma 4.5 we can choose an open covering ω of P such that any two ω-near G-maps over B into P are G-homotopic over B. Choose ω' according to (R_2). Consider the fiber product (pull-back) $P \times_B P \subseteq P \times P$ which naturally becomes a G-space over B. The maps $h_0 p_\lambda$ and $h_1 p_\lambda$ determine a G-map $f : E \to P' = P \times_B P$ over B such that

$$g_0 f = h_0 p_\lambda \qquad \text{and} \qquad g_1 f = h_1 p_\lambda, \tag{4.10}$$

where $g_0, g_1 : P \times_B P \to P$ are the projections (which are G-maps over B) on the first and second factor respectively. By (8) and (10) we have

$$g_0 f \simeq_G g_1 f \quad \text{over} \quad B. \tag{4.11}$$

Using the property $P \in G\text{-}ANR_B$, it can be easily proved (see [18, p. 240]) that then $P' = P \times_B P$ is a $G\text{-}ANR_B$. By Lemma 4.7 there is a $P'' \in G\text{-}ANR_B$ and there are G-maps over B, $f' : E \to P''$ and $h : P'' \to P'$ such that

$$hf' = f \qquad \text{and} \qquad g_0 h \simeq_G g_1 h \quad \text{over} \quad B. \tag{4.12}$$

Let ω'' be an open covering of P'' which refines the coverings $(g_0 h)^{-1}(\omega')$ and $(g_1 h)^{-1}(\omega')$. Applying the property (R_1) we find a $\lambda'' \in \Lambda$ and a G-map $\psi : E_{\lambda''} \to P''$ over B such that $\psi p_{\lambda''}$ and f' are ω''-near. Clearly, one can assume that $\lambda'' \geq \lambda$.

Consequently, $g_0 h \psi p_{\lambda''}$ and $g_0 h f'$ are ω'-near maps. However, by (10) and (12) we have

$$g_0 h f' = g_0 f = h_0 p_\lambda, \tag{4.13}$$

so that $g_0 h \psi p_{\lambda''}$ and $h_0 p_{\lambda\lambda''} p_{\lambda''}$ are ω'-near maps. Therefore, by (R_1) there is an index $\lambda_0 \geq \lambda''$ such that the maps $g_0 h \psi p_{\lambda'' \lambda_0}$ and $h_0 p_{\lambda\lambda_0}$ are ω-near and thus

$$g_0 h \psi p_{\lambda''\lambda_0} \simeq_G h_0 p_{\lambda\lambda_0} \quad \text{over} \quad B. \tag{4.14}$$

Similarly there is an index $\lambda_1 \geq \lambda''$ such that

$$g_1 h \psi p_{\lambda''\lambda_1} \simeq_G h_1 p_{\lambda\lambda_1} \quad \text{over} \quad B. \tag{4.15}$$

Now let $\lambda' \geq \lambda_0, \lambda_1$. Then we have by (14), (12) and (15)

$$h_0 p_{\lambda\lambda'} \simeq_G g_0 h \psi p_{\lambda''\lambda'} \simeq_G g_1 h \psi p_{\lambda''\lambda'} \simeq_G h_1 p_{\lambda\lambda'} \quad \text{over} \quad B,$$

which is the desired homotopy (9).

Theorem 4.2 and Theorem 4.4 immediately imply the following

COROLLARY 4.8. *Every G-space E over B admits a $G\text{-}ANR_B$-expansion, i.e., $[G\text{-}ANR_B]$ is a dense subcategory of $[G\text{-}Top_B]$.*

We will now define the G-shape category over B, denoted by $G\text{-}SH_B$, as a shape category $SH(\mathcal{T}, \mathcal{P})$ with $\mathcal{T} = [G\text{-}TOP_B]$, $\mathcal{P} = [G\text{-}ANR_B]$. One also has a G-shape functor over B, $G\text{-}Sh_B \colon [G\text{-}TOP_B] \to G\text{-}SH_B$. When B is a one-point G-space, clearly $G\text{-}SH_B$ is naturally isomorphic to $G\text{-}SH$ the G-shape category constructed in [7] (see also [3]). In the case of G the trivial group and E a metric spaces over a given metric base B, the fiberwise shape category SH_B was considered by T. Yagasaki [26]. Using the method of resolutions, V.H. Baladze [8] later constructed a fiberwise shape category for arbitrary spaces over a given base B. Fiberwise shape of compact metric spaces over a compact metric base has been previously considered by H. Kato [19] and by M. Clapp and L. Montejano [10].

THEOREM 4.9. *Let* $G\text{-}Sh_B\colon G\text{-}TOP_B \to G\text{-}SH_B$ *denotes the* G*-shape functor over the base* B *and let* $\pi\colon G\text{-}TOP_B \to G/N\text{-}TOP_{B/N}$ *denotes the* N*-orbit functor for any closed normal subgroup* $N \subseteq G$. *Suppose that* B *is a* G*-ANE over* B/N. *Then there is a unique functor* $\mu\colon G\text{-}SH_B \to G/N\text{-}SH_{B/N}$ *such that* $\mu \circ G\text{-}Sh_B = G/N\text{-}Sh_{B/N} \circ \pi$.

