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A NOTE ON QUASI-ISOMETRIES

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Abstract. The paper aims at investigating some basic properties of a quasi isometry which is defined to be a bounded linear operator T on a Hilbert space such that $T^{*2}T^2 = T^*T$.

1. Introduction

Partial isometries provide an extensively studied extension of isometries. They have played significant role in structural study of Hilbert space operators. Agler and Stankus [1] studied another extension called m-isometries. In the present note, we study quasi-isometries the definition of which is given below.

Definition 1.1. *A bounded linear operator T is called a quasi-isometry if $T^{*2}T^2 = T^*T$.*

Clearly every isometry is a quasi-isometry; whereas an idempotent operator is a quasi-isometry but need not be an isometry. On the other hand, a quasi-isometry which is an m-isometry turns out to be an isometry. Thus the classes of partial isometries, m-isometries and quasi-isometries which are extensions of the class of isometries are independent.

1.1. *Notation and terminologies.* For a bounded linerar operator T on a complex Hilbert space H, we write $\sigma(T)$, $a(T)$ and $\sigma_p(T)$ to designate the spectrum, the approximate point spectrum and the point spectrum of T respectively. Notations $N(T)$, $R(T)$ and $r(T)$ are used for the null space, the range space, and the spectral readius of T respectively. An operator T is called

1. quasinormal if T commutes with T^*T;

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2. hyponormal if $T^*T \geq TT^*$;
3. k-paranormal if $\left\|T^k x\right\| \|x\|^{k-1} \geq \|Tx\|^k$ for all $x \in H$;
4. k-quasihyponormal if $\left\|T^{k+1}x\right\| \geq \left\|T^*T^k x\right\|$ for every x in H;
5. m-isometry if

$$\sum_{k=0}^{m}(-1)^k \left[\begin{array}{c} n \\ k \end{array} \right] T^{*m-k}T^{m-k}$$

6. dominant if $R(T-\alpha I) \subset R(T^* - \bar{\alpha} I)$ for all complex numbers α.

2. Results

Theorem 2.1. *Let T be an operator with the right-handed polar decomposition $T = UP$. Then T is a quasi-isometry iff PU is a partial isometry with $N(PU) = N(U)$.*

Proof. Suppose T is a quasi-isometry. Then $PU^*PU^*UPUP = P^2$ or $PU^*P^2UP = P^2$. Since $N(P) = N(U)$, $UU^*P^2UP = UP$. A premultiplication by U^* yields $U^*P^2UP = P$ or $PU^*P^2U = P$. Another application of the relation $N(P) = N(U)$ gives $UU^*P^2U = U$ or $U^*P^2U = U^*U$. This shows that PU is a partial isometry with $N(PU) = N(U)$. Conversely, assume that PU is a partial isometry and $N(PU) = N(U)$. Then $U^*P^2U = U^*U$ because U^*P^2U and U^*U are projections having the common range space. Clearly, $U = UU^*P^2U$ or $P^2 = PU^*P^2UP$; thus T is a quasi-isometry.

For a quasi-isometry, the inequality $\|T\| \geq 1$ is obvious. Further improvement is not possible as can be seen from the following result.

Theorem 2.2. *If T is a quasi-isometry and if $\|T\| = 1$, then T is hyponormal.*

Proof. By hypothesis,

$$\begin{aligned} \left\|Tx - T^*T^2x\right\|^2 &= \|Tx\|^2 + \left\|T^*T^2x\right\|^2 - 2Re\left\langle Tx, T^*T^2x\right\rangle \\ &= \|Tx\|^2 + \left\|T^*T^2x\right\|^2 - 2\|Tx\|^2 \\ &\leq \|Tx\|^2 + \|Tx\|^2 - 2\|Tx\|^2 = 0 \end{aligned}$$

Thus

$$T = T^*T^2 \tag{2.1}$$

Hence $T^* = T^{*2}T$. This gives $N(T) \subset N(T^*)$ or $N(U) \subset N(U^*)$. Clearly $U^*U \geq UU^*$. Since $P^2 \leq I$, we find $U^*P^2U = U^*U \geq UU^* \geq UP^2U^*$. This leads to

$$PU^*(T^*T)UP \geq P(TT^*)P \tag{2.2}$$

Since $P^2(TT^*) = TT^*$ by (2.1), P commutes with TT^*. This along with (2.2) will yield

$$T^*T = T^{*2}T^2 \geq P(TT^*)P = P^2(TT^*) = TT^*.$$

Thus T is hyponormal.

COROLLARY 2.3. *Let T be a quasi-isomery. Then T is quasi-normal iff it is a partial isometry.*

PROOF. The result follows from Theorem 2.1 and Theorem 2.2.

