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SYMMETRIC (66,26,10) DESIGNS HAVING $Frob_{55}$ AS AN AUTOMORPHISM GROUP

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Abstract. Up to isomorphism there are three symmetric $(66, 26, 10)$ designs with automorphism group isomorphic to $Frob_{55}$. Among them there is one self-dual and one pair of dual designs. Full automorphism groups of dual designs are isomorphic to $Frob_{55}$, and full automorphism group of the self-dual design is isomorphic to $Frob_{55} \times D_{10}$. For those three designs corresponding derived and residual designs with respect to a block are constructed.

1. Introduction and preliminaries

A symmetric (v, k, λ) design is a finite incidence structure $(\mathcal{P}, \mathcal{B}, I)$, where $\mathcal{P}$ and $\mathcal{B}$ are disjoint sets and $I \subseteq \mathcal{P} \times \mathcal{B}$, with the following properties:

1. $|\mathcal{P}| = |\mathcal{B}| = v$,
2. every element of $\mathcal{B}$ is incident with exactly k elements of $\mathcal{P}$,
3. every pair of elements of $\mathcal{P}$ is incident with exactly λ elements of $\mathcal{B}$.

Mapping $g = (\alpha, \beta) \in S(\mathcal{P}) \times S(\mathcal{B})$ with the following property:

$$(P, x) \in I \Leftrightarrow (P\alpha, x\beta) \in I, \ for \quad all\ P \in \mathcal{P}, x \in \mathcal{B},$$

is an automorphism of a symmetric design $(\mathcal{P}, \mathcal{B}, I)$. The set of all automorphism of the design $\mathcal{D}$ is a group called the full automorphism group of $\mathcal{D}$, and it is denoted by $Aut\mathcal{D}$. Each subgroup of the $Aut\mathcal{D}$ is an automorphism group of $\mathcal{D}$.

Let $\mathcal{D} = (\mathcal{P}, \mathcal{B}, I)$ be a symmetric (v, k, λ) design and $G \leq Aut\mathcal{D}$. Group G has the same number of point and block orbits. Let us denote the number of $G-$orbits by t, point orbits by $\mathcal{P}_1, \ldots, \mathcal{P}_t$, block orbits by $\mathcal{B}_1, \ldots, \mathcal{B}_t$, and put $|\mathcal{P}_r| = \omega_r$, $|\mathcal{B}_i| = \Omega_i$. We shall denote points of the orbit $\mathcal{P}_r$ by $r_0, \ldots, r_{\omega_r - 1}$,

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(i.e. $\mathcal{P}_r = \{r_0, \ldots, r_{\omega_r - 1}\}$). Further, denote by γ_{ir} the number of points of $\mathcal{P}_r$ which are incident with the representative of the block orbit $\mathcal{B}_i$. For those numbers the following equalities hold:

$$\sum_{r=1}^{t} \gamma_{ir} = k\,, \tag{1.1}$$

$$\sum_{r=1}^{t} \frac{\Omega_j}{\omega_r} \gamma_{ir}\gamma_{jr} = \lambda\Omega_j + \delta_{ij} \cdot (k - \lambda)\,. \tag{1.2}$$

Definition 1. *The $(t \times t)$-matrix (γ_{ir}) with entries satisfying properties* (1) *and* (2) *is called the orbit structure for parameters (v, k, λ) and orbit distribution $(\omega_1, \ldots, \omega_t)$, $(\Omega_1, \ldots, \Omega_t)$.*

The first step of the construction of designs is to find all orbit structures (γ_{ir}) for some parameters and orbit distribution. The next step, called indexing, is to determine for each number γ_{ir} exactly which points from the point orbit $\mathcal{P}_r$ are incident with representative of the block orbit $\mathcal{B}_i$. Because of the large number of possibilities, it is often necessary to involve a computer in both steps of the construction.