For the proof we need the following propositions:

PROPOSITION 4.10. *Let* $N \subseteq G$ *be a closed normal subgroup and let* $[\underline{p}] = ([p_\lambda]) : E \to \underline{E} = \{E_\lambda, [p_{\lambda\lambda'}], \Lambda\}$ *be a fiberwise* G*-expansion of* E *over* B. *Then using the notations of* §2, $[\underline{p}/N] = ([p_\lambda/N]) : E/N \to \underline{E}/N = (E_\lambda/N, [p_{\lambda\lambda'}/N], \Lambda)$ *is a fiberwise* G/N*-expansion of* E/N *over* B/N.

PROOF. In this proof we will shortly denote by φ' the map $\pi(\varphi) = \varphi/N$ induced by a G-map φ (see §2).

We must check the conditions (E_1) and (E_2).

(E_1): For let $\overline{h}\colon E/N \to P$ be a G/N-map over B/N where P is a G/N-$ANR_{B/N}$. Let $r\colon E \to B$, $r'\colon E/N \to B/N$, $\alpha\colon P \to B/N$, $\beta_E\colon E \to E/N$ and $\beta_B\colon B \to B/N$ be the involving maps and projections.

We have $r'\beta_E = \beta_B r$ and $\alpha\overline{h} = r'$. Now consider $Q \subseteq P \times B$ the pull-back of maps $\alpha\colon P \to B/N$ and $\beta_B\colon B \to B/N$. If $\kappa_P\colon Q \to P$ and $\kappa_B\colon Q \to B$ are the natural projections, then $\alpha\kappa_P = \beta_B\kappa_B$. If we consider the G/N-spaces P and B/N as G-spaces then the G/N-map α can be regarded as a G-map (see Remark 2.1), and therefore Q naturally becomes a G-space over B with the projection κ_B. It is well-known [18, p. 240] and easy to prove that Q is a G-ANR_B. Define the map $h\colon E \to Q$ by putting $h(x) = (\overline{h}\beta_E(x), r(x))$. One easily verifies that h is a well-defined G-map over B. Since $[\underline{p}]$ is a G-expansion over B, there exist $\lambda \in \Lambda$ and a G-map $f\colon E_\lambda \to Q$ over B such that $fp_\lambda \simeq_G h$ over B. Then f induces a G/N-map $f'\colon E_\lambda/N \to Q/N$ over B/N. Denote by $\xi_\lambda\colon E_\lambda \to E_\lambda/N$ the N-orbit projection, then $\kappa_P f = f'\xi_\lambda$. Clearly, the property $fp_\lambda \simeq_G h$ over B implies $f'p'_\lambda \simeq_{G/N} \overline{h}$ over B/N. As $Q/N = P$ (see [15, §4.1]) we conclude that (E_1) holds.

(E_2): Let $\overline{f}_0, \overline{f}_1\colon E_\lambda/N \to P$ be G/N-maps over B/N such that

$$\overline{f}_0 p'_\lambda \simeq_{G/N} \overline{f}_1 p'_\lambda \qquad \text{over} \qquad B/N, \tag{4.16}$$

where P is an arbitrary G/N-$ANR_{B/N}$ with the projection $\alpha : P \to B/N$.

Now let Q be the pull-back of maps $\alpha\colon P \to B/N$ and $\beta_B\colon B \to B/N$, as before Q is a G-ANR_B. For $i = 0, 1$ define the G-map $f_i\colon E_\lambda \to Q$ by $f_i(x) = (\overline{f_i}\xi_\lambda(x), s_\lambda(x))$, $x \in E_\lambda$, where $s_\lambda\colon E_\lambda \to B$ is the projection. One easily verifies that f_i is a G-map over B. We claim that $f_0 p_\lambda \simeq_G f_1 p_\lambda$ over B. Indeed, let $F_t\colon E/N \to P$, $0 \le t \le 1$ be a G/N-homotopy over B/N from $\overline{f}_0 p'_\lambda$ to $\overline{f}_1 p'_\lambda$ (see [16]). This G-homotopy can be lifted to a G-homotopy Φ_t :

$E \to Q$ over B by putting $\Phi_t(y) = (F_t\beta_E(y), r(y))$ for all $y \in E$. Let us check that $\Phi_i = f_i p_\lambda$, $i = 0, 1$. In fact $\Phi_i(y) = (F_i\beta_E(y), r(y)) = (\overline{f_i}p'_\lambda\beta_E(y), r(y))$. Since $p'_\lambda\beta_E = \xi_\lambda p_\lambda$ and $r = s_\lambda p_\lambda$, we have $\Phi_i(y) = (\overline{f_i}\xi_\lambda p_\lambda(y), s_\lambda p_\lambda(y)) = f_i p_\lambda(y)$, i.e., $\Phi_i = f_i p_\lambda$.

