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AN IMPROVED INEQUALITY FOR k-TH DERIVATIVE OF A POLYNOMIAL

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Abstract. For a polynomial $p(z)$ of degree n, we have obtained

$$|p^{(k)}(\beta)| \leq \frac{n(n-1)(n-2)....(n-k+1)}{|\beta|^k}\Big[\frac{l}{2^k}\{|p(\beta)| + \max_{1\leq i\leq n}|p(\beta z_i)|\}+ (1-\frac{l}{2^{k-1}})\max_{1\leq t\leq 2n}|p(\beta b_t)|\Big], \beta \neq 0 \ \&\ k \geq 1,$$

a refinement ot the well known Bernstein's inequatily

$$\max_{|z|=1}|p^{(k)}(z)| \leq n(n-1)(n-2)\cdots(n-k+1)\max_{|z|=1}|p(z)|,$$

$z_1, z_2....., z_n$ being the zeros of z^n+1 and $b_1, b_2, ..., b_{2n}$ the zeros of $z^{2n}-1$. The inequality is sharp.

1. Introduction and statement of results

Let $p(z)$ be a polynomial of degree n. It easily follows from the well known Bernstein's theorem [3] that

$$\max_{|z|=R}|p'(z)| \leq \frac{n}{R}\max_{|z|=R}|p(z)|, R > 0, \tag{1.1}$$

with equality in (1.1) for $p(z) = \alpha z^n$. On applying inequality (1.1) again and again, we get

$$\max_{|z|=R}|p^{(k)}(z)| \leq \frac{n(n-1)...(n-k+1)}{R^k}\max_{|z|=R}|p(z)|, \quad R > 0 \ \&\ k \geq 1, \tag{1.2}$$

with equality in (1.2) for $p(z) = \alpha z^n$.

We have been able to improve the inequality (1.2) and obtain a new inequality, which is sharp. More precisely, we prove

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THEOREM 1.1. *Let $p(z)$ be a polynomial of deqree n. Then*

$$|p^{(k)}(\beta)| \leq \frac{n(n-1)...(n-k+1)}{|\beta|^k}[\frac{1}{2^k}\{|p(\beta)| + \max_{1\leq i\leq n}|p(\beta z_i)|\}$$
$$+\left(1-\frac{1}{2^{k-1}}\right)\max_{1\leq t\leq 2n}|p(\beta b_t)|], \beta\neq 0,\ \&\ k\geq 1, \tag{1.3}$$

where $z_1, z_2, ..., z_n$ are the zeros of z^n+1 and $b_1, b_2, ..., b_{2n}$ are the zeros of $z^{2n}-1$. The inequality is sharp and the extremal polynomial is $p(z)=\alpha z^n$.

2. LEMMAS

For the proof of Theorem 1.1, we require the following lemmas.

LEMMA 2.1. *Let $p(z)$ be a polynomial of degree n and $z_1'', z_2'', ..., z_n''$ be the zeros of z^n+a, $a\neq 0$. Then for any complex number β such that $\beta^n+a\neq 0$, we have*

$$p'(\beta) = \frac{n\beta^{n-1}}{a+\beta^n}p(\beta) + \frac{a+\beta^n}{na}\sum_{i=1}^{n} p(z_i'')\frac{z_i''}{(z_i''-\beta)^2} \tag{2.4}$$

$$\frac{1}{na}\sum_{i=1}^{n}\frac{z_i''\beta}{(z_i''-\beta)^2} = -\frac{n\beta^n}{(\beta^n+a)^2}. \tag{2.5}$$

This lemma is due to Aziz [1].

LEMMA 2.2. *Let $p(z)$ be a polynomial of degree n. Then*

$$\max_{|z|=1}|p'(z)| \leq n\max_{1\leq t\leq 2n}|p(b_t)|, \tag{2.6}$$

where $b_1, b_2, \ldots, b_{2n}$ are as in Theorem 1.1.

This lemma is due to Frappier, Rahman and Ruscheweyh [2].