Definition 2. *The set of indices of points of the orbit $\mathcal{P}_r$ indicating which points of $\mathcal{P}_r$ are incident with the representative of the block orbit $\mathcal{B}_i$ is called the index set for the position (i, r) of the orbit structure.*

First symmetric $(66, 26, 10)$ design is constructed by Tran van Trung (see [10]). Full automorphism group of that design is isomorphic to $Frob_{55} \times D_{10}$. M.-O. Pavčević and E. Spence (see [9]) have proved that there are at least 588 symmetric (66,26,10) designs. Only one of them, namely the one constructed by Tran van Trung, has $Frob_{55}$ as an automorphism group.

2. Centralizer of an automorphism group

During construction of symmetric designs we shall use elements of a normalizer of an automorphism group in the group $S = S(\mathcal{P}) \times S(\mathcal{B})$ to avoid construction of mutually isomorphic designs (see [3]). Therefore, we need to determine which permutations belong to this normalizer. Moreover, we shall determine some elements of a centralizer of an automorphism group in the group S.

Theorem 1. *Let X be a finite set, $G \leq S(X)$, $x_1, x_2 \in X$ and $G_{x_2}^{\bar{g}} = G_{x_1}$, for $\bar{g} \in G$. There exists a bijective mapping $\alpha : x_1G \to x_2G$, such that*

$$(x\alpha)g = (xg)\alpha, \text{ for all } g \in G,\ x \in x_1G.$$

Proof. Mapping $\alpha : x_1G \to x_2G$, defined as follows:

$$x\alpha = (x_1g')\alpha = x_2\bar{g}g', \text{ for all } x \in x_1G,$$

has the property from the theorem. ∎

Corollary 1. *Let X be a finite set, $G \leq S(X)$, $x_1, x_2 \in X$ and $G_{x_2}^{\bar{g}} = G_{x_1}$, for $\bar{g} \in G$. There exists $\alpha \in C_{S(X)}(G)$, such that $(x_1 G)\alpha = x_2 G$.*

Proof. Permutation $\alpha : X \to X$ is defined as follows:

$$x\alpha = \begin{cases} (x_1 g')\alpha = x_2 \bar{g} g', & for\ x \in x_1 G, \\ (x_2 \bar{g} g')\alpha = x_1 g', & for\ x \in x_2 G, \\ x, & else. \end{cases}$$

□

Theorem 2. *Let $\mathcal{D} = (\mathcal{P}, \mathcal{B}, I)$ be a symmetric design, $G \leq Aut\mathcal{D}$, and let $\Delta = (\gamma_{i,j})$ be an orbit structure of the design $\mathcal{D}$ with respect to the group G. Furthermore, let Δ be $(t \times t)-$matrix and let $g = (\alpha, \beta)$ be an element of the $S_t \times S_t$ with following properties:*

1. *if $i\alpha = j$, then stabilizers G_{x_i} and G_{x_j} are conjugated, where $\mathcal{B}_i = x_i G$, $\mathcal{B}_j = x_j G$,*
2. *if $r\beta = s$, then G_{P_r} and G_{P_s} are conjugated, where $\mathcal{P}_r = P_r G$, $\mathcal{P}_s = P_s G$.*

Then there exists permutation $g^* \in C_S(G)$, such that $i\alpha = j$ if and only if $\mathcal{B}_i g^* = \mathcal{B}_j$, and $r\beta = s$ if and only if $\mathcal{P}_r g^* = \mathcal{P}_s$.

Proof. If α and β are identity mappings, then g^* is identity mapping. If α i β are not identity mappings, then $\alpha = \alpha_1 \ldots \alpha_m$, $\beta = \beta_1 \ldots \beta_n$, $m, n \in N$, where $\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n$ are transpositions. From the corollary 1 follows that for each α_k, $1 \leq k \leq m$, there exists $\alpha_k^* \in C_S(G)$, which fixes all points, such that

$$i\alpha_k = j, \Leftrightarrow \mathcal{B}_i \alpha_k^* = \mathcal{B}_j, \text{ za } 1 \leq k \leq m,$$

and for each permutation β_l, $1 \leq l \leq n$, there exists permutation $\beta_l^* \in C_S(G)$, which fixed all blocks, such that

$$r\beta_l = s, \Leftrightarrow \mathcal{P}_r \beta_l^* = \mathcal{P}_s, \text{ za } 1 \leq l \leq n.$$

