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ON THE STRICT TOPOLOGY IN NON-ARCHIMEDEAN SPACES OF CONTINUOUS FUNCTIONS

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Abstract. The strict topology, on the space $C(X, E)$ of all continuous functions on a topological space X with values in a non-Archimedean locally convex space E, is introduced and several of its properties are investigated. The dual spaces of $C(X, E)$, under the strict topology and under the bounding convergence topology, turn out to be certain spaces of E'-valued measures.

1. Introduction

The strict topology was for the first time defined by Buck [3] on the space $C_b(X, E)$ of all bounded continuous functions on a locally compact space X with values in a normed space E. Several other authors have extended Buck's results by taking as X a completely regular space or an arbitrary topological space and as E either the scalar field or a locally convex space or even an arbitrary topological vector space. In the case of non-Archimedean valued functions Prolla [17], p.198, has defined on $C_b(X, E)$ the strict topology β assuming that X is locally compact zero-dimensional and E a non-Archimedean normed space. In [9] the author has defined the strict topology β_o on $C_b(X, E)$ taking as X a topological space and as E a non-Archimedean locally convex space. In case X is locally compact zero-dimensional and E a non-Archimedean normed space, β_o coincides with β by [9], Proposition 2.5. As is shown in [14], Theorem 3.2, the strict topology β_o is a weighted topology.

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In this paper we introduce in section 5 the strict topology β_b on the space $C(X,E)$ of all continuous functions from a topological space X to a non-Archimedean locally convex space E. We show in Proposition 5.5 that β_b is the finest weighted topology ω_V such that $CV_o(X,E)=C(X,E)$ algebraically. We prove in section 6 that the dual space of $(C(X,E),\beta_b)$ is a certain space of E'-valued measures. We show that β_b has almost all of the properties that β_o has. We also characterize in Theorem 6.3 the dual space of $C(X,E)$ under the topology of uniform convergence on the so called bounding subsets of X.

2. Preliminaries

Throughout this paper, $\mathbb{K}$ is a complete non-Archimedean valued field whose valuation is non-trivial. By a seminorm, on a vector space E over $\mathbb{K}$, we mean a non-Archimedean seminorm. Similarly by a locally convex space or a normed space we mean non-Archimedean such spaces. For E a locally convex space over $\mathbb{K}$, we denote by $cs(E)$ the collection of all continuous seminorms on E. By E' we denote the topological dual space of E, while, for E Hausdorff, $\widehat{E}$ is the completion of E. For E,F locally convex spaces over $\mathbb{K}$, $E\otimes F$ is the projective tensor product of E,F. For any unexplained terms, concerning non-Archimedean spaces, we refer to [18].

Let now X be a topological space and E a Hausdorff locally convex space over $\mathbb{K}$. The space of all continuous E-valued functions on X is denoted by $C(X,E)$. The subspace of all bounded members of $C(X,E)$ is denoted by $C_b(X,E)$. In case E is the scalar field $\mathbb{K}$, we write $C_b(X)$ and $C(X)$ instead of $C_b(X,\mathbb{K})$ and $C(X,\mathbb{K})$, respectively. If f is a function from X to E, A a subset of X and p a seminorm on E, we define the extended real number $p_A(f)$ by

$$p_A(f)=\sup\{p(f(x)):x\in A\}.$$

In case E is a normed space, we define

$$\omega_A(f)=\sup\{\|f(x)\|:x\in A\},\quad \|f\|=\omega_X(f)$$

The strict topology β_0 on $C_b(X,E)$ (see [9]) is the locally convex topology on $C_b(X,E)$ generated by the seminorms $p_\phi(f)=p_X(\phi f)$ where $p\in cs(E)$ and $\phi\in\mathbb{K}^X$ bounded and vanishing at infinity. The support of a function $f\in E^X$ or $f\in\mathbb{K}^X$ is the closure of the set $\{x:f(x)\neq 0\}$. We denote by τ_c the topology of uniform convergence on the compact subsets of X. In case X is zero-dimensional, β_oX and υ_oX is the Banaschewski compactification and the $\mathbf{N}$-repletion of X, respectively ($\mathbf{N}$ is the set of all positive integers).

Let $K(X)$ denote the algebra of all clopen (i.e closed and open) subsets of X. We denote by $M(X,E')$ (see [11]) the space of all finitely-additive E'-valued measures m on $K(X)$ for which $m(K(X))$ is an equicontinuous subset of E'. For every $m\in M(X,E')$ there exists $p\in cs(E)$ such that

$\|m\|_p = m_p(X) < \infty$, where, for $A \in K(X)$,

$$m_p(A) = \sup\{|m(B)s| : B \in K(X), B \subset A, p(s) \leq 1\}.$$

As is shown in [11],

$$m_p(A \cup B) = \max\{m_p(A), m_p(B)\}.$$

We denote by $M_p(X, E')$ the set of all $m \in M(X, E')$ for which $m_p(X) < \infty$. An element m of $M(X, E')$ is called tight if there exists $p \in cs(E)$ such that $m_p(X) < \infty$ and, for each $\epsilon > 0$, there exists a compact subset D of X such that $m_p(A) < \epsilon$ if A is disjoint from D. In this case we also say that m_p is tight.

Let now $m \in M(X, E')$ and let $A \in K(X)$. Consider the collection Ω_A of all $\alpha = \{A_1, A_2, ..., A_n; x_1, x_2, ..., x_n\}$ where $\{A_1, ..., A_n\}$ is a clopen partition of A and $x_i \in A_i$. The collection Ω_A becomes a directed set by defining $\alpha_1 \geq \alpha_2$ iff the partition of A in α_1 is a refinement of the one in α_2. If f is an E-valued function on X and $\alpha = \{A_1, ..., A_n; x_1, ..., x_n\}$ in Ω_A, we define

$$\omega_\alpha(f, m) = \sum_{i=1}^{n} m(A_i) f(x_i).$$

If the limit

$$\lim_{\alpha \in \Omega_A} \omega_\alpha(f, m)$$

exists, then we say that f is m-integrable over A and we denote this limit by $\int_A f dm$. The integral of f over the empty set is taken to be zero. We say that f is m-integrable if it is m-integrable over every $A \in K(X)$. We write

$$m(f) = \int f dm \quad for \quad \int_X f dm.$$

(see [11]). By [9], Proposition 3.2, if m is tight, then every $f \in C_b(X, E)$ is m-integrable and the mapping $f \mapsto m(f)$ is an element of the dual space of $(C_b(X, E), \beta_o)$. Conversely, every β_o-continuous linear form on $C_b(X, E)$ is given by a unique tight element of $M(X, E')$.

Next we recall the definition of a non-Archimedean weighted space. A Nachbin family on X is a collection V of non-negative upper semicontinuous (u.s.c) functions on X such that:

1) For each $x \in X$, there exists $v \in V$ with $v(x) > 0$.
2) For $v_1, v_2 \in V$ and d a positive number, there exists $v \in V$ with $dv_1, dv_2 \leq v$.

We say that a Nachbin family V_1 is coarser than another one V_2, or that V_2 is stronger than V_1 , and write $V_1 \preceq V_2$, if for every $v \in V_1$ there exists $w \in V_2$ such that $v \leq w$. If V_1 is both coarser and stronger than V_2 , then we say that V_1 is equivalent to V_2 and write $V_1 \cong V_2$. For a non-negative function

v on X, $f \in E^X$ and p a seminorm on E, we define the extended real number $p_v(f)$ by

$$p_v(f) == \sup\{v(x)p(f(x)) : x \in X\}.$$

In case f is $\mathbb{K}$-valued, we define

$$\omega_v(f) = \sup\{v(x)|f(x)| : x \in X\}.$$

The weighted space $CV(X, E)$ is defined to be the space of all $f \in C(X, E)$ for which $p_v(f) < \infty$ for each $v \in V$ and each $p \in cs(E)$. The corresponding weighted topology ω_V on $CV(X, E)$ is the locally convex topology defined by the seminorms $p_v, p \in cs(E), v \in V$. As usual, we denote by $CV_o(X, E)$ the subspace of $CV(X, E)$ which consists of all f such that, for each $v \in V$ and each $p \in cs(X, E)$, the function $x \mapsto v(x)p(f(x))$ vanishes at infinity on X. We write $CV(X)$ and $CV_o(X)$ when $E = \mathbb{K}$.

Throughout the paper, X is a topological space and E a Hausdorff non-Archimedean locally convex space over $\mathbb{K}$.

3. Bounding sets

Following Govaerts [8] we say that a subset A of X is bounding if every $f \in C(X)$ is bounded on A.

The following Proposition characterizes the bounding sets.

Proposition 3.1. *Assume that X is a zero-dimensional Hausdorff topological space.. For a subset A of X, the following are equivalent:*
(1) Every f in $C(X, E)$ is bounded on A.
(2) A is bounding.
(3) The closure $B = cl_{\upsilon_o X} A$ of A, in the $\mathbf{N}$-repletion of X, is compact.

