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# SOME SYMMETRIC $(71, 15, 3)$ DESIGNS WITH AN INVOLUTORY ELATION

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Abstract. Symmetric designs for $(71, 15, 3)$ with the semi-standard automorphism group $G \cong E_8 \cdot F_{21}$ have been investigated. There were constructed exactly three nonisomorphic designs, two of them with an involutory elation.

## 1. Introduction

In [1], we have proved that exactly six nonisomorphic triplanes for $(71, 15, 3)$ exists having the automorphism group $G \cong E_8 \cdot F_{21}$ acting non-semistandardly. The group $G$ is a faithful extension of an elementary abelian group $E_8$ of order 8 by a Frobenius group of order 21. Here we shall confine our attention to the semi-standard automorphism group situation. We use the method from [2] and this group $G$ to "construct" the triplanes for $(71, 15, 3)$ under this assumption. We have found that there exist triplanes with these parameters nonisomorphic to the triplanes in [1], admitting an elation of order 2 and also another class of triplanes without that property. Hence, the property of admitting an involutory elation yields a new approach to the construction and the classification of symmetric design for $(71, 15, 3)$.

## 2. Notation and basic definitions

Suppose $G$ is an authomorphism group of a symmetric $D = (v, k, \lambda)$ design. Actually, $G$ has orbits of lengths $a_1, a_2, \ldots, a_c$ on points and orbits of lengths $b_1, b_2, \ldots, b_c$ on blocks of $D$ respectively.

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DEFINITION 1. *An automorphism group $G$ is said to be semi-standard if, after possibly renumbering the orbits we have $a_i \leq a_j$ and $b_i \leq b_j$ for $i \leq j$ and if $a_i = b_i$ for all $i = 1, 2, \ldots, c$.*

We also say that a group $G$ acts semistandardly. In the oposite case, we say $G$ acts non-semistandardly.

DEFINITION 2. *Let $\alpha \neq 1$ be an automorphism on a symmetric $2-(v,k,\lambda)$ design. We say that $\alpha$ is an elation if there exists an $\alpha$-fixed block $b$ containing exactly $k$ $\alpha$-fixed points and if $\alpha$ fixes a set of $k$ blocks having exactly one $\alpha$-fixed point of $b$ in common.*

Notation and other definitions can be found in [1].

## 3. Results of the investigation in the semi-standard group situation

Now, suppose the group $G \cong E_8 \cdot F_{21}$ is an automorphism group of a symmetric $2-(71,15,3)$ design. This group $G$ is generated by the set of three elements of order two, three and seven respectively. (For details of the action of generators see [1].) It was proved in [1] that an involution had seven or fifteen points, an automorphism of order three fixed five points and an automorphism of order seven fixed exactly one point. In the case where $G$ acts semistandardly it was proved that $G$ had three orbits on the points(blocks) having lengths $1, 14, 56$. Here we have constructed symmetric designs $D_5$, $D_6$ and $D_7$, all of them nonisomorphic. (See the method in [2].) Without loss of generality the constructed designs have the following block orbit representatives:

The first block orbit representative of $D_5$, $D_6$ and $D_7$:

$$1_1\ 2_1\ 2_2\ 2_3\ 2_4\ 2_5\ 2_6\ 2_7\ 2_8\ 2_9\ 2_{10}\ 2_{11}\ 2_{12}\ 2_{13}\ 2_{14}$$

The second block orbit representative of $D_5$, $D_6$ and $D_7$:

$$1_1\ 2_1\ 2_2\ 3_8\ 3_9\ 3_{11}\ 3_{18}\ 3_{19}\ 3_{21}\ 3_{25}\ 3_{33}\ 3_{38}\ 3_{45}\ 3_{51}\ 3_{53}$$