If mappings α^* and β^* are defined as follows:

$$\alpha^* = \alpha_1^* \ldots \alpha_m^*, \qquad \beta^* = \beta_1^* \ldots \beta_n^*,$$

then mapping $g^* = \alpha^* \beta^*$ satisfies the required property. □

Definition 3. *Let $A = (a_{i,j})$ be a $(m \times n)-$matrix and $g = (\alpha, \beta) \in S_m \times S_n$. Matrix $B = Ag$ is $(m \times n)-$matrix $B = (b_{i,j})$, where $b_{i\alpha, j\beta} = a_{i,j}$. If $Ag = A$, we shall say that g is an automorphism of a matrix A. All automorphisms of a matrix A form full automorphism group of A, which will be denoted by $AutA$.*

Definition 4. *Let $\Delta = (\gamma_{i,r})$ be an orbit structure of a symmetric (v, k, λ) design $\mathcal{D}$ with respect to the group $G \leq Aut\mathcal{D}$, and Ω_i, ω_r, $1 \leq i, r \leq$*

t,, lengths of $G-$orbits of blocks and points. Mapping $g=(\alpha,\beta)\in S_t\times S_t$ is called an automorphism of the orbit structure Δ, if following conditions hold:

1. *g is an automorphism of the matrix Δ,*
2. *if $i\alpha=j$, then $\Omega_i=\Omega_j$, for all i,j, $1\le i,j\le t$,*
3. *if $r\beta=s$, then $\omega_r=\omega_s$, for all r,s, $1\le r,s\le t$.*

All automorphisms of an orbit structure Δ form full automorphism group of Δ, which will be denoted by $Aut\Delta$.

Remark. During construction of orbit structures, for elimination of isomorphic copies one can use all permutations from $S_t\times S_t$ which satisfy conditions from the theorem 2, but during indexing of an orbit structure Δ, one can use just automorphisms from $Aut\Delta$.

3. $Frob_{55}$ acting on a symmetric (66,26,10) design

Let G be a Frobenius group of order 55. Since there is only one isomorphism class of such groups, we may write

$$G\cong\langle\rho,\sigma\mid\rho^{11}=1,\,\sigma^5=1,\,\rho^\sigma=\rho^3\rangle.$$

Let α be an automorphism of a symmetric design. We shall denote by $F(\alpha)$ the number of points fixed by α. In that case, the number of blocks fixed by α is also $F(\alpha)$.

Lemma 1. *Let ρ be an automorphism of a symmetric $(66,26,10)$ design. If $|\rho|=11$, then $F(\rho)=0$.*

Proof. It is known that $F(\rho)<k+\sqrt{k-\lambda}$ and $F(\rho)\equiv v(mod\ |\rho|)$. Therefore, $F(\rho)\in\{0,11,22\}$. One can not construct fixed blocks for $F(\rho)\in\{11,22\}$. ∎

Lemma 2. *Let G be a Frobenius automorphism group of order 55 of a symmetric $(66,26,10)$ design $\mathcal{D}$. G acts semistandardly on $\mathcal{D}$ with orbit distribution $(11,11,11,11,11,11)$ or $(11,55)$.*

Proof. Frobenius kernel $\langle\rho\rangle$ acts on $\mathcal{D}$ with orbit distribution $(11,11,11,11,11,11)$. Since $\langle\rho\rangle\triangleleft G$, σ maps $\langle\rho\rangle$-orbits on $\langle\rho\rangle$-orbits. Therefore, only possibilities for orbit distributions are $(11,11,11,11,11,11)$ and $(11,55)$. Since automorphism group of a symmetric design has the same number of orbits on sets of blocks and points, G acts semistandardly on $\mathcal{D}$.