Proof. It is clear that (1) implies (2). Also (2) implies (3) by [8] Proposition 1. (3) $\Rightarrow$ (1). Assume that, for some continuous E-valued function g on X and some continuous seminorm p on E, the function $x \mapsto p(g(x))$ is not bounded on A. Let $\mathbb{R}^+$ be the set of nonnegative real numbers and consider on $\mathbb{R}^+$ the ultrametric $d(a, b) = max\{a, b\}$ if $a \neq b$ and $d(a, a) = 0$. Then $\mathbb{R}^+$ is ultranormal, i.e. every two disjoint closed subsets of $\mathbb{R}^+$ are separated by disjoint clopen sets. Since $\mathbb{R}^+$ is metrizable with nonmeasurable cardinal, it is realcompact (see [7] 15.24). Also $\mathbb{R}^+$ is complete and noncompact. Thus by [2], Theorem 9, the $\mathbb{R}^+$-repletion of X coincides with $\upsilon_o X$. Since the function $h : X \to \mathbb{R}^+, h(x) = p(g(x))$ is continuous, there exists a continuous extension $\bar{h} : \upsilon_0 X \to \mathbb{R}^+$. Since $B = cl_{\upsilon_o X} A$ is compact, we get that $h(A)$ is bounded in $\mathbb{R}^+$, a contradiction.

The following Proposition refers to an arbitrary topological space (not necessarily zero-dimensional).

Proposition 3.2. *For a subset A of a topological space X, the following are equivalent:*
(1) Every $f \in C(X,E)$ is bounded on A.
(2) A is bounding.

Proof. Let τ_0 be the zero-dimensional topology generated by the clopen subsets of X (we refer to τ_0 as the zero-dimensional topology corresponding to the topology of X). Since a function f, from X to a zero-dimensional space, is continuous iff it is τ_0-continuous, we may assume that X is zero-dimensional (not necessarily Hausdorff). If now X is Hausdorff, then (1) is equivalent to (2) by the preceeding Proposition. If X is not Hausdorff, consider the equivalence relation $\sim$ on X defined by : $x \sim y$ iff $f(x) = f(y)$ for each $f \in C(X,E)$. Let $Y = X/\sim$ and consider on Y the quotient topology. If $Q : X \to Y$ is the quotient map, then Q maps clopen sets onto clopen sets. Indeed, let $V \subset X$ be clopen and let $D = Q(V)$. If $x \in Q^{-1}(D)$, then $x \sim y$ for some $y \in V$. But then, if ϕ is the $\mathbb{K}$-characteristic function of V, we have $\phi(x) = \phi(y) = 1$ and so $x \in V$, i.e. $Q^{-1}(D) = V$, which implies that D is open. Also, if V^c is the complement of V, then $Q(V^c)$ is open and hence D is clopen. It follows now that Y is zero-dimensional. Also Y is Hausdorff. Indeed, if $Q(x) \neq Q(y)$, then $f(x) \neq f(y)$, for some $f \in C(X,E)$. Since E is Hausdorff and zero-dimensional, there are clopen disjoint neighborhoods W_1, W_2 of $f(x)$ and $f(y)$ respectively. If $V_i = f^{-1}(W_i), i = 1,2$, then $Q(V_1)$ and $Q(V_2)$ are disjoint neighborhoods of $Q(x)$ and $Q(y)$, respectively. Assume now (2). Then $D = Q(A)$ is a bounding subset of Y. By the preceeding Proposition, every $g \in C(Y,E)$ is bounded on D. If $u \in C(X,E)$ and if $g : Y \to E, g(Q(x)) = u(x)$, then g is bounded on D and so u is bounded on A. Since (1) clearly implies (2), the result follows.

Proposition 3.3. *Assume that either X or E has non-measurable cardinal. If A is a bounding subset of X, then $f(A)$ is totally bounded in E for every $f \in C(X,E)$.*

Proof. Taking on X the corresponding zero-dimensional topology, we may assume that X is zero-dimensional (not necessarily Hausdorff). Assume first that X is Hausdorff. Then $B = cl_{\upsilon_o X} A$ is compact. Let $p \in cs(E)$ and let $E_p = E/ker\ p$ with the corresponding norm-topology. Let $\phi \to \hat{E}_p$ be the canonical map and let $h = \phi \circ f$. We claim that $h(A)$ is totally bounded in $\hat{E}_p$. Assume the contrary. Denoting by $|Z|$ the cardinal number of a set Z, we have that $|h(X)| \leq |X|$ and $|h(X)| \leq |E_p| \leq |E|$. Our hypothesis implies that $h(X)$ has nonmeasurable cardinal. Also, the closure G of $h(A)$ in $\hat{E}_p$ has nonmeasurable cardinal since $h(X)^{\mathbb{N}}$ has nonmeasurable cardinal and every element of G is the limit of a sequence in $h(X)$. Thus G is a realcompat, noncompact ultranormal space and hence the G-repletion of X coincides with $\upsilon_o X$ by [2], Theorem 7. Let $\bar{h} : \upsilon_0 X \to G$ be a continuous extension of h. Since B is compact, $\bar{h}(B)$ is compact and so $h(A)$ is totally bounded, a contradiction. So, $h(A)$ is totally bounded in E_p and therefore $f(A)$ is p-totally bounded in E. This proves the result when X is Hausdorff. In case X is not Hausdorff, let Y, Q be as in the proof of Proposition 3.2. If $g : Y \to E, g(Qx) = f(x)$ and if $D = Q(A)$, then D is bounding in Y and so $g(D) = f(A)$ is totally bounded in E. This clearly competes the proof.

4. The Topology of Uniform Convergence on Bounding Sets

For $p \in cs(E)$ and A a bounding subset of X, p_A (as it is defined in Sec. 2) is a seminorm on $C(X, E)$. We denote by $\tau_{u,b}$ the locally convex topology on $C(X.E)$ generated by the seminorms p_A, $p \in cs(E)$, A a bounding subset of X. We refer to $\tau_{u,b}$ as the topology of uniform convergence on the bounding subsets of X.

For the rest of this section, we assume that either X or E has non-measurable cardinal.

Theorem 4.1. *Assume that E is complete and consider the following condition:*
$(*)$ *If $f : X \to E$ is such that $f|_A$ is continuous if A is bounding and $f(A)$ is totally bounded in E, then f is continuous on X.*
Then: (a) The space $(C(X, E), \tau_{u,b}) = G$ is complete when $(*)$ is satisfied.
(b) If X is ultranormal and E is a Fréchet space, then completeness of G implies that $(*)$ holds.

Proof. (a) Assume that $(*)$ is satisfied and let (f_α) be a Cauchy net in G. For $x \in X$, $(f_\alpha(x))$ is a Cauchy net in E and thus the limit $f(x) = lim f_\alpha(x)$ exists. If A is a bounding subset of X, then $f_\alpha \to f$ uniformly on A and thus the restriction of f to A is continuous. Also, given $p \in cs(X, E)$, there exists α_0 such that $p_A(f_\alpha - f_{\alpha_0}) \leq 1$ for all $\alpha \geq \alpha_0$. Since $f_{\alpha_0}(A)$ is totally bounded, there exists a finite suset S of E such that

$$f_{\alpha_0}(A) \subset S + W, \quad W = \{s \in E : \quad p(s) \leq 1\}.$$

It follows now that $f(A) \subset S + W$, which proves that $f(A)$ is totally bounded. By our hypothesis, f is continuous and clearly $f_\alpha \to f$.

(b) Suppose that G is complete and that X is ultranormal and E a Fréchet space. Let $p \in cs(E)$ and let A be a closed bounding subset of X. If $g : A \to E$ is continuous and $g(A)$ is totally bounded in E, then there exists a continuous function $h : X \to E$ with $h(X) \subset g(A) \cup \{0\}$ and $p_A(g - h) \leq 1$. Indeed, there are $x_1, x_2, \ldots, x_n$ in A such that the sets $V_1, \ldots, V_n$, $V_k = \{s \in E : p(s - g(x_k)) \leq 1\}$, are pairwise disjoint and cover $g(A)$. The sets $W_k = g^{-1}(V_k), k = 1, \ldots, n$, are closed in A (and thus in X) and cover A. Since X is ultranormal, there are pairwise disjoint clopen sets $A_1, \ldots, A_n$ in X with $W_k \subset A_k$. Now it suffices to take as h the function $\sum_{k=1}^n \phi_k g(x_k)$, where ϕ_k is the $\mathbb{K}$-characteristic function of A_k.

Let now $f : X \to E$ be such that, for each bounding subset A of X, $f(A)$ is totally bounded and $f|_A$ is continuous. Let (p_n) be an increasing sequence of continuous seminorms on E, generating its topology, and let $A \subset X$ be bounding. Then $\bar{A}$ is bounding. As we have shown above, there exists $g_1 \in C(X, E)$, with $g_1(X) \subset f(\bar{A}) \cup \{0\}$, such that $(p_1)_{\bar{A}}(g_1 - f)) \le 1$. Clearly $(f - g_1)(\bar{A})$ is totally bounded in E. Proceeding by induction, we get a sequence (g_n) in $C(X, E)$ such that, for each n, $(p_n)_{\bar{A}}(h_n - f) \le 1/n$ and $g_n(X) \subset (f - h_{n-1})(\bar{A}) \cup \{0\}$, where $h_n = \sum_{k=1}^{n} g_k$. Clearly $(p_n)_X(g_{n+1}) \le 1/n$. Now, for each $x \in X$, the series $\sum_{n=1}^{\infty} g_n(x)$ converges. Define $h = \sum_{n=1}^{\infty} g_n$. Then $h_n \to h$ uniformly and so h is continuous on X. Also, $h = f$ on A. In fact, given m, we have that

$$(p_m)_A(f - h) \le \max\{(p_m)_A(f - h_n), (p_m)_A(h_n - h)\}.$$

For $n \ge m$, we have $(p_m)_A(f - h_n) \le 1/n$ and

$$(p_m)_A(h_n - h) = (p_m)_A(\sum_{k>n} g_k) \le 1/n.$$

It follows that $(p_m)_A(f - h) = 0$ and so $f = h$ on A since E is Hausdorff.