Hence one can apply the property (E_2) to $\underline{p}\colon E \to \underline{E}$. Since Q is a G-ANR_B there exists an index $\mu \geq \lambda$ such that

$$f_0 p_{\lambda\mu} \simeq_G f_1 p_{\lambda\mu} \qquad \text{over} \qquad B.$$

This G-homotopy induces a G/N-homotopy $\overline{f_0}p'_{\lambda\mu} \simeq_{G/N} \overline{f_1}p'_{\lambda\mu}$ over B/N (note that $Q/N = P$). This verifies (E_2) and completes the proof.

In a similar way one can prove the following

Proposition 4.11. *Let $N \subseteq G$ be a closed normal subgroup and let $\underline{p} = (p_\lambda)\colon E \to \underline{E} = (E_\lambda, p_{\lambda\lambda'}, \Lambda)$ be a fiberwise G-resolution of E over B. Then with the notations of* §2, *$(\underline{p}/N) = (p_\lambda/N)\colon E/N \to \underline{E}/N = (E_\lambda/N, p_{\lambda\lambda'}/N, \Lambda)$ is a fiberwise G/N-resolution of E/N over B/N.*

Theorem 3.1 and Proposition 4.10 immediately imply the following

Proposition 4.12. *Let $N \subseteq G$ be a closed normal subgroup and let $[\underline{p}]\colon E \to [\underline{E}] = (E_\lambda, [p_{\lambda\lambda'}], \Lambda)$ be a fiberwise G-ANR_B-expansion of E over B. Suppose that B is a G-ANE over B/N. Then $[\underline{p}/N] = ([p_\lambda/N]) : E/N \to \underline{E}/N = (E_\lambda/N, [p_{\lambda\lambda'}/N], \Lambda)$ is a G/N-$ANR_{B/N}$-expansion of E/N over B/N.*

Proof of Theorem 4.9. According to [22, Ch. I, §2.3] the objects of G-SH_B are G-spaces over B and the morphisms of G-SH_B between G-spaces E and F over B are given by triples $([\underline{p}], [\underline{q}], [\underline{f}])$ where $[\underline{p}] : E \to [\underline{E}]$ and $[\underline{q}]\colon F \to [\underline{F}] = (F_\lambda, [q_{\lambda\lambda'}], A)$ are G-ANR_B-expansions of E and F respectively, and $[\underline{f}] : [\underline{E}] \to [\underline{F}]$ is a morphism of $pro - [G\text{-}TOP_B]$ (see [22, Ch. I, §1.1]). In particular, one can take for $[\underline{p}]$ and $[\underline{q}]$ morphisms induced by G-ANR_B-resolutions $\underline{p}$ and $\underline{q}$ (Theorems 4.2 and 4.4). According to Proposition 4.12, $[\underline{p}/N] : E/N \to \underline{E}/N$ and $[\underline{q}/N] : F/N \to \underline{F}/N$ are G-$ANR_{B/N}$-expansions.

Let $[\underline{f}] = (\phi, [f_\alpha])$ where $\phi : A \to \Lambda$ is a function and $f_\alpha\colon E_{\phi(\alpha)} \to F_\alpha$ is a G-map. Clearly $[\underline{f}/N] = (\phi, [f_\alpha/N])$ is a morphism of $pro - [G/N\text{-}TOP_{B/N}]$. Now we define $\mu : G\text{-}SH_B \to G/N\text{-}SH_{B/N}$ by putting $\mu(E) = E/N$ for objects of G-SH_B and $\mu([\underline{p}], [\underline{q}], [\underline{f}]) = ([\underline{p}/N], [\underline{q}/N], [\underline{f}/N])$ for morphisms of G-SH_B. One easily verifies (by virtue of Proposition 4.12) that μ is the desired functor. The uniqueness of μ is also easy to check.

Theorem 4.9 has the following immediate corollaries:

Corollary 4.13. *Let $N \subseteq G$ be a closed normal subgroup and E, F be G-spaces over B with $G\text{-}Sh_B(E) = G\text{-}Sh_B(F)$. Suppose that B is a G-ANE over B/N. Then we have $G/N\text{-}Sh_{B/N}(E/N) = G/N\text{-}Sh_{B/N}(F/N)$.*

In particular $Sh_{B/G}(E/G) = Sh_{B/G}(F/G)$ *whenever* B *is a* G-ANE *over* B/G.

Corollary 4.14. *Let* $N \subseteq G$ *be a closed normal subgroup,* E *and* F *be any* G*-spaces. If* G-$Sh(E) = G$-$Sh(F)$ *then* G/N-$Sh(E/N) = G/N$-$Sh(F/N)$*. In particular* $Sh(E/G) = Sh(F/G)$.

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