Stabilizer of each block from the block orbit of length 11 is conjugated to $\langle\sigma\rangle$. Therefore, entries of orbit structures corresponding to point and block orbits of length 11 must satisfy the condition $\gamma_{ir}\equiv 0,1\ (mod\ 5)$. Solving equations (1) and (2) we got one orbit structure for orbit distribution $(11,11,11,11,11,11)$, namely

OS1	11	11	11	11	11	11
11	1	5	5	5	5	5
11	5	5	5	5	5	1
11	5	5	5	5	1	5
11	5	5	5	1	5	5
11	5	5	1	5	5	5
11	5	1	5	5	5	5

and one orbit structure OS2

OS2	11	55
11	1	25
55	5	21

for orbit distribution $(11, 55)$. ∎

4. Orbit distribution (11,11,11,11,11,11)

Theorem 3. *Up to isomorphism there is only one symmetric* $(66, 26, 10)$ *design with automorphism group* $Frob_{55}$ *acting with orbit distribution* $(11, 11, 11, 11, 11, 11)$. *This design is self-dual. Full automorphism group of that design is isomorphic to* $Frob_{55} \times D_{10}$, *and* 2*-rank is* 31.

Proof. We shall denote points by $1_i, \ldots, 6_i,\ i = 0, 1, \ldots, 10$, and assume that automorphisms ρ and σ act on the set of points as follows:

$\rho = (I_0, I_1, \ldots, I_{10}),\ I = 1, 2, \ldots, 6,$

$\sigma = (K_0)(K_1, K_3, K_9, K_5, K_4)(K_2, K_6, K_7, K_{10}, K_8),\ K = 1, 2, 3, 4, 5, 6.$

As representatives of block orbits we shall choose blocks fixed by $\langle\sigma\rangle$. Therefore, index sets which could occur are:

$$0 = \{0\},\ 1 = \{1, 3, 4, 5, 9\},\ 2 = \{2, 6, 7, 8, 10\}.$$

We have constructed, up to isomorphism, only one design, which is presented in terms of index sets as follows:

0	1	1	1	1	1
2	1	1	2	2	0
2	1	2	1	0	2
2	2	1	0	1	2
2	2	0	1	2	1
2	0	2	2	1	1

For elimination of isomorphic copies of the constructed design we have used full automorphism group of the orbit structure OS1, and an element α of the normalizer $N_S(G)$ which acts on the points as follows:

$$r_i\alpha = r_j,\ j \equiv 2i\ (mod\ 11),\ \textit{for}\ 1 \le r \le 6,\ 0 \le i \le 10.$$

This permutation induces following permutation of index sets:

$$(0)(1, 2).$$

Using the computer program by V. Tonchev we have found out that the order of the full automorphism group of the constructed design is 550. This program gave us generators of the full automorphism group, which enables us to determine that this group is isomorphic to $Frob_{55} \times D_{10}$. Another program by V. Tonchev have been used for computation of 2-rank. □

5. Orbit distribution (11,55)

It would be difficult to proceed with indexing of orbit structure OS2. For example, there are $\binom{55}{25}$ possibilities for index sets for the position $(1,2)$ in the OS2. Therefore, we shall use the principal series $\langle 1 \rangle \lhd \langle \rho \rangle \lhd G$ of the automorphism group G. Our aim is to find all orbit structures for the group $\langle \rho \rangle$ corresponding to the structure OS2. We shall construct designs from those orbit structures for $\langle \rho \rangle$, having in mind the action of the permutation σ on $\langle \rho \rangle$-orbits.

Theorem 4. *Up to isomorphism there are three symmetric* $(66,26,10)$ *designs with automorphism group* $Frob_{55}$ *acting with orbit distribution* $(11,55)$. *Let us denote them by* $\mathcal{D}_1$, $\mathcal{D}_2$ *and* $\mathcal{D}_3$. *Full automorphism groups of designs* $\mathcal{D}_1$ *and* $\mathcal{D}_2$ *are isomorphic to* $Frob_{55}$, *and* $2-$*ranks are* 31. *Designs* $\mathcal{D}_1$ *and* $\mathcal{D}_2$ *are dually isomorphic. Design* $\mathcal{D}_3$ *is isomorphic to the design from theorem 3.*