LEMMA 2.3. *Let $p(z)$ be a polynomial of degree n. Then for $s\geq 1$ and $|\beta|=1$,*

$$|p^{(s)}(\beta)| \leq \frac{n-s+1}{2}\{|p^{(s-1)}(\beta)| + \max_{1\leq m\leq n-s+1}|p^{(s-1)}(u_m')|\}, \tag{2.7}$$

where $u_1', u_2', ..., u_{n-s+1}'$ are the roots of

$$z^{n-s+1}+a=0 \tag{2.8}$$
$$a = e^{i\gamma(n-s+1)} \tag{2.9}$$

and

$$\gamma = \arg\beta$$

PROOF OF LEMMA 2.3. As

$$\beta^{n-s+1} + a \neq 0,$$

we have on applying Lemma 2.1 to the polynomial $p^{(s-1)}(z)$,

$$p^{(s)}(\beta) = \frac{(n-s+1)\beta^{n-s}}{\beta^{n-s+1}+a} p^{(s-1)}(\beta) + \frac{\beta^{n-s+1}+a}{(n-s+1)a} \sum_{m=1}^{n-s+1} p^{(s-1)}(u'_m) \frac{u'_m}{(u'_m-\beta)^2},$$

which implies

$$|p^{(s)}(\beta)| \leq \frac{n-s+1}{2} |p^{(s-1)}(\beta)| + \frac{2}{n-s+1} \{ \sum_{m=1}^{n-s+1} |\frac{u'_m}{(u'_m-\beta)^2}| \}. \tag{2.10}$$

$$\begin{aligned} \max_{1\leq m\leq n-s+1} |p^{(s-1)}(u'_m)| &= \frac{n-s+1}{2} |p^{(s-1)}(\beta)| \\ &+ \frac{2}{n-s+1} \{ \sum_{m=1}^{n-s+1} |\frac{u'_m\beta}{(u'_m-\beta)^2}| \} \max_{1\leq m\leq n-s+1} |p^{(s-1)}(u'_m)| \\ &= \frac{n-s+1}{2} |p^{(s-1)}(\beta)| + \frac{2}{n-s+1} \{ -\sum_{m=1}^{n-s+1} \frac{u'_m\beta}{(u'_m-\beta)^2} \}. \\ &\max_{1\leq m\leq n-s+1} |p^{(s-1)}(u'_m)|, \end{aligned}$$

as $\frac{u'_m\beta}{(u'_m-\beta)^2}$ is a negative real number for $m = 1, 2, ..., n-s+1$. Now by the second part of Lemma 2.1, (2.10) can be written as

$$\begin{aligned} |p^{(s)}(\beta)| \leq & \frac{n-s+1}{2} |p^{(s-1)}(\beta)| + \frac{2}{n-s+1} \{ \frac{(n-s+1)^2\beta^{n-s+1}a}{(\beta^{n-s+1}+a)^2} \}. \\ & \max_{1\leq m\leq n-s+1} |p^{(s-1)}(u'_m)| = \frac{n-s+1}{2} |p^{(s-1)}(\beta)| \\ & + \frac{2}{n-s+1}(n-s+1)^2 \frac{1}{4} \max_{1\leq m\leq n-s+1} |p^{(s-1)}(u'_m)| \end{aligned}$$

and this completes the proof of Lemma 2.3.

3. PROOF OF THEOREM 1.1

Let

$$T(z) = p(\beta z). \tag{3.11}$$

Then

$$|p^{(k)}(\beta)| = \frac{1}{|\beta|^k} |T^{(k)}(1)| \tag{3.12}$$

Now by Lemma 2.3, we have for $k \geq 2$.

$$|T^{(k)}(1)| \leq \frac{n-k+2}{2}\{|T^{(k-1)}(1)| + \max_{1\leq m\leq n-k+1}|T^{(k-1)}(u_m)|\} \tag{3.13}$$

where $u_1, u_2, ..., u_{n-k+1}$ are the roots of $z^{n-k+1}+1=0$. Again by Lemma 2.3, we have

$$|T^{(k-1)}(1) \leq \frac{n-k+2}{2}\{|T^{(k-2)}(1)| + \max_{1\leq j\leq n-k+2}|T^{(k-2)}(w_j)|\}, \tag{3.14}$$

where $w_1, w_2, ..., w_{n-k+2}$ are the roots of $z^{n-k+2}+1=0$. Combining (3.13) and (3.14), we obtain

$$\begin{aligned}|T^{(k)}(1)| \leq{}& \frac{n-k+1}{2}\frac{n-k+2}{2}|T^{(k-2)}(1)|+\\ & \frac{n-k+1}{2}\frac{n-k+2}{2}\max_{1\leq j\leq n-k+2}|T^{(k-2)}(w_j)|+\\ & \frac{n-k+1}{2}\max_{1\leq m\leq n-k+1}|T^{(k-1)}(u_m)|.\end{aligned}$$