Consider next the family Φ of all bounding subsets of X. For each $A \in \Phi$ there exists $f_A \in C(X, E)$ such that $f_A = f$ on A. Directing Φ by set inclusion, we get a net $(f_A)_{A \in \Phi}$ in G. It is easy to see that this net is Cauchy in G and hence converges in G to some $g \in C(X, E)$. Since $g(x) = \lim f_A(x) = f(x)$, we have that $f = g$ and the result follows.

Next we look at the dual space of $(C(X, E), \tau_{u,b})$.

Proposition 4.2. *For every non-empty bounding subset A of X, every $p \in cs(E)$ and every $f \in C(X, E)$, there are pairwise disjoint clopen subsets $A_1, \dots, A_n$, covering A, and $x_i \in A_i$ such that $p(f(x) - f(x_i)) \le 1$ for all $x \in A_i$. Thus, for $h = \sum_1^n \phi_i f(x_i)$, where ϕ_i is the $\mathbb{K}$-characteristic function of A_i, we have that $p_A(f - h) \le 1$.*

Proof. Since $f(A)$ is totally bounded, there are $x_1, \dots, x_n$ in A such that

$$f(A) \subset \{f(x_1), \dots, f(x_n)\} + \{s \in E : p(s) \le 1\}.$$

We may assume that the sets $Z_k = \{s : p(s - f(x_k)) \le 1\}, k = 1, \dots, n$, are pairwise disjoint and cover $f(A)$. Now it suffices to take $A_k = f^{-1}(Z_k)$.

Definition A subset A of X is said to be a support of an $m \in M(X, E')$ if $m(U) = 0$ for every clopen set U disjoint from A.

Recall that m is said to be τ-additive (see [11], Definition 3.1) if, for every net (V_α) of clopen sets with $V_\alpha \downarrow \emptyset$, we have that $m(V_\alpha) \to 0$ in the topology $\sigma(E', E)$. If m is τ-additive, then the set

$$\text{supp } m = \bigcap \{V \in K(X) : m(U) = 0 \text{ if } U \cap V = \emptyset\}$$

is a support for m ([11], Theorem 3.5). If in addition X is zero-dimensional, then supp m is the smallest closed support of m. Every tight element of $M(X, E')$ is τ-addditive. Indeed, let $p \in cs(E)$ be such that m_p is tight and let (V_α) be a net of clopen sets with $V_\alpha \downarrow \emptyset$. Given $\epsilon > 0$, there exists a compact subset D of X such that $m_p(V) < \epsilon$ if V is disjoint from D. Since $V_\alpha \downarrow \emptyset$ and D is compact, there exists α_1 such that $D \subset V_{\alpha_1}^c$. If now $\alpha \geq \alpha_1$, then $D \subset V_\alpha^c$ and so $m_p(V_\alpha) < \epsilon$. It is now clear that $m(V_\alpha) \to 0$ weakly in E'.

We say that an $m \in M(X, E')$ has bounding support if one of its support sets is bounding. Now for $p \in cs(E)$, we denote by $M_{b,p}(X, E')$ the space of all $m \in M_p(X, E')$ which have bounding support. Set

$$M_b(X, E') = \bigcup_{p \in cs(E)} M_{b,p}(X, E').$$

Proposition 4.3. *If $m \in M_{b,p}(X, E')$, then every $f \in C(X, E)$ is m-integrable. Moreover, if A is a bounding support of m and if $|\lambda| > 1$, then for every $f \in C(X, E)$ we have*

$$|\int f dm| \leq |\lambda| m_p(X) p_A(f).$$

Thus m defines an element L_m of the dual space of $G = (C(X, E), \tau_{u,b})$, $L_m(f) = \int f dm$. If the valuation of $\mathbb{K}$ is dense or if it is discrete and $p(E) \subset |\mathbb{K}| = \{|\mu| : \mu \in \mathbb{K}\}$, then

$$|\int f dm| \leq m_p(X) p_A(f).$$

Proof. Let $\mu \in \mathbb{K}, \mu \neq 0$. Given $f \in C(X, E)$, there exist $x_1, \ldots, x_n$ in a bounding support A of m and pairwise disjoint clopen sets $A_1, \ldots, A_n$ covering A such that $x_k \in A_k$ and $p(f(x) - f(x_k)) \leq |\mu|$ if $x \in A_k$. Let A_{n+1} be the complement in X of the set $\bigcup_{k=1}^n A_k$ and choose $x_{n+1} \in A_{n+1}$ if $A_{n+1} \neq \emptyset$. If now $\{B_1, \ldots, B_N\}$ is a refinement of $\{A_1, \ldots, A_{n+1}\}$ and if $y_j \in B_j$, then

$$|\sum_{j=1}^N m(B_j) f(y_j) - \sum_{i=1}^n m(A_i) f(x_i)| \leq |\mu| m_p(X).$$

This proves that f is m-integrable over X. Clearly f is m-integrable over every clopen subset of X. Choose now $\gamma \in \mathbb{K}$ with $|\gamma| \leq p_A(f) \leq |\lambda\gamma|$. Given $\epsilon > 0$, there exist (by the above argument) pairwise disjoint clopen sets $A_1, \ldots, A_n$ and $x_k \in A_k \cap A$ such that $| \int f dm - \sum_{i=1}^n m(A_i) f(x_i| < \epsilon$. Since

$$|m(A_i) f(x_i| \leq |\lambda\gamma| m_p(X) \leq |\lambda| p_A(f) m_p(X),$$

we have

$$|m(f)| \leq \max\{\epsilon, |\lambda| p_A(f) m_p(X)\}.$$

Taking $\epsilon \to 0$, we get that $|\int f dm| \leq |\lambda| p_A(f) m_p(X)$. In case of a dense valuation, we get the last assertion by taking $|\lambda| \to 1$. Also, if the valuation is discrete and $p(E) \subset |\mathbb{K}|$, then $p_A(f) = |\varrho|$, for some $\varrho \in \mathbb{K}$. As above we get that $|\int f dm| \leq |\varrho| m_p(X)$, and this completes the proof.

Proposition 4.4. *If $L \in (C(X,E), \tau_{u,b})'$, then there exists $m \in M_b(X,E')$ such that $L(f) = m(f)$ for all $f \in C(X,E)$.*

Proof. Let $p \in cs(E)$ and let A be a closed bounding subset of X such that

$$\{f : p_A(f) \leq 1\} \subset \{f : L(f) \leq 1\}.$$

For each clopen subset D of X, define $m(D)$ on E by $m(D)s = L(\phi_D s)$, where ϕ_D is the $\mathbb{K}$-characteristic function of D. Since $|m(D)s| \leq 1$ if $p(s) \leq 1$, it follows that $m \in M_p(X,E')$ and that $m_p(X) \leq 1$. Moreover, as it is easy to see, $m(D) = 0$ if D is disjoint from A. Since now both L and L_m are $\tau_{u,b}$-continuous. it follows that $L = L_m$ since they coincide on a $\tau_{u,b}$-dence subset of $C(X,E)$ (by Proposition 4.2). This clearly completes the proof.

Combining Propositions 4.3 and 4.4, we get the following

Theorem 4.5. *The mapping $m \mapsto L_m$, from $M_b(X,E')$ to the dual space of $(C(X,E), \tau_{u,b})$, is an algebraic isomorphism.*

The next Theorem characterizes the equicontinuous subsets of the dual space of $G = (C(X,E), \tau_{u,b})$.

Theorem 4.6. *For a subset H of the dual space $M_b(X,E')$ of G, the following are equivalent:*
(1) H is equicontinuous.
(2) (a) There exists $p \in cs(E)$ such that $\sup_{m \in H} m_p(X) < \infty$.
(b) There exists a bounding subset A of X such that, for every $m \in H$ and every clopen subset D of X disjoint from A, we have $m(D) = 0$.

Proof. If H is equicontinuous, then there exists $p \in cs(E)$ and a bounding subset A of X such that $\{f : p_A(f) \leq 1\} \subset H^o$. It is easy to see that, for all $m \in H$ and all D disjoint from A, we have $m(D) = 0$ and $m_p(X) \leq 1$. Conversely, assume that (2) is satisfied. We may assume that $m_p(X) \leq 1$ for all $m \in H$. Let now $f \in C(X,E)$ with $p_A(f) \leq 1$. The set $D = \{x : p(f(x)) \leq 1\}$ is clopen and contains A. Now, for $m \in H$, we have $|\int f dm| = |\int_D f dm| \leq 1$ and so $f \in H^o$. This completes the proof.