Proof. OS1 is the only orbit structure for the group $\langle \rho \rangle$ corresponding to the orbit structure OS2 for the group $\langle \rho, \sigma \rangle \cong Frob_{55}$. We shall proceed with indexing of orbit structure OS1, knowing that σ acts on the set of six $\langle \rho \rangle$-orbits of blocks as a permutation $(1)(2,3,4,5,6)$, and on the set of point orbits as $(1)(2,6,5,4,3)$. It is sufficient to determine index sets for the first and second row of orbit structure OS1. Also, in the first row we have to determine index sets only for positions $(1,1)$ and $(1,2)$. We shall assume that automorphisms ρ and σ act on the set of points as follows:

$\rho = (I_0, I_1, \dots, I_{10}),\ I = 1, 2, \dots, 6,$

$\sigma = (1_0)(1_1, 1_3, 1_9, 1_5, 1_4)(1_2, 1_6, 1_7, 1_{10}, 1_8), (2_i, 3_{3i}, 4_{9i}, 5_{5i}, 6_{4i}),$

$i = 0, 1, \dots, 10.$

Operation with indices is multiplication modulo 11. As representatives of the first block orbit we shall choose blocks fixed by $\langle \sigma \rangle$. Therefore, index sets for position $(1,1)$ is $\{0\}$. As representatives of the second block orbit we shall choose first blocks in this orbit, with respect to the lexicographical order. Index sets which could occur in designs are:

$$0 = \{0\},\ \dots\ ,\ 10 = \{10\},\ 11 = \{0,1,2,3,4\},\ \dots\ ,\ 472 = \{6,7,8,9,10\}.$$

To eliminate isomorphic structures during the indexing, we have been using the permutation α, automorphisms of the orbit structure OS1 which commute with permutation representation of the σ on the set of $\langle \rho \rangle$-orbits, and permutations $\beta_k \in N_S(G),\ 1 \le k \le 10$, defined as follows:

$1_i\beta_k = 1_i,\ for\ i = 0, 1, \ldots, 10,$
$2_i\beta_k = 2_j, j \equiv (i + k)(mod\ 11),\ for\ i = 0, 1, \ldots, 10,$
$6_i\beta_k = 6_j, j \equiv (i + 3k)(mod\ 11),\ for\ i = 0, 1, \ldots, 10,$
$5_i\beta_k = 5_j, j \equiv (i + 9k)(mod\ 11),\ for\ i = 0, 1, \ldots, 10,$
$4_i\beta_k = 4_j, j \equiv (i + 5k)(mod\ 11),\ for\ i = 0, 1, \ldots, 10,$
$3_i\beta_k = 3_j, j \equiv (i + 4k)(mod\ 11),\ for\ i = 0, 1, \ldots, 10.$

We have constructed, up to isomorphism, three designs, denoted by $\mathcal{D}_1$, $\mathcal{D}_2$ and $\mathcal{D}_3$. The statistics of intersection of any three blocks proves that those designs are mutually nonisomorphic. With the help of the computer program by V. Tonchev, we have computed that orders of full automorphism groups of designs $\mathcal{D}_1$ and $\mathcal{D}_2$ are 55. Using the computer program Nauty (see [5], [8]), it was determined that $\mathcal{D}_1$ and $\mathcal{D}_2$ are dually isomorphic, and $\mathcal{D}_3$ is isomorphic to the design from theorem 3. Designs $\mathcal{D}_1$ and $\mathcal{D}_2$ are presented in terms of index sets as follows:

		$\mathcal{D}_1$			
0	288	312	384	369	311
13	21	55	163	333	9
177	175	27	423	5	13
214	151	331	4	177	193
194	229	1	214	18	70
21	3	194	40	49	354

		$\mathcal{D}_2$			
0	422	294	225	279	321
13	31	448	129	25	4
177	416	30	235	1	166
214	106	447	3	204	406
194	410	9	83	392	218
21	5	98	269	213	456

$\square$

It is clear that the design $\mathcal{D}_3$ is isomorphic to the one constructed by Tran van Trung. Designs $\mathcal{D}_1$ and $\mathcal{D}_2$ were not known before. Therefore, there are at least 590 symmetric $(66, 26, 10)$ designs.