COROLLARY 2.4. *If T is a quasi-isometry and quasiniplotent then $T = 0$.*

PROOF. As $r(T) = 0$, $\|T^n\| \leq 1$ for some positive integer n. Since T^n is also a quasi-isometry, $\|T^n\| = 1$. By Theorem 2, T^n is hyponormal. The desired assertion follows from the relation $\|T^n\| = r(T^n)$.

In the next theorem, we collect some spectral properties of quasi-isometries.

THEOREM 2.5. *Let T be a quasi-isometry. Then*

1. $a(T) \sim \{0\}$ *is a subset of the unit circle,*
2. $\bar{\alpha} \in \sigma_p(T^*)$ *whenever* $\alpha \in \sigma_p(T)$,
3. $\bar{\alpha} \in a(T^*)$ *whenever* $\alpha \in a(T)$,
4. *the eigenspaces corresponding to distinct non-zero eigenvalues of T are mutually orthogonal,*
5. *isolated points of $\sigma(T)$ are eigen values of T.*

PROOF. (1) A simple calculation proves the assertion.
(2) Let $\alpha \in \sigma_p(T)$. Supoose first that $\alpha = 0$. If $0 \in \mathbb{C} \setminus \sigma_p(T^*)$, then from $T^{*2}T^2 = T^*T$, $T^*T^2 = T$ or $T^{*2}T = T^*$. Consequently T turns out to be an isometry. But this will contradict the fact that $0 \in \sigma_p(T)$. Now consider the case when α is non-zero. Choose a non-zero vector x such that $Tx = \alpha x$. Since $T^{*2}T^2 = T^*T$, we find $\alpha T^*x = \alpha^2 T^{*2}x$. In view of (1), $|\alpha| = 1$ and therefore $(T^* - \bar{\alpha}I)T^*x = 0$. To establish that $\bar{\alpha} \in \sigma_p(T^*)$, we need to show that T^*x is non zero. If $T^*x = 0$ then $0 = \langle x, T^*x \rangle = \langle Tx, x \rangle = \alpha \langle x, x \rangle$ and hence $\alpha = 0$ because x is nonzero. This contradicts the fact that $|\alpha| = 1$.
(3) Let $\alpha \in a(T)$. If $\alpha = 0$, then as argued above, one can show that $0 \in a(T^*)$. Assume that α is non-zero. Choose a sequence (x_n) of unit vectors such that $(T - \alpha I)x_n \to 0$. Then

$$-\alpha^2 T^{*2}x_n + \alpha T^*x_n = T^{*2}(T^2x_n - \alpha^2 x_n) - T^*(Tx_n - \alpha x_n) \to 0$$

as $n \to \infty$ or $(\alpha T^* - I)T^*x_n \to 0$. Since

$$\alpha = lim \langle Tx_n, x_n \rangle = lim \langle x_n, T^*x_n \rangle$$

and $\alpha \neq 0$, (T^*x_n) does not converge to zero. Choose a subsequence $(T^*x_{n_k})$ of (T^*x_n) so that

$$\|T^*x_{n_k}\| \geq M$$

for some positive number M. Set

$$y_k = \frac{T^* x_{n_k}}{\|T^* x_{n_k}\|}$$

Then (y_k) is a sequence of unit vectors such that $(\alpha T^* - I)y_k \to 0$ or $(T^* - \bar{\alpha} I)y_k \to 0$ as $|\alpha| = 1$.

(4) Let α and β be distinct nonzero eigen-values of T. If $Tx = \alpha x$ and $Ty = \beta y$ then $0 = \langle T^2 x, T^2 y\rangle - \langle Tx, Ty\rangle = \alpha\bar{\beta}(\alpha\bar{\beta} - 1)\langle x, y\rangle$. Since $\alpha \neq 0$ and $\beta \neq 0$ $\alpha\bar{\beta} \neq 0$ and $|\beta| = 1$. Also, $\alpha \neq \beta$. Therefore, all these will give $\alpha \neq \dfrac{1}{\bar{\beta}}$ or $\alpha\bar{\beta} = 1$. Thus we infer that $\langle x, y\rangle = 0$. This proves the assertion.

(5) Let z_0 be an isolated point of $\sigma(T)$. Then there exists $R > 0$ such that $\{z : |z - z_0| < R\} \cap \sigma(T) = \{z_0\}$. Define

$$E = \frac{1}{2\pi} \int_{|z - z_0| = R} (zI - T)^{-1} dz.$$

Then E is a non-zero idempotent operator commuting with T and $E(H)$ is invariant under T. Also $T/E(H)$ is a quasi-isometry and $\sigma(T/E(H)) = \{z_0\}$. If $z_0 = 0$, then $T/E(H) = 0$ by Corollary 2. If $z_0 \neq 0$, then $T/E(H)$ is an invertible quasi-isometry and so must be unitary. Consequently $T/E(H) = z_0 I/E(H)$. In either case, $z_0 \in \sigma_p(T)$ which completes the proof.