5. The Strict Topology β_b

In this section we will introduce the strict topology β_b on $C(X,E)$. It will turn out that β_b is the finest of all Nachbin topologies ω_V such that $CV_0(X,E) = C(X,E)$ (algebraically). We will need some preliminary results.

Lemma 5.1. *Assume that E is non-trivial. Let v be a non-negative function on X and consider the following properties:*
(1) $p_v(f) < \infty$ for every $f \in C(X,E)$ and every $p \in cs(E)$.
(2) $\omega_v(f) < \infty$ for every $f \in C(X)$.
(3) $A_v = \{x \in X : v(x) \neq 0\}$ is a bounding subset of X and v is bounded on X.
(4) For every $f \in C(X,E)$ and every $p \in cs(E)$, $p_v(f) < \infty$ and the function $x \mapsto v(x)p(f(x))$ vanishes at infinity.
(5) For each $g \in C(X)$, we have that $\omega_v(g) < \infty$ and the function $x \mapsto v(x)|g(x)|$ vanishes at infinity.
(6) v is bounded, A_v is bounding and v vanishes at infinity.
Then (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) *and* (4) $\Leftrightarrow$ (5) $\Leftrightarrow$ (6).

Proof. Clearly (1) implies (2).

(2) $\Rightarrow$ (3) Taking as f the constant function 1, we get that v is bounded. Assume that A_v is not bounding and let $g \in C(X)$ be not bounded on A_v. Then, there exists a sequence (λ_n) of non-zero elements of $\mathbb{K}$, with $|\lambda_n| \to \infty$, and $x_n \in A_v$ such that $|\lambda_n| < |g(x_n)| < |\lambda_{n+1}|$ for all n. Let $W_n = \{x : |\lambda_n| \leq |g(x)| < |\lambda_{n+1}|\}$. Let $|\lambda| > 1$ and choose, for each n, a $\mu_n \in \mathbb{K}$ such that $|\mu_n| \leq v(x_n) < |\lambda\mu_n|$. Take $f = \sum_{n=1}^{\infty} \mu_n^{-1}\lambda_n\phi_n$, where ϕ_n is the $\mathbb{K}$-characteristic function of W_n. Then, f is continuous and $v(x_n)|f(x_n)| \geq |\lambda_n|$, and so $\|f\|_v = \infty$, a contradiction.

(3) $\Rightarrow$ (1). Let $f \in C(X,E)$ and $p \in cs(E)$. Since A_v is bounded, there exists $d > \sup_{x\in A_v} p(f(x))$. Now $p_v(f) \leq d\|v\|$. This proves the equivalence of $(1), (2), (3)$.

Next we observe that (4) implies (5). Also, it is easy to see that (5) implies (6). Finally, assume that (6) holds. From the equivalence of (3) and (1), we get that $p_v(f) < \infty$ for each $p \in cs(E)$ and each $f \in C(X,E)$. If $d > \sup_{x\in A_v} p(f(x))$, choose a compact set D such that $v(x) < \epsilon/d$ if x is not in D. Now for $x \notin D$ we have that $v(x)p(f(x)) < \epsilon$. This completes the proof.

Lemma 5.2. *If v is a bounded non-negative u.s.c. function on X and $0 < |\lambda| < 1$, then there exists $\phi : X \to \mathbb{K}$ bounded such that $|\phi|$ is u.s.c. and $|\phi| \leq v \leq |\lambda^{-1}\phi|$. If v vanishes at infinity, so does ϕ.*

Proof. We may assume that $\|v\| < |\lambda|$. Set

$$D_n = \{x : v(x) \geq |\lambda^n|\}, \quad A_n = D_n \setminus D_{n-1},$$

and let ϕ_n be the $\mathbb{K}$-characteristic function of A_n. Let $\phi = \sum_{n=1}^{\infty} \lambda^n\phi_n$. Then $|\phi|$ is u.s.c.. Indeed, for ϵ real, set $B_\epsilon = \{x : |\phi(x)| \geq \epsilon\}$. If $\epsilon > |\lambda|$, then $B_\epsilon = \emptyset$, while for $\epsilon \leq 0$ we have $B_\epsilon = X$. If $0 < \epsilon \leq |\lambda|$, there exists positive integer n such that $|\lambda|^{n+1} < \epsilon \leq |\lambda|^n$. It is easy to see that $B_\epsilon = D_n$. This proves that $|\phi|$ is u.s.c.. Let now $x \in X$. If $\phi(x) \neq 0$, then $x \in A_n$ for some n, and so $|\phi(x)| = |\lambda|^n \leq v(x)$. Also, $x \notin D_{n-1}$ and so $v(x) < |\lambda|^{n-1}$, which implies that $v(x) \leq |\lambda^{-1}\phi(x)|$. In case $\phi(x) = 0$ we have $v(x) = 0$. This proves that $|\phi| \leq v \leq |\lambda^{-1}\phi|$. It is also clear that $|\phi|$ vanishes at infinity when v does.

COROLLARY 5.3. *If V is a Nachbin family consisting of bounded functions, then there exists a family Φ of bounded $\mathbb{K}$-valued functions on X such that $|\Phi| = \{|\phi| : \phi \in \Phi\}$ is a Nachbin family equivalent to V.*

LEMMA 5.4. *Let $S_0(X)$ be the family of all $\mathbb{K}$-valued functions ϕ on X such that $|\phi|$ is u.s.c., vanishes at infinity and has bounding support. Then $|S_0(X)|$ is a Nachbin family on X*

PROOF. For ϕ_1, ϕ_2 in $S_0(X)$, let

$$\phi : X \to \mathbb{K}, \quad \phi(x) = \begin{cases} \phi_1(x) + \phi_2(x) & \text{if } |\phi_1(x)| \neq |\phi_2(x)| \\ \phi_1(x) & \text{otherwise} \end{cases}$$

is in $S_0(X)$ and $|\phi| = \max\{|\phi_1|, |\phi_2|\}$. It is easy to see that $|S_0(X)|$ is a Nachbin family.

Using the preceeding Lemmas, we get the following

PROPOSITION 5.5. *Assume that E is non-trivial and let V be a Nachbin family on X The following are equivalent:*
(1) $CV_0(X, E) = C(X, E)$ *(algebraically).*
(2) $CV_0(X) = C(X)$ *(algebraically).*
(3) $V \leq |S_0(X)|$.

By the preceeding Propositions, $|S_0(X)|$ is the finest (up to equivalence) of all Nachbin families V on X such that $CV_0(X, E) = C(X, E)$ algebraically.

DEFINITION. The strict topology on $C(X, E)$ is the locally convex topology β_b generated by the seminorms $p_\phi, \phi \in S_0(X), p \in cs(E)$, where

$$p_\phi(f) = \sup\{|\phi(x)|p(f(x)) : x \in X\}.$$

PROPOSITION 5.6. *If E is a polar space, then β_b is a polar topology.*

PROOF. Let $\phi \in S_0(X)$, p a polar seminorm on E and $f \in C(X, E)$. If $p_\phi(f) > \theta > 0$, then $p(\phi(x)f(x)) > \theta$ for some $x \in X$. Since p is polar, there exists $u \in E', |u| \leq p$, such that $|u(\phi(x)f(x))| > \theta$. The function $\omega : C(X, E) \to \mathbb{K}, \omega(g) = u(\phi(x)g(x))$ is linear and $|\omega| \leq p_\phi$. Moreover $|\omega(f)| > \theta$. This proves that p_ϕ is polar.

PROPOSITION 5.7. *Let G be the space spanned by the functions gs, where g is a characteristic function of a clopen subset of X and $s \in E$. Then G is β_b-dense in $C(X, E)$.*

PROOF. Let $f \in C(X, E), \phi \in S_0(X), p \in cs(E)$. Without loss of generality we may assume that $\|\phi\| \leq 1$. Since the function ϕf vanishes at infinity, there exists a compact subset D of X such that $p(\phi(x)f(x)) \leq 1$ if $x \notin D$. By the compactness of D, there are $x_1, \ldots, x_n \in D$ and pairwise disjoint clopen sets $A_1, \dots, A_n$ covering D such that $p(f(x) - f(x_i)) \leq 1$ if $x \in A_i$ Let now g_i be the $\mathbb{K}$-characteristic function of A_i and let $h = \sum_{i=1}^n g_i f(x_i)$. Then $p_\phi(f - h) \leq 1$. This completes the proof.

For $p \in cs(E), \beta_{b,p}$ (resp. $\tau_{u,b,p}$) is the topology on $C(X, E)$ generated by the seminorms $p_\phi, \phi \in S_0(X)$ (resp. by p_A, A a bounding subset of X). Analogously, $\tau_{c,p}$ is the topology generated by the seminorms p_A, A a compact subset of X. Clearly a subset of $C(X, E)$ is a β_b-neighborhood of zero iff it is a $\beta_{b,p}$-neighborhood for some $p \in cs(E)$. Analogous properties have the topology $\tau_{u,b}$ and the toplogy τ_c of compact convergence.