6. Designs 2-(26,10,9) derived from symmetric (66,26,10) designs

As we proved, there are exactly three mutually nonisomorphic symmetric (66,26,10) designs having $Frob_{55}$ as an automorphism group. Excluding one block and all points that do not belong to that block from symmetric (66,26,10) design, one can obtain 2-(26,10,9) design (see [6]). According to [2], there are at least nineteen 2-(26,10,9) designs.

Theorem 5. *Up to isomorphism there are six* $2-(26, 10, 9)$ *designs derived from symmetric* $(66, 26, 10)$ *designs having* $Frob_{55}$ *as an automorphism group. Let us denote them by* $\mathcal{D}_{i1}$ *and* $\mathcal{D}_{i2}$*,* $i = 1, 2, 3$*, where indices i corespond to those from theorem* 4*. Designs* $\mathcal{D}_{11}$ *and* $\mathcal{D}_{21}$ *have trivial full automorphism groups. Full automorphism groups of designs* $\mathcal{D}_{12}$ *and* $\mathcal{D}_{22}$ *are isomorphic to* Z_5*. Full automorphism group of design* $\mathcal{D}_{31}$ *is isomorphic to* Z_{10} *and full automorphism group of design* $\mathcal{D}_{32}$ *is isomorphic to* $D_{10} \times Z_5$*.*

Proof. Full automorphism groups of designs $\mathcal{D}_1$, $\mathcal{D}_2$ and $\mathcal{D}_3$ act on those designs with orbit distributions (11,55), i.e. in two block orbits. Designs 2-(26,10,9) derived from one symmetric (66,26,10) design are mutually isomorphic if corresponding blocks that were excluded belong to the same block orbit and mutually non-isomorphic if corresponding blocks belong to different block orbits. Therefore, from each design $\mathcal{D}_i$, $i = 1, 2, 3$, one can obtain two non-isomorphic derived 2-(26,10,9) designs, $\mathcal{D}_{i1}$ and $\mathcal{D}_{i2}$. Those six designs are mutually non-isomorphic. Using the computer program by V. Tonchev, we have determinated orders of their full automorphism groups. This program gave us generators of full automorphism groups, which enables us to determine the full automorphism groups for designs $\mathcal{D}_{31}$ and $\mathcal{D}_{32}$ as well . As an example, we present design $\mathcal{D}_{11}$.

$\mathcal{D}_{11}$

1 2 3 6 7 11 17 19 22 23	0 2 7 8 14 15 16 18 19 24	1 3 4 5 6 7 10 15 21 23
2 3 7 8 10 12 14 15 23 24	0 1 3 8 9 12 13 17 18 19	2 4 5 6 7 9 11 13 14 22
3 4 5 8 11 12 15 16 20 24	1 2 4 5 8 10 13 15 19 25	3 6 7 8 13 20 21 23 24 25
4 6 8 9 10 13 17 18 21 24	2 3 6 11 13 14 15 16 20 25	4 7 8 10 12 13 14 16 21 22
0 4 7 11 12 16 18 19 20 22	0 3 4 5 7 9 14 15 17 25	0 6 7 9 11 15 19 20 21 25
1 4 9 13 14 16 17 19 20 23	1 6 9 10 12 13 14 16 18 25	0 1 4 7 8 15 18 20 21 22
2 5 8 9 12 13 15 17 20 21	2 4 7 8 9 10 11 16 17 19	1 2 4 8 12 15 19 22 23 25
0 3 6 9 10 14 15 18 22 24	0 5 8 11 14 17 22 23 24 25	0 2 3 5 8 9 16 18 21 23
0 1 5 7 11 13 14 19 21 23	0 1 4 6 8 10 14 20 23 25	1 3 6 15 16 17 19 21 22 24
0 1 2 5 6 10 16 18 21 22	0 1 2 7 9 10 11 12 21 24	2 7 9 16 17 21 22 23 24 25
0 3 4 5 10 11 14 16 17 21	1 2 3 9 10 11 13 20 22 24	0 3 4 5 9 13 15 17 22 23
1 5 6 8 11 12 15 17 18 22	0 2 3 8 9 11 12 18 23 25	0 1 4 6 8 9 16 20 23 24
0 2 5 6 7 10 12 13 19 23	1 3 5 9 12 14 19 20 21 25	1 2 5 7 14 17 18 20 24 25
1 3 4 6 7 9 11 12 18 25	0 2 4 5 6 12 13 20 22 24	2 3 4 5 6 16 18 19 24 25