Continuing similarly, we obtain for $k \geq 2$

$$\begin{aligned}|T^{(k)}(1)| \leq{}& \frac{n-k+1}{2}\frac{n-k+2}{2}\cdots\frac{n}{2}|T(1)|\\ &+\frac{n-k+1}{2}\frac{n-k+2}{2}\cdots\frac{n}{2}\max_{1\leq i\leq n}|T(z_i)|\\ &+\frac{n-k+1}{2}\frac{n-k+2}{2}\cdots\frac{n-1}{2}\max_{1\leq h\leq n-1}|T'(x_h)|\\ &+\frac{n-k+1}{2}\frac{n-k+2}{2}\cdots\frac{n-2}{2}\max_{1\leq g\leq n-2}|T''(d_g)|+\cdots\\ &\cdots+\frac{n-k+1}{2}\frac{n-k+2}{2}\max_{1\leq j\leq n-k+2}|T^{(k-2)}(w_j)|\\ &+\frac{n-k+1}{2}\max_{1\leq m\leq n-k+1}|T^{(k-1)}(u_m)|,\end{aligned} \tag{3.15}$$

where $z_1, z_2, ..., z_n$ are the roots of $z^n+1=0$, $x_1, x_2, ..., x_{n-1}$ are the roots of $z^{n-1}+1=0$, $d_1, d_2, ..., d_{n-2}$ are the roots of $z^{n-2}+1=0$, and so on.

Now, by Lemma 2.2 and Bernstein's theorem [3], we have from (3.15), for $k \geq 2$

$$\begin{aligned}|T^{(k)}(1)| \leq{}& \frac{n-k+1}{2}\frac{n-k+2}{2}\cdots\frac{n}{2}|T(1)|\\ &+\frac{n-k+1}{2}\frac{n-k+2}{2}\cdots\frac{n}{2}\max_{1\leq i\leq n}|T(z_i)|\end{aligned}$$

$$
\begin{aligned}
&+\frac{n-k+1}{2}\frac{n-k+2}{2}\cdots\frac{n-1}{2}n\max_{1\le t\le 2n}|T(b_t)| \\
&+\frac{n-k+1}{2}\frac{n-k+2}{2}\cdots\frac{n-2}{2}(n-1)n\max_{1\le t\le 2n}|T(b_t)| \\
&\cdots+\frac{n-k+1}{2}\frac{n-k+2}{2}(n-k+3)\cdots n\max_{1\le t\le 2n}|T(b_t)| \\
&+\frac{n-k+1}{2}(n-k+2)(n-k+3)\cdots n\max_{1\le t\le 2n}|T(b_t)| \\
&=(n-k+1)(n-k+2)\cdots(n-1)n[\frac{1}{2^k}\{|T(1)|+\max_{1\le i\le n}|T(z_i)|\} \\
&+(1-\frac{1}{2^{k-1}})\max_{1\le t\le 2n}|T(b_t)|] \qquad (3.16)
\end{aligned}
$$

Further, for $k=1$, we have by Lemma 2.3

$$|T'(1)| \le \frac{n}{2}\{|T(1)|+\max_{1\le i\le n}|T(z_i)|\}, \qquad (3.17)$$

On combining (3.16) and (3.17), we get for $k \ge 1$

$$|T^{(k)}(1) \le (n-k+1)(n-k+2)...n[\frac{1}{2^k}\{|T(1)| \qquad (3.18)$$

$$+\max_{1\le i\le n}|T(z_i)|\}+(1-\frac{1}{2^{k-1}})\max_{1\le t\le 2n}|T(b_t)|] \qquad (3.19)$$

which, by (3.11) & (3.12), implies

$$|p^{(k)}(\beta)| \le \frac{1}{|\beta|^k}(n-k+1)(n-k+2)...n[\frac{1}{2^k}\{|p(\beta)| \qquad (3.20)$$

$$+\max_{1\le i\le n}|p(\beta z_i)|\}+(1-\frac{1}{2^{k-1}})\max_{1\le t\le 2n}|p(\beta b_t)|] \qquad (3.21)$$

thereby proving Theorem 1.1.

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