For a sequence (K_n) of compact subsets of X and a sequence (d_n) of positive numbers, we denote by $W_p(K_n, d_n)$ the set $\bigcap_{n=1}^\infty \{f \in C(X, E) : p_{K_n}(f) \leq d_n\}$. The proof of the following Proposition is analogous to the one of Proposition 2.6 in [9].

Proposition 5.8. *The collection of all sets of the form $W_p(K_n, |\lambda_n|)$, where $0 < |\lambda_n| < |\lambda_{n+1}|, |\lambda_n| \to \infty$, (K_n) an increasing sequence of compact subsets of X such that $\bigcup K_n$ is bounding in X, is a base at zero for the topology $\beta_{b,p}$.*

Proposition 5.9. *An absolutely convex subset W of $C(X, E)$ is a $\beta_{b,p}$-neighborhood of zero iff the following is satisfied: There exists a bounding subset A of X such that, for each $d > 0$, there is a compact subset D of A and $\delta > 0$ such that $V \cap W_d \subset W$, where*

$$V = \{f : p_D(f) \leq \delta\}, \quad W_d = \{f : p_A(f) \leq d\}.$$

Proof. Assume that W is a $\beta_{b,p}$-neighborhood of zero. We may assume that $W = \{f : p_\phi(f) \leq 1\}$ for some $\phi \in S_0(X)$. Let A be the bounding support of ϕ. Given $d > 0$, choose $n > \max\{d, \|\phi\|\}$. There exists a compact subset D of X such that $|\phi(x)| \leq 1/n$ if $x \notin D$. Taking $D \cap A$ instead of D, we may assume that $D \subset A$. (Note that $A = \overline{\{x : \phi(x) \neq 0\}}$). If now $G = \{f : p_D(f) \leq 1/n\}$, then $W_d \cap G \subset W$. Conversely, assume that the condition is satisfied for some bounding subset A of X. Let $|\lambda| > 1$ and let $V = \{f : p_A(f) \leq 1\}$. There exist a decreasing sequence (δ_n) of positive numbers and an increasing sequence (K_n) of compact subsets of X contained in A such that $V_n \bigcap \lambda^n V \subset W$, where $V_n = \{f : p_{K_n}(f) \leq \delta_n\}$. Set

$$W_1 = V_1 \bigcap (\bigcap_{n=1}^\infty (V_{n+1} + \lambda^n V).$$

With an argument analogous to the one used in [9], Theorem 2.8, we show that $W_1 \subset W$. Also, if $0 < |\lambda_1| < \min\{1, \delta_1\}$ and $\lambda_n = \lambda^{n-1}$ for $n > 1$, we show that $W_p(K_n, |\lambda_n|) \subset W_1$. Thus the result follows from Proposition 5.8.

By the next Proposition, β_b agrees with τ_c on $\tau_{u,b}$-bounded sets.

Proposition 5.10. *(1)* $\tau_c \leq \beta_b \leq \tau_{u,b}$.
(2) $\beta_b = \tau_c$ *on* $\tau_{u,b}$*-bounded sets.*

Proof. (1) It is obvious.
(2) Let H be $\tau_{u,b}$-bounded. We want to show that $\beta_b = \tau_c$ on H. We may assume that H is absolutely convex. It is then enough to show that every $\beta_{b,p}$-neighborhood of zero in H is also a $\tau_{c,p}$-neighborhood. So, let W be a $\beta_{b,p}$- neighborhood of zero in $C(X,E)$. There exists $\phi \in S_0(X)$ such that $W_1 = \{f : p_\phi(f) \leq 1\} \subset W$. Since H is $\tau_{u,b}$-bounded, there exists $d > 0$ such that $H \subset \{f : p_A(f) \leq d\}$ where $A = supp\phi$. By the preceeding Proposition, there exists $\delta > 0$ and a compact set D such that

$$\{f \in C(X,E) : p_A(f) \leq d\} \bigcap \{f : p_D(f) \leq \delta\} \subset W_1$$

and so $H \bigcap \{f : p_D(f) \leq \delta\} \subset W_1$. This completes the proof

As the following Proposition states, the topologies β_b and $\tau_{u,b}$ have the same bounded sets.

Proposition 5.11. β_b *and* $\tau_{u,b}$ *have the same bounded sets.*

Proof. Assume that a subset H of $C(X,E)$ is β_b-bounded but not $\tau_{u,b}$-bounded. Let $p \in cs(E)$ and A a bounding subset of X such that $\sup\{p_A(f) : f \in H\} = \infty$. For $|\lambda| > 1$ we choose inductively a sequence (f_n) in H and a sequence (x_n) in A such that $p(f_1(x_1)) > |\lambda^2|$ and

$$p(f_k(x_k)) > \max\{|\lambda|^{2k}, \sup\{p(f(x_i)) : f \in H, 1 \leq i < k\}\}$$

for $k > 1$. Let ϕ_n be the $\mathbb{K}$-charascteristic function of $\{x_1, \ldots, x_n\}$ and set $\phi = \sum_{n=1}^{\infty} \lambda^{-n}\phi_n$. It is easy to see that $\phi \in S_0(X)$. Since $|\phi(x_n)| = |\sum_{k=n}^{\infty} \lambda^{-k}| = |\lambda|^{-n}$, we have that $p(\phi(x_n)f_n(x_n)) \geq |\lambda|^n$ and so $\sup_{f\in H} p_\phi(f) = \infty$, a contradiction.

Since β_0 is defined on $C_b(X,E)$ by the seminorms p_ϕ, $p \in cs(E)$ and ϕ a $\mathbb{K}$-valued function on X such that $|\phi|$ is u.s.c and vanishes at infinity (see [12]), is clear that β_0 is finer than the topology induced on $C_b(X,E)$ by β_b. The next Proposition refers to the question of when these two topologies coincide on $C_b(X,E)$.

Proposition 5.12. *If* X *is zero-dimensional, then the following are equivalent:*
(1) $C_b(X,E) = C(X,E)$.
(2) $C_b(X) = C(X)$.
(3) $\upsilon_0 X$ *is compact.*
(4) Every countable subset of X *is bounding.*
(5) $\beta_b = \beta_0$ *on* $C_b(X,E)$.

Proof. By Proposition 3.1, (1), (2) and (3) are equivalent. Also it is clear that (3) implies (4) and it is easy to see that (4) implies (2).

(2) $\Rightarrow$ (5) It follows from the definitions of β_b and β_0 since (by (2)) every subset of X is bounding.
(5) $\Rightarrow$ (4). Let (x_n) be a sequence of distinct elements of X and let ϕ_n be the $\mathbb{K}$-characteristic function of $\{x_1, \dots, x_n\}$ Take $0 < |\lambda| < 1$ and consider the function $\phi = \sum_{n=1}^{\infty} \lambda^n \phi_n$. Then $|\phi|$ is u.s.c. and vanishes at infinity. Thus, if $p \in cs(E)$, then $W = \{f \in C_b(X, E) : p_\phi(f) \leq 1\}$ is a β_0-neighborhood of zero. By our hypothesis, there exists $q \in cs(E)$ and $\omega \in S_0(X)$ such that

$$V = \{f \in C_b(X, E) : q_\omega(f) \leq 1\} \subset W.$$

Since X is zero-dimensional, is easy to see that every x_n is in the support A of ω. This completes the proof.

Proposition 5.13. *If every bounding subset of X is relatively compact, then $\beta_b = \tau_{u,b}$. The converse is also true if X is Hausdorff and zero-dimensional.*

Proof. The condition is clearly sufficient since, in this case, $\tau_c = \tau_{u,b}$. Conversely, assume that $\beta_b = \tau_{u,b}$ and that X is zero-dimensional. Let A be a bounding subset of X and choose a non-zero $p \in cs(E)$. By our hypothesis, there exist $\phi \in S_0(X)$ and $q \in cs(E)$ such that

$$\{f : q_\phi(f) \leq 1\} \subset Z = \{f : p_A(f) \leq 1\}.$$

Choose $s \in E$ with $p(s) > 1$ and a non-zero $\mu \in \mathbb{K}$ with $q(\mu s) \leq 1$. There exists a compact subset D of X such that $|\phi(x)| < |\mu|$ if $x \notin D$. Now $A \subset D$. If this is not the case, then there exists a clopen neighborhood V of an element of A which is disjoint from D. If now ψ is the $\mathbb{K}$-characteristic function of V, then $f = \psi s$ is not in Z which is a contradiction since $q_\phi(f) \leq 1$.

Proposition 5.14. *If every bounding σ-compact subset of X is relatively compact, then $\beta_b = \tau_c$. The converse is also true if we assume that X is Hausdorff and zero-dimensional.*

Proof. Assume that the condition is sastisfied and let W be $\beta_{b,p}$-neighborhood of zero. There exist an increasing sequence (K_n) of compact subsets of X, such that $A = \bigcup K_n$ is bounding, and an increasing sequence (d_n) of positive real numbers, with $d_n \to \infty$, such that $W_p(K_n, d_n) \subset W$. By our hypothesis, $\bar{A}$ is compact and

$$\{f : p_{\bar{A}}(f) \leq d_1\} \subset W_p(K_n, d_n).$$

Conversely, let $\beta_b = \tau_c$ and assume that X is zero-dimensional. Let (A_n) be a sequence of compact subsets of X such that $A = \bigcup A_n$ is bounding. We may assume that (A_n) is increasing. Let $|\lambda| > 1$ and set $V = W_p(A_n, |\lambda|^n)$. Since V is a β_b-neighborhood of zero, there exists (by our hypothesis) a compact subset Z of X and $q \in cs(E)$ such that $\{f : q_Z(f) \leq 1\} \subset V$. Now $A \subset Z$ and the result follows.