3 5 6 7 8 10 17 18 19 20
0 1 2 10 15 16 17 20 21 25
1 2 3 8 10 11 14 17 20 22
2 3 4 11 13 16 18 20 21 23
0 3 10 12 13 16 17 19 22 25
1 4 11 12 14 17 18 21 23 24
2 4 9 14 15 18 19 21 22 25
3 4 10 12 19 21 22 23 24 25
0 10 11 13 15 18 20 22 23 25
1 5 11 13 15 16 18 19 23 24
0 2 4 6 12 14 17 19 20 24
0 1 3 7 12 13 14 15 16 24
0 6 8 11 13 17 19 21 24 25
1 5 7 8 9 13 16 22 24 25
2 6 12 13 14 15 17 18 21 23
3 7 8 13 14 18 19 20 21 22
5 9 10 14 18 19 20 22 23 24
6 8 9 10 11 14 15 16 19 23
4 7 10 11 13 15 17 18 24 25
5 9 10 11 12 15 19 20 21 24
5 6 8 11 12 14 16 21 22 25
6 7 9 12 15 16 17 20 22 23
5 7 10 12 16 17 18 20 23 25

□

7. Some new 2-(40,16,10) designs

Residual design with respect to a block (see [6]) of a symmetric (66,26,10) design is 2-(40,16,10) design. The existence of such design follows from the existence of symmetric (66,26,10) design (see [10]).

Theorem 6. *Up to isomorphism there are six* $2-(40,16,10)$ *residual designs of symmetric* $(66,26,10)$ *designs having* $Frob_{55}$ *as an automorphism group. Let us denote them by* $\mathcal{D}'_{i1}$ *and* $\mathcal{D}'_{i2}$*,* $i=1,2,3$*, where indices* i *corespond to those from theorem 4. Designs* $\mathcal{D}'_{11}$ *and* $\mathcal{D}'_{21}$ *have trivial full automorphism groups. Full automorphism groups of designs* $\mathcal{D}'_{12}$ *and* $\mathcal{D}'_{22}$ *are isomorphic to* Z_5*. Full automorphism group of design* $\mathcal{D}'_{31}$ *is isomorphic to* Z_{10} *and full automorphism group of design* $\mathcal{D}'_{32}$ *is isomorphic to* $D_{10}\times Z_5$*.*

Proof. We proceed similarly like in the proof of theorem 5. As an example, we present design $\mathcal{D}'_{11}$.

$$\mathcal{D}'_{11}$$

0 2 6 7 10 12 13 16 17 18 20 23 25 28 29 39
0 1 3 6 8 11 12 14 19 20 21 23 24 26 28 30
0 1 4 6 7 9 13 15 17 18 21 22 27 28 29 31
0 1 2 5 7 8 12 13 14 18 19 22 24 25 29 32
1 2 3 7 8 9 10 14 15 16 19 20 24 26 29 33
2 3 4 6 8 9 11 12 13 15 21 23 24 25 27 34
2 3 4 5 9 10 14 16 17 20 22 23 25 26 28 35
3 4 5 7 10 11 13 15 16 18 20 21 25 26 27 36