It is easy to show that if T is an idemepotent operator with $N(T^*) \subset N(T)$ then T is a projection. The following gives a partial extension of this to quasi-isometries.

Theorem 2.6. *If T is a quasi-isometry for which $N(T^*) \subset N(T)$, then T is a normal partial isometry.*

Proof. By hypothesis, $TT^*T^2 = T^2$ or $T^{*2} = T^{*2}TT^*$ and so $(TT^*) = (TT^*)^2$. This shows that T is a partial isometry. By Corollary 1, T must be quasinormal. This alongwith the given condition $N(T^*) \subset N(T)$ forces $N(T) = N(T^*)$ or $R(T^*T) = R(TT^*)$. Using the fact that T is a partial isometry, one can conclude that T is normal.

Remark 2.7. The above theorem raises the following question: Is a quasi-isometry T normal if $N(T) \subset N(T^*)$? In case T is idempotent, it is obvious that T is a projection.

Corollary 2.8. *A quasi-isometry whose adjoint is a dominant operator is a normal partial isometry.*

Theorem 2.9. *Let T be a quasi-isometry. Then T is normal if either*

1. *T^* is k-paranormal, or*
2. *T^* is k-quasihyponormal*

PROOF. Suppose (1) holds. Then $\|T\| = r(T)$. By Theorem 3, $r(T) = 1$; thus $\|T\| = 1$. As seen in the proof of Theorem 2, $T = T^*T^2$ or $T^* = T^{*2}T$. Since $\|T^{*k}x\| \|x^{k-1}\| \geq \|T^*x\|^k$, the relation $T^{*n}T^n = T^*T$, $(n = 1, 2, 3, ...)$ yields

$$\begin{aligned} \|T^{*2}Tx\| \|Tx\|^{k-1} &\geq \|T^*T^{k-1}x\|^k \\ &\geq \|T^{*k-1}T^{k-1}x\|^k \\ &= \|T^*Tx\|^k \end{aligned}$$

Since $T^* = T^{*2}T$, above inequality gives $\|T^*x\| \|Tx\|^{k-1} \geq \|T^*Tx\|^k$. In particular, $N(T^*) \subset N(T)$. Now the result follows from Theorem 4. Next we assume (2). Then $\|T^{*k+1}T^kx\| \geq \|TT^{*k}T^kx\|$ for all x in H. Since $T^{*k}T^k = T^*T$, we find $\|T^{*2}Tx\| \geq \|TT^*Tx\|$ or $\langle T^2T^{*2}Tx, Tx\rangle \geq \langle (TT^*)^2Tx, Tx\rangle$; thus $\langle T^2T^{*2}Tx, x\rangle \geq \langle (TT^*)^2x, x\rangle$; for all x in $\overline{R(T)}$. Therefore, since $\langle T^2T^{*2}x, x\rangle = 0 = \langle (TT^*)^2x, x\rangle$ for x in $N(T^*)$, we have

$$\langle T^2T^{*2}x, x\rangle \geq \langle (TT^*)^2x, x\rangle \tag{2.3}$$

for all x in H. In particular, $\|T^2\| = \|T\|^2$. Combining the relation with the assumption that T is quasi-isometry, we find $\|T\| = 1$. By Theroem 2, T turns out to be hyponormal, and so $N(T) \subset N(T^*)$. Now from (3), it is not difficult to show that T^* is hyponormal. This completes the arguments.

THEOREM 2.10. *Let $T = UP$ be a quasi-isometry. Let $S = PUP$. If $N(T) \subset N(T^*)$ and if S is normal, then T is normal.*

PROOF. As seen in the proof of Theroem 1. $U^*P^2U = U^*U$. This will imply that $S^*S = PU^*UP = P^2$. Since PU is a partial isometry with $N(PU) = N(P)$, PUP turns out to be the polar decomposition of S. Now the normality of S will give $PUP = P^2U$ and so $(UP - PU)x \in R(T^*)$ for each x in H. Thus we have $UP = PU$. Next we show that U is normal. Since S is normal, PU turns out to be normal. Therefore if $U^*x = 0$, then $PUx = U^*Px = P^*Ux = 0$ and hence $Ux = 0$ as $N(PU) = N(U)$; thus $N(U^*) \subset N(U)$. Since $UP = PU$, T is quasi-normal. Consequently, we have $N(U^*) = N(U)$ or $R(U^*U) = R(UU^*)$. This shows that U is normal. From this we derive $T^*T = P^2 = U^*UP^2 = UU^*P^2 = UP^2U^* = TT^*$. This completes the proof.

REMARK 2.11. Above result need not hold unless $N(T) \subset N(T^*)$. To see this, consider the operator

$$T = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

on $\mathbb{C}^2$. Then T is a quasi-isometry with polar decomposition

$$T = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{bmatrix}.$$

Although PUP is normal, T fails to be normal.

References

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