We get easily the following

PROPOSITION 5.15. *If β_b is bornological or barrelled, then $\beta_b = \tau_{u,b}$.*

PROPOSITION 5.16. *If X is Hausdorff and zero dimensional, then the following are equivalent:*
(1) β_b is metrizable.
(2) E is metrizable, every bounding subset of X is relatively compact and there exists a fundamental sequence (K_n) of compact subsets of X, i.e. every compact subset of X is contained in some K_n.

PROOF. (1) $\Rightarrow$ (2). Let (ϕ_n) be a sequence in $S_0(X)$ and let (p_n) be an increasing sequence of continuous seminorms on E such that the sets $W_n = \{f : (p_n)_{\phi_n}(f) \leq 1\}, n = 1, 2, \ldots$, is a β_b-base at zero. It is easy see that the topology of E is generated by the sequence of seminorms (p_n) and so E is metrizable. Also, by the preceeding Proposition, $\beta_b = \tau_{u,b}$ and so every bounding subset of X is relatively compact, which implies that $\tau_c = \beta_b$. Let now (q_n) be a sequence of continuous seminorms on E and (D_n) an increasing sequence of compact subsets of X such that the sets $Z_n = \{f : (q_n)_{D_n}(f) \leq 1\}, n = 1, 2, \ldots$, is a base at zero for $\tau_c = \beta_b$. It is now easy to show that every compact subset of X is contained in some D_n.

(2) $\Rightarrow$ (1). Let (p_n) be an increasing sequence of continuous seminorms on E, generating its topology, and let (K_n) be an increasing fundamental sequence of compact subsets of X. Set $O_n = \{f : (p_n)_{K_n}(f) \leq 1/n\}$. Then (O_n) is a base at zero for τ_c. Since our hypothesis and Proposition 5.13 imply that $\tau_c = \tau_{u,b} = \beta_b$, the result follows.

We look next at the question of when the space $(C(X,E), \beta_b)$ is a semi-Montel space (SM-space). We need the following Lemma whose proof is analogous to the one of the Lemma 2.1 in [16].

LEMMA 5.17. *A subset H of $C(X,E)$ is β_b-compactoid iff it is $\tau_{u,b}$-bounded and τ_c-compactoid.*

PROPOSITION 5.18. *If X is Hausdorff and zero-dimensional, then the following are equivalent:*
(1) $(C(X), \tau_c)$ is an SM-space.
(2) $(C(X), \tau_c)$ is nuclear.
(3) Every compact subset of X is finite.
(4) $(C(X), \beta_b)$ is an SM-space.

PROOF. The equivalence of (1),(2), (3) is proved in [6], Proposition 3.2.

(1) $\Rightarrow$ (4). It follows from the preceeding Lemma since β_b and $\tau_{u,b}$ have the same bounded sets.

(4) $\Rightarrow$ (1). Let D be an absolutely convex subset of $C(X)$, which is τ_c-bounded, M a compact subset of X and $d > 0$. Set $W = \{f : \omega_M(f) \leq d\}$ and let $|\mu| \geq \sup_{f \in D} \omega_M(f)$. For each $f \in D$, set $V_f = \{x : |f(x)| \leq |\mu|\}$ and let $g_f = \phi_f f$, where ϕ_f is the $\mathbb{K}$-characteristic function of V_f. The set $H = \{g_f :$

$f \in D\}$ is $\tau_{u,b}$-bounded and hence β_b-bounded. By our hypothesis, H is β_b-compactoid and hence τ_c-compactoid. Thus, for $|\lambda| > 1$, there are $f_1, \ldots, f_n$ in D such that $H \subset \lambda co(g_{f_1}, \ldots, g_{f_n}) + W$. Now $D \subset \lambda co(f_1, \ldots, f_n) + W$ and so D is τ_c-compactoid. This completes the proof.

The following Theorem is analogous to Theorem 2.5 in [16] which refers to β_0.

Theorem 5.19. *If X is Hausdorf and zero-dimensional, then the following are equivalent:*
(1) E is an SM-space and every compact subset of X is finite.
(2) $(C(X), \tau_c)$ and E are SM-spaces.
(3) $(C_b(X), \beta_0)$ and E are SM-spaces.
(4) $(C(X,E), \tau_c)$ is an SM-space.
(5) $(C_b(X,E), \beta_0)$ is an SM-space.
(6) $(C(X), \beta_b)$ and E are SM-spaces.
(7) $(C(X,E), \beta_b)$ is an SM-space.

Proof. By [16], Theorem 2.5, (1) - (5) are equivalent.
(4) $\Rightarrow$ (7) It follows from Lemma 5.17 since $\tau_{u,b}$ and β_b have the same bounded sets.
(7) $\Rightarrow$ (6) It is a consequence of the fact that both E and $(C(X), \beta_b)$ are topologially isomorphic to certain subspaces of $(C(X,E), \beta_b)$.
Finally (6) is equivalent to (2) in view of the preceeding Proposition.

Concerning the nuclearity of $(C(X,E), \beta_b)$, we have the following

Theorem 5.20. *$(C(X,E), \beta_b) = G$ is nuclear iff both $(C(X), \beta_b)$ and E are nuclear.*

Proof. Assume that G is nuclear. Since E is topologically isomorphic to a subspace of G and since G is polar, it follows that E is polar. Since, for $V = |S_0(X)|, (C(X,E), \beta_b) = CV_0(X,E)$ and $(C(X), \beta_b) = CV_0(X)$, and since $CV_0(X) \otimes E$ is topologically isomorphic to a dense subspace M of $CV_0(X,E)$ (by [13], Proposition 4.2), it follows that $(C(X), \beta_b) \otimes E$ is topologically isomorphic to a dense subspace of $(C(X,E), \beta_b)$. Now the result folows from the fact that a dense subspace, of a locally convex space H, is nuclear iff H is nuclear and from the fact that the projective tensor product of two locally convex spaces is nuclear iff each of the two spaces is nuclear ([6], Theorem 2.10).

6. The Dual Space of $(C(X,E), \beta_b)$

For p a continuous seminorm on E, let $M_{t,b,p}(X, E')$ denote the space of all $m \in M_p(X, E')$ with the property that there exists a bounding subset A of X such that: (1) A is a support set for m. (2) For each $\epsilon > 0$ there exists a compact subset D of A such that $m_p(V) < \epsilon$ for each clopen set V disjoint from D.

Proposition 6.1. *If $m \in M_{t,b,p}(X, E')$, then:*
(a) Every $f \in C(X, E)$ is m-integrable.
(b) The linear map $L_m : C(X, E) \to \mathbb{K}$, $L_m(f) = \int f dm = m(f)$ is $\beta_{b,p}$-continuous.

Proof. Let A be a bounding subset of X such that (1) and (2) above hold. Without loss of generality, we may assume that $m_p(X) \leq 1$. Let $d > 0$ and let $f \in C(X, E)$ with $p_A(f) \leq d$. Without loss of generality, we may assume that $d = |\gamma|$ for some $\gamma \in \mathbb{K}$. Given $\mu \in \mathbb{K}, \mu \neq 0$, choose a compact subset D of A such that $m_p(V) < |\mu| \ /d$ if V is disjoint from D. The set $Z = \{x : p(f(x)) \leq d\}$ is clopen and contains A. By the compactness of D, there are $x_1, \ldots, x_n$ and pairwise disjoint clopen sets $A_1, \ldots, A_n$, contained in Z and covering D, such that $x_i \in A_i \cap D$ and $p(f(x) - f(x_i)) < |\mu|$ if $x \in A_i$. Let

$$A_{n+1} = Z \cap A_1^c \cap \ldots \cap A_n^c \quad \text{and} \quad A_{n+2} = \left(\bigcup_{k=1}^{n+1} A_k \right)^c .$$

Choose $x_{n+1} \in A_{n+1}$ and $x_{n+2} \in A_{n+2}$ if these sets are non-empty (if one of these sets is empty, we leave it out). If now $\{B_1, \ldots, B_N\}$ is a clopen partition of X which is a refinement of $\{A_1, \ldots . A_{n+2}\}$ and if $y_j \in B_j$, then

$$|\sum_{j=1}^{N} m(B_j) f(y_j) - \sum_{i=1}^{n+2} m(A_i) f(x_i)| \leq |\mu|.$$

This proves that $\int f dm$ exists. If moreover $p_D(f) \leq 1$, then

$$|\sum_{i=1}^{n} m(A_i) f(x_i)| \leq 1$$

and this implies that $|\int f dm| \leq \max\{|\mu|, 1\}$. Taking $0 < |\mu| < 1$ we get that

$$\{f : p_A(f) \leq d, p_D(f) \leq 1\} \subset W = \{f : |\int f dm| \leq 1\}.$$

This (by Propositioon 5.9) implies that L_m is $\beta_{b,p}$-continuous.