0 4 5 8 10 11 14 17 18 19 21 22 26 27 29 37

1 5 6 9 11 12 15 16 17 19 22 23 24 27 28 38

2 4 6 7 10 11 12 14 20 22 23 30 31 32 33 36

0 2 3 5 8 11 12 15 17 20 21 31 32 33 34 37

1 3 4 7 9 12 13 18 20 21 22 32 33 34 35 38

4 5 6 8 13 14 16 19 21 22 23 28 33 34 35 36

0 2 5 6 9 10 12 13 14 15 22 34 35 36 37 39

1 3 6 7 11 14 15 17 18 23 29 30 35 36 37 38

0 4 7 8 10 13 15 16 19 20 23 24 31 36 37 38

1 2 5 7 8 9 11 14 16 18 21 23 32 37 38 39

0 3 8 9 12 15 16 17 18 19 22 25 30 33 38 39

0 1 2 4 6 9 10 17 18 19 20 26 30 31 34 39

1 3 5 10 11 13 16 17 19 21 27 30 31 32 35 39

1 2 4 5 6 7 8 13 16 17 24 26 30 34 35 38

3 5 7 8 9 12 14 17 20 27 28 29 31 35 36 39

2 4 7 8 9 15 17 21 24 25 28 30 32 36 37 39

3 5 6 8 9 10 12 13 22 26 29 30 31 33 37 38

0 4 9 10 11 12 14 16 24 25 27 29 31 32 34 38

0 1 5 7 10 11 12 13 15 24 26 28 32 33 35 39

0 1 8 10 11 13 14 17 23 25 27 30 33 34 36 39

1 2 9 11 13 14 15 16 25 26 29 30 31 34 35 37

0 2 3 6 14 15 18 24 26 27 28 31 32 35 36 38

1 2 3 4 6 10 12 15 16 17 19 27 32 33 36 37

0 3 4 5 6 7 11 16 25 28 29 33 34 37 38 39

0 1 3 5 6 7 20 22 23 24 25 27 30 31 34 37

1 4 6 8 10 12 18 21 23 25 26 31 32 35 38 39

2 5 6 7 9 11 18 19 22 25 26 27 30 32 33 36

2 3 7 8 13 18 19 23 26 27 28 31 33 34 37 39

0 3 4 7 8 9 10 14 19 27 28 30 32 34 35 38

0 1 4 5 8 9 11 15 18 20 23 29 31 33 35 36

1 5 6 9 10 19 20 21 24 28 29 32 34 36 37 39

0 2 10 11 16 18 20 21 22 24 28 30 33 35 37 38

1 2 3 7 10 11 19 21 22 25 28 29 31 34 36 38

3 4 6 8 11 17 20 22 24 25 26 29 32 35 37 39

0 2 4 5 9 21 23 24 26 27 29 30 33 36 38 39

0 3 6 13 14 16 17 18 21 24 26 29 32 33 34 36

1 4 14 15 18 19 20 22 24 25 27 33 34 35 37 39

0 5 7 12 15 17 19 20 21 25 26 30 34 35 36 38

0 1 2 8 12 16 21 22 25 26 27 28 31 35 36 37

0 1 3 9 13 16 20 22 23 26 27 32 36 37 38 39

1 2 4 12 13 14 17 20 21 27 28 29 30 33 37 38

2 3 5 10 14 15 17 18 21 22 23 24 31 34 38 39

0 2 3 4 11 13 15 19 22 23 28 29 30 32 35 39
1 3 4 5 12 14 16 18 24 25 28 30 31 33 36 39
4 5 12 15 16 18 19 23 26 28 29 30 31 32 34 37
2 5 13 17 19 20 23 24 25 27 29 31 32 33 35 38
6 9 11 13 14 15 19 20 21 25 26 28 31 33 38 39
7 12 14 15 16 20 21 22 23 26 27 29 30 32 34 39
6 7 8 10 15 16 21 22 24 27 29 30 31 33 35 39
8 9 11 13 16 17 18 20 22 24 28 30 31 32 34 36
0 6 7 9 14 16 17 19 21 23 25 31 32 33 35 37
1 8 10 15 17 22 23 25 26 28 29 32 33 34 36 38
7 9 10 11 12 17 18 23 24 26 27 28 33 34 35 37
2 6 8 11 12 16 18 19 20 27 29 34 35 36 38 39
3 9 10 12 13 18 19 21 23 24 25 29 30 35 36 37
4 7 11 12 13 14 17 19 22 24 26 31 36 37 38 39
5 6 8 10 13 14 15 18 20 25 27 28 30 32 37 38

□

In [2] there is only information about existence of 2-(40,16,10) designs. According to this information, at least five of constructed designs are new.

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