The third block orbit representatives for

$$D_5:\quad 2_5\ 2_7\ 2_8\ 3_8\ 3_9\ 3_{11}\ 3_{15}\ 3_{16}\ 3_{26}\ 3_{27}\ 3_{30}\ 3_{35}\ 3_{48}\ 3_{52}\ 3_{55}$$

$$D_6:\quad 2_5\ 2_7\ 2_8\ 3_8\ 3_9\ 3_{11}\ 3_{15}\ 3_{27}\ 3_{28}\ 3_{32}\ 3_{37}\ 3_{42}\ 3_{46}\ 3_{49}\ 3_{52}$$

$$D_7:\ 2_5\ 2_7\ 2_8\ 3_{10}\ 3_{12}\ 3_{13}\ 3_{16}\ 3_{25}\ 3_{26}\ 3_{30}\ 3_{33}\ 3_{35}\ 3_{48}\ 3_{51}\ 3_{55}$$

Here the intersection numbers of all triples of blocks for $D_5$,$D_6$ and $D_7$ have been calculated but the constructed designs $D_6$ and $D_7$ do not differ in this characteristic. Hence we have used the method of spreads (see [1]) to prove the nonisomorphism of designs $D_5$, $D_6$ and $D_7$. We proved that all of them are mutually nonisomorphic as well as nonisomorphic to the designs constructed in [1]. Compare the results of investigation in the *Table* 1 with the results in [1].

*Table* 1:

| $D_i$ | $b_*$ | $(r_1, r_2, r_3)$ | $(s_1, s_2, s_3)$ |
|---|---|---|---|
| $D_5$ | 1 | $(35, 63, 69)$ | $(56, 52, 100)$ |
| $D_6$ | 1 | $(35, 51, 63)$ | $(56, 160, 86)$ |
| $D_7$ | 1 | $(35, 51, 63)$ | $(56, 160, 70)$ |

Here, $b_*$ is the number of special blocks of a design $D_i$, $r_i$ means the number of blocks of a reduced structure $I_b/R$ where $b$ is an element of the $i$-th orbit of blocks of $D_i$ and $s_i$ means the number of spreads with respect to the block $b$ in the $i$-th orbit of blocks of $D_i$.

It was interesting to investigate the acting of involution $f = \rho^*$ fixing fifteen points of the constructed designs $D_5$ ,$D_6$ and $D_7$. (See the permutation $\rho^*$ in [1]). Here we have proved that $f$ was not an automorphism on $D_5$ but $f$ acted on $D_6$ and $D_7$ as an elation.

Now the first orbit of blocks of $D_6$ and $D_7$ contains just one $G$-fixed block containing exactly fifteen $f$-fixed points. The second orbit consists of fourteen blocks which all have the point $1_1$ in common. We find that the blocks incident with the point $1_1$ are all the $f$-fixed blocks. Here a pair of points of these blocks is either fixed or interchanged by $f$. The third orbit of blocks of $D_6$ and $D_7$ contains further fifty-six blocks of these designs. By verification it turns out that this permutation $f$ breaks up this orbit of blocks in twenty-eight pairs of blocks interchanged by $f$ for both $D_6$ and $D_7$. As one can see, the permutation $f$ is an elation on the symmetric designs $D_6$ and $D_7$ respectively. This automorphism $f$ is different from, and commutes with the automorphisms provided by the group $G$ and it was found under this assumption. We have seen that the permutation $f$ was not an automorphism on $D_1$, it acted as an automorphism on the symmetric designs $D_2$ and $D_3$ (see in [1]) but it was not an elation on designs $D_2$ and $D_3$.

We have investigated only the symmetric $(71, 15, 3)$ designs with the automorphism group $G \cong E_8 \cdot F_{21}$. We did not check whether it was possible to produce still more symmetric $(71, 15, 3)$ designs with an involutory elation in another case. Thus we have the following result:

THEOREM 3.1. *There exist at least two non-isomorphic symmetric* $(71, 15, 3)$ *designs admitting an involutory elation. Both of them are non-self-dual.*

## REFERENCES

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