Set

$$M_{t,b}(X, E') = \bigcup_{p \in cs(E)} M_{t,b,p}(X, E').$$

By the preceeding Proposition, every $m \in M_{t,b}(X, E')$ defines a β_b-continuous linear functional L_m on $C(X, E)$. By the next Proposition, every β_b-continuous linear functional on $C(X, E)$ is of the form L_m for some $m \in M_{t,b}(X, E')$.

Proposition 6.2. *If L is a β_b-continuous linear functional on $C(X,E)$, then $L = L_m$ for some $m \in M_{t,b}(X,E')$.*

Proof. The restriction of L to $C_b(X,E)$ is β_0-continuous. Thus, by [9], Theorem 3.4, there exists $m \in M(X,E')$ such that $L(f) = m(f)$ for all f in $C_b(X,E)$. Let $p \in cs(E)$ and $\phi \in S_0(X))$ be such that

$$W = \{f \in C(X,E) : p_\phi(f) \leq 1\} \subset \{f : |L(f)| \leq 1\}.$$

If V is clopen, $p(s) \leq 1$, $|\mu| \geq \|\phi\|$ and if g is the $\mathbb{K}$-characteristic function of V, then $f = \mu^{-1} g s \in W$ and so $|m(V)s| \leq |\mu|$. Thus $m_p(X) \leq |\mu|$. Also, it is easy to see that $m(V) = 0$ if V is disjoint from the support A of ϕ. Next we observe that, for $\gamma \neq 0$, there exists a compact set D such that $|\phi(x)| < |\gamma|$ if $x \notin D$. We may take D contained in A. It is now easy to see that, for V disjoint from D, we have $m_p(V) \leq |\gamma|$. This proves that $m \in M_{t,b}(X,E')$. Now, since for $f = gs$, g a characteristic function of a clopen set, we have that $L(f) = m(f)$, it folows that $L = L_m$ by Proposition 5.7 since both L and L_m are β_b-continuous.

Combining Propositions 6.1 and 6.2, we have the following

Theorem 6.3. *The map $m \mapsto L_m$, from $M_{t,b}(X,E')$ to the dual space of $(C(X,E),\beta_b)$ is an algebraic isomorphism.*

Proposition 6.4. *A subset H of the dual space $M_{t,b}(X,E')$ of $(C(X,E),\beta_b) = G$ is $\beta_{b,p}$-equicontinuous iff $\sup_{m\in H} m_p(X) < \infty$ and there exists a bounding subset A of X, which is a common support for all $m \in H$, such that for every $\epsilon > 0$ there exists a compact subset D of A with $m_p(V) < \epsilon$ for all $m \in H$ and all clopen V disjoint from D.*

Proof. Assume that H is $\beta_{b,p}$-eqicontinuous and let $\phi \in S_0(X)$ be such that $W = \{f : p_\phi(f) \leq 1\} \subset H^0$. It is easy to see that $m_p(X) \leq \|\phi\|$ for all $m \in H$ and that the support A of ϕ is a support set for every $m \in H$. Let now $\mu \neq 0$ and let D be a compact subset of X such that $|\phi(x)| < |\mu|$ if $x \notin D$. Clearly we may take $D \subset A$. Conversely, assume that the condition is satisfied. Without loss of generality, we may assume that $m_p(X) \leq 1$ for every $m \in H$. Let now $d > 0$ and choose μ with $|\mu| \geq d$. Let D be a compact subset of A such that $m_p(V) < |\mu|^{-1}$, for all $m \in H$, if V is disjoint from D and let

$$Z = \{f : p_D(f) \leq 1, p_A(f) \leq d\}$$

Let $f \in Z$ and set

$$U = \{x : p(f(x)) \leq 1\}, V = \{x : p(f(x)) \leq |\mu|\}.$$

For $m \in H$, we have $|\int_{U\cap V} f dm| \leq 1$ and $|\int_{V\cap U^c} f dm| \leq 1$ and so $|m(f)| = |\int_V f dm| \leq 1$. Now the result follows from Proposition 5.9.

Proposition 6.5. *Let X be locally compact zero-dimensional and let $m \in M_p(X, E')$ with a closed bounding support A such that m_p is tight. Then $m \in M_{t,b}(X, E')$.*

Proof. Given $\epsilon > 0$, there exists a compact subset D of X such that $m_p(V) < \epsilon$ if V is disjoint from D. Since X is locally compact and zero-dimensional, there exists a clopen compact set Y containing D. We will finish the proof by showing that $m_p(V) < \epsilon$ for every clopen set V disjoint from $Y \cap A$. So let V be such a set. Since $V \cap Y$ is disjoint from A, we have that $m_p(V \cap Y) = 0$. Thus, $m_p(V) = m_p(V \cap Y^c) < \epsilon$. This completes the proof.

The following Proposition will be needed in the next section.

Proposition 6.6. *Let H be a subset of $M(X, E')$ consisting of measures which are τ-additive, have a bounding support and with respect to which every $f \in C(X, E)$ is integrable. If the set*

$$S(H) = \overline{\bigcup_{m \in H} \textit{supp } m}$$

is not bounding, then for every sequence (a_n) in $\mathbb{K}$ there exist $f \in C(X, E)$ and a sequence (m_n) in H such that $m_n(f) = a_n$ for all n.

Proof. Let $g \in C(X)$ be not bounded on $S(H)$. Let $|\lambda_1| > 1$. The set $A = \{x : |g(x)| > |\lambda_1|\}$ must intersect the set $D = \bigcup_{m \in H} \text{supp } m$. Hence there exists $m_1 \in H$ for which supp m_1 intersects A. Let $|\lambda_2| > \max\{2, |\lambda_1|\}$ be such that supp $m_1 \subset \{x : |g(x)| < |\lambda_2|\}$. Now there exists a clopen set U_1 contained in $\{x : |\lambda_1| < |g(x)| < |\lambda_2|\}$ and $s_1 \in E$ with $m_1(U_1)s_1 = 1$. Assume that we have already chosen $m_1, \dots, m_n$ in H, clopen sets $U_1, \dots, U_n$, $\lambda_1, \dots, \lambda_{n+1}$ in $\mathbb{K}$ and $s_1, \dots, s_n$ in E. There exist $m_{n+1} \in H$, $\lambda_{n+2} \in \mathbb{K}$ with $|\lambda_{n+2}| > \max\{n+2, |\lambda_{n+1}|\}$, a clopen set U_{n+1} contained in $\{x : |\lambda_{n+1}| < |g(x)| < |\lambda_{n+2}|\}$ and $s_{n+1} \in E$ such that supp $m_{n+1} \subset \{x : |g(x)| < |\lambda_{n+2}|\}$ and $m_{n+1}(U_{n+1})s_{n+1} = 1$. Let (γ_n) be any sequence in $\mathbb{K}$ and consider the function $f = \sum_{n=1}^{\infty} \gamma_n \phi_n s_n$ where ϕ_n is the $\mathbb{K}$-characteristic function of U_n. It is easy to see that f is continuous. Moreover,

$$m_n(f) = m_n\Big(\sum_{k=1}^{n} \gamma_k \phi_k s_k\Big) = \sum_{k=1}^{n} \gamma_k m_n(U_k) s_k.$$

Thus

$$m_1(f) = \gamma_1 m_1(U_1) s_1 = \gamma_1, \quad m_{n+1}(f) = \sum_{k=1}^{n} \gamma_k m_{n+1}(U_k) s_k + \gamma_{n+1}.$$

It is now clear that we can choose (γ_n) so that $m_n(f) = a_n$ for all n.

7. The Case of a Normed Space E

In this section we assume that E is a non-Archimedean normed space. For $f \in C(X, E)$, we set

$$B_f = \{g \in C(X, E) : \|g(x)\| \leq \|f(x)\| \ \text{ for all } \ x \in X\}.$$

Cearly B_f is $\tau_{u,b}$-bounded.

Proposition 7.1. *Let L be a linear functional on $C(X, E)$ such that $L|_{B_f}$ is τ_c-continuous for every $f \in C(X, E)$. Then, there exists a tight element m of $M(X, E')$, with bounding support, such that $L(f) = m(f)$ for all $f \in C(X, E)$.*

Proof. For n a positive integer, let $s \in E$ with $\|s\| \geq n$. If $f(x) = s$ for all $x \in X$, then $D_n = \{g : \|g\| \leq n\} \subset B_f$. Thus $L|_{D_n}$ is τ_c-continuous. Since, for E a normed space, β_0 is the finest locally convex topology on $C_b(X, E)$ which coincides with τ_c on the sets D_n (by [9], Corollary 2.9), it follows that L is β_0-continuous on $C_b(X, E)$ and hence there exists $m \in M_p(X, E)$ (for some $p \in cs(E)$) such that m_p is tight and $L(f) = m(f)$ when $f \in C_b(X, E)$ ([9], Theorem 3.4).
Claim I: supp m is bounding. Indeed, assume the contrary and let $g \in C(X)$ be not bounded on $A = \text{supp } m$. There exists a sequence (λ_n) in $\mathbb{K}$ such that $0 < |\lambda_1| < |\lambda_2| < \ldots < |\lambda_n| \to \infty$ and $A \cap A_k \neq \emptyset$ where $A_k = \{x : |\lambda_k| \leq |g(x)| < |\lambda_{n+1}|\}$. There exist a clopen subset V_k, contained in A_k, and $s_k \in E$, with $\|s_k\| \leq 1$ and $m(V_k)s_k = \mu_k \neq 0$. Set $f = \sum_{k=1}^{\infty} \mu_k^{-1} \lambda_k \phi_k s_k$, where ϕ_k is the $\mathbb{K}$-characteristic function of V_k. Then, f is continuous. Moreover, if $f_n = \sum_{k=1}^{n} \mu_k^{-1} \lambda_k \phi_k s_k$, then $f_n \in B_f$ and $f_n \to f$ with respect to τ_c. Hence, $L(f) = \lim L(f_n)$. But, since f_n is bounded, we have

$$L(f_n) = \sum_{k=1}^{n} \mu_k^{-1} \lambda_k m(V_k) s_k = \sum_{k=1}^{n} \lambda_k$$

and so $|L(f_n)| = |\lambda_n|$. Thus $|L(f)| = \lim |\lambda_n| = \infty$, a contradiction.
Claim II: $L(f) = m(f)$ for all $f \in C(X, E)$. Indeed, for $\alpha = \{A_1, \ldots, A_n; x_1, \ldots, x_n\} \in \Omega_X$, set $f_\alpha = \sum_{i=1}^{n} \phi_{A_i} f(x_i)$ where ϕ_{A_i} is the $\mathbb{K}$-characteristic function of A_i. Then $m(f) = \lim m(f_\alpha)$. On the other hand, $f_\alpha \to f$ with respect to τ_c. Indeed, let $\epsilon > 0$ and let D be a compact subset of X. There are pairwise disjoint clopen sets $A_1, \ldots, A_n$ covering D and $x_i \in A_i$ such that $\|f(x) - f(x_i)\| < \epsilon$ if $x \in A_i$. Let A_{n+1} be the complement of the set $\bigcup_{i=1}^{n} A_i$ and, in case A_{n+1} is not empty, let $x_{n+1} \in A_{n+1}$. Then $\alpha_0 = \{A_1, \ldots, A_{n+1}; x_1, \ldots, x_{n+1}\} \in \Omega_X$. If $\alpha \geq \alpha_0$, then $\omega_D(f - f_\alpha) \leq \epsilon$, which proves that $f_\alpha \to f$ with respect to τ_c. Since $f_\alpha \in B_f$, we get that $L(f) = \lim L(f_\alpha) = \lim m(f_\alpha) = m(f)$. This completes the proof.

COROLLARY 7.2. *If X is locally compact zero-dimensional and if L is a linear functional on $C(X,E)$, then L is β_b-continuous iff $L|_{B_f}$ is τ_c-continuous for every $f \in C(X,E)$.*

PROOF. If L is β_b-continuous, then $L|_{B_f}$ is τ_c-continuous, since B_f is $\tau_{u,b}$-bounded and thus $\tau_c = \beta_b$ on B_f . On the other hand, if $L|_{B_f}$ is τ_c-continuous for every $f \in C(X,E)$, then (by the preceeding Proposition) there exists $m \in M(X,E')$, which is tight and has a bounding support, such that $L(f) = m(f)$ for all $f \in C(X,E)$. But then (by Proposition 6.5) $m \in M_{t,b}(X,E')$ and so is L is β_b-continuous.

Let $G_{t,b}(X,E')$ be the space of all $m \in M(X,E')$ which are tight and have bounding support.

PROPOSITION 7.3. *If $m \in G_{t,b}(X,E')$, then every $f \in C(X,E)$ is m-integrable.*

PROOF. Let A=supp m. Given $f \in C(X,E)$, choose $d \geq \sup_{x\in A} \|f(x)\|$ and set $W = \{x : \|f(x)\| \leq d\}$. Let ϕ be the $\mathbb{K}$-characteristic function of W and set $g = \phi f, h = f - g$. It is easy to see that h is m-integrable with $\int hdm = 0$. Also g is m-integrable since it is bounded and m is tight. Thus $f = g + h$ is m-integrable.

Let now τ_1 (resp τ_2) be the finest locally convex topology (resp. the finest polar topology) on $C(X,E)$ which coincides with τ_c on each of the sets $B_f, f \in C(X,E)$. Since τ_2 is the polar topolgy which corresponds to τ_1, the two topologies have the same dual space. This common dual space is contained in $G_{t,b}(X,E')$ by Proposition 7.1. On the other hand, let $m \in G_{t,b}(X,E')$ and let p be the norm of E. If A=supp m and $f \in C(X,E)$, then there exists $\mu \in \mathbb{K}$ with $|\mu| \geq \sup_{x\in A} \|f(x)\|$. There is a compact set D such that $m_p(U) < |\mu|^{-1}$ if U is disjoint from D. Let $\gamma \in \mathbb{K}$ be such that $m_p(X) \leq |\gamma|^{-1}$. We claim that

$$\{g : g \in B_f, \omega_D(g) \leq |\gamma|\} \subset W = \{g : |m(g)| \leq 1\}.$$

Indeed let $g \in B_f, \omega_D(g) \leq |\gamma|$, and set

$$U = \{x : \|g(x)\| \leq |\gamma|\}, \quad V = \{x : \|f(x)\| \leq |\mu|\}.$$

Since

$$|\int_{V\cap U} gdm| \leq 1 \text{ and } |\int_{V\cap U^c} gdm| \leq 1$$

we have that $|m(g)| = |\int_V gdm| \leq 1$. This clearly proves that W is a τ_1-neighborhood of zero and so L_m is τ_1-continuous . So we have the following

PROPOSITION 7.4. *$(C(X,E),\tau_i)' = G_{t,b}(X,E')$, for $i = 1, 2$.*

Theorem 7.5. *Let X be locally compact zero-dimensional. Then:*
(1) $(C(X,E),\beta_b)' = (C(X,E),\tau_i)' = M_{t,b}(X,E')$, *for* $i = 1,2$.
(2) A subset H of $M_{t,b}(X,E')$ is β_b-equicontinuous iff it is τ_i-equicontinuous.
(3) If E is polar, then $\beta_b = \tau_2$.
(4) In case E is a polar space, β_b coincides with the finest polar topology on $C(X,E)$ which agrees with τ_c on $\tau_{u,b}$- bounded sets.

Proof. (1) It follows from Propositions 6.5 and 7.4.
(2) Clearly $\beta_b \le \tau_1$ and so every β_b-equicontinuous is τ_1-equicontinuous. On the other hand, assume that H is τ_1-equicontinuous.
Claim I: The set

$$S(H) = \overline{\bigcup_{m\in H} \text{supp } m}$$

is bounding. Assume the contrary. Then, by Proposition 6.6, there exist $f \in C(X,E)$ and a sequence (m_n) in H such that $m_n(f) = \lambda^n$ for all n, where $|\lambda| > 1$. But then f is not absorbed by the polar H^0 of H in $C(X,E)$, a contradiction.
Claim II: $\sup_{m\in H} \|m\| < \infty$, where $\|m\| = m_p(X)$. Indeed, there exists a compact subset D of X and $\gamma \in \mathbb{K}$, $0 < |\gamma| \le 1$, such that

$$\{f : \omega|_D(f) \le |\gamma|, \|f\| \le 1\} \subset H^0$$

From this we get easily that $\|m\| \le |\gamma|^{-1}$ for all $m \in H$.
Claim III For each $\gamma \ne 0$, there exists a compact subset D of $S(H)$ such that $m_p(U) \le |\gamma|$, for all $m \in H$, if U is disjoint from D. Indeed, there exist a compact subset Y and $\mu \ne 0$, such that

$$O = \{f : \omega_Y(f) \le |\mu|, \|f\| \le 1\} \subset \gamma H^0.$$

We may choose Y clopen. Let now U be a clopen set disjoint from $Y \cap S(H) = D$. Since $U \cap Y$ is disjoint from $S(H)$, we have that $m_p(U \cap Y) = 0$ for all $m \in H$ and so $m_p(U) = m_p(U \cap Y^c)$. If now ϕ is the $\mathbb{K}$-characteristic function of $U \cap Y^c$, then for each $s \in E$, with $\|s\| \le 1$, we have that $\phi s \in O$ and so $|m(U \cap Y^c)s| \le |\gamma|$. This implies that $m_p(U) = m_p(U \cap Y^c) \le |\gamma|$. Now claims I, II, III above imply that H is β_b-equicontinuous by Proposition 6.4
(3) It follows from (2) since the topology of a polar space coincides with the topology of uniform convergence on the equiocontiuous subsets of its dual space.
(4) Let τ_3 be the finest polar topology which agrees with τ_c on $\tau_{u,b}$-bounded sets. Since every $B_f, f \in C(X,E)$ is $\tau_{u,b}$-bounded, it follows that $\tau_3 \le \tau_2$ and so $\tau_3 = \beta_b = \tau_2$ since $\beta_b \le \tau_3$.

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