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INTEGRATION OF MULTIFUNCTIONS WITH RESPECT TO A MULTIMEASURE

H. E. Brink and P. Maritz
Technikon Free State and University of Stellenbosch, South Africa

Abstract: We define the bilinear integral of a measurable multifunction with respect to a multimeasure and study the properties of the resulting set-valued set function.

1. Introduction

The need for multimeasures first arised in mathematical economics when Vind [28] studied equilibrium theory for exchange economies with production, where the coalitions and not the individual agents are the basic economic units. Since then the subject of multimeasures has received much attention and, as it turned out, developed to be the set-valued analogue of the classical theory of vector measures. Significant contributions to the theory and applications (control systems, statistics, mathematical economics, game theory, etc) of multimeasures were made by, among others, Artstein [3], Debreu and Schmeidler [10], Schmeidler [27], Wenxiu, Jifeng and Aijie [29] for $\mathbb{R}^n$-valued multimeasures, by Aló, de Korvin and Roberts [1,2], Costé [7], Hiai [16], Papageorgiou [22-25] and Kandilakis [18] for Banach space-valued multimeasures and by Castaing [5], Costé and Pallu de la Barriére [8,9] and Godet-Thobie [13,15] for multimeasures with values in a locally convex vector space.

Various developments in mathematical economics and optimal control have led to the study of the measurability of multifunctions. Also, integrals of multifunctions have been studied in connection with statistical problems (see Kudō [19] and Richter [26]). Accordingly, many papers dealt with the basic theory of integration of multifunctions and several approaches were established. A beginning of what might be called a calculus of multifunctions can be found in [4]. In [4], Aumann considered integration of selectors of the multifunction and his integral turned out to be the appropriate analytic tool in the applied fields mentioned before. However, when it comes to integration with respect to a multimeasure, only two approaches can be distinguished. Kandilakis [18] defined his integral in terms of the Bochner integral while Papageorgiou [22] considered the bilinear integral of Dinculeanu [11]. It is the purpose of this paper to study some of the properties of the set-valued bilinear integral of a multifunction with respect to a multimeasure.

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2. Preliminaries

Let T be a non-empty point set on which no topological structure is required and let X be a linear topological space with topological dual X'. We denote by $\mathcal{P}(X)$ the class of all nonempty subsets of X. Furthermore, by $\mathcal{P}_f(X)$ (respectively, $\mathcal{P}_k(X)$) we will denote the closed (respectively, compact) sets in $\mathcal{P}(X)$. A c after f or k will mean that the set is in addition convex. A w in front of $f(b)$ (respectively, k) means that the closedness (respectively, compactness) is with respect to the weak topology $w(X, X')$.

We now let (X, d) be a metric space. Then the distance between a point $x \in X$ and a non-empty set $A \subseteq X$ is defined as $d(x, A) = \inf\{d(x, a) \mid a \in A\}$. Furthermore, for any $A, B \in \mathcal{P}_k(X)$, we define their *Hausdorff semi-metric* by $d(A, B) = \sup\{d(a, B) \mid a \in A\}$, and their *Hausdorff metric* by $H(A, B) = \max\{d(A, B), d(B, A)\}$. In addition, we put $\|A\| = H(A, \{0\})$ (the norm of the set A). Whenever we refer to the metric space $\mathcal{P}_k(X)$, it must be understood that $\mathcal{P}_k(X)$ is equipped with the Hausdorff metric H.

For $A \in \mathcal{P}(X)$, we let $\overline{A}$ denote the closure of A and $\overline{co}A$ denote the closed convex hull of A. For all $x' \in X'$, we set $\sigma(x', A) = \sup\{(x', x) \mid x \in A\}$ (the support function of A). Furthermore, if $A, B \in \mathcal{P}(X)$, then we put

$$A + B = \{a + b \mid a \in A, b \in B\}.$$

For the rest of this section we consider $(T, \mathcal{S})$ where $\mathcal{S}$ is a σ-ring of subsets of T.

Definition 2.1. *If Y is a linear topological space, then a set-valued set function $M : \mathcal{S} \to \mathcal{P}(Y)$ is called a* **multimeasure** *if*

(a) *$M(\emptyset) = \{0\}$ and $M(A \cup B) = M(A) + M(B)$ for every pair $A, B \in \mathcal{S}$ of disjoint sets.*

(b) *for every $y_k \in M(A_k)$ the series $\Sigma_{k=1}^{\infty} y_k$ is unconditionally convergent and*

$$M\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} M(A_k) = \left\{y \in Y \mid y = \sum_{k=1}^{\infty} y_k,\ y_k \in M(A_k)\right\}.$$

As for single-valued measures we have the notion of total variation of a multimeasure. Let Y be a normed space and suppose that $M : \mathcal{S} \to \mathcal{P}(Y)$ is a multimeasure. For every $A \subseteq T$ we define the *variation of M on A*, denoted by $v(M, A)$, by

$$v(M, A) = \sup_I \sum_{i \in 1} \|M(A_i)\|,$$

where the supremum is taken for all the families $(A_i)_{i \in I} \subseteq \mathcal{S}$ of mutually disjoint sets contained in A. The set function $v(M)$ is called the *variation of M* and the restriction of $v(M)$ to the class $\mathcal{S}$ will again be denoted by $v(M)$. We say that $M : \mathcal{S} \to \mathcal{P}(Y)$ is of *bounded variation* (with respect to $\mathcal{S}$) if $v(M, A) < \infty$ for every $A \in \mathcal{S}$. Note that the variation $v(M)$ of a multimeasure M is a positive measure (see Proposition 1.1 on page 98 of [16]).

A set $A \in \mathcal{S}$ is said to be an *atom* for a multimeasure $M : \mathcal{S} \to \mathcal{P}(Y)$ if $M(A) \neq \{0\}$ and if either $M(B) = \{0\}$ or $M(A \backslash B) = \{0\}$ holds for every $B \subseteq A, B \in \mathcal{S}$. We say that M is *atomic* if there exists at least one atom in $\mathcal{S}$, and that M is *non-atomic* if there are no atoms in $\mathcal{S}$. If $\mu : \mathcal{S} \to Y$ is a positive measure on $\mathcal{S}$, then we say that M is *μ-continuous* on $\mathcal{S}$ if and only if for any $A \in \mathcal{S}$ with $\mu(A) = 0$ we have that $M(A) = \{0\}$. We call a measure $m : \mathcal{S} \to Y$ a *selector* of M if $m(A) \in M(A)$ for all $A \in \mathcal{S}$. We denote by S_M the set of selectors of M.

Let Y be a fixed Banach space and let $m : \mathcal{S} \to Y$ be a vector measure with finite variation $v(m)$. Then we denote by $\mathcal{M}(v(m))$ the σ-ring of all $v(m)$-measurable subsets of T and $\Sigma(v(m))$ is the δ-ring of all $v(m)$-integrable subsets of T. The extensions of m and $v(m)$ to $\Sigma(v(m)$ will again be denoted by m and $v(m)$. Consider the class

$$\mathcal{E}_X(v(m)) = \left\{ f : T \to X \mid \ f = \sum_{i=1}^{n} x_i \chi_{A_i},\ x_i \in X,\ A_i \in \Sigma(v(m)), i = 1, 2, \ldots, n \right\},$$

let Z be another Banach space and suppose that $(x, y) \mapsto xy$ is a bilinear mapping of $X \times Y$ into Z such that $\|xy\| \leq \|x\| \, \|y\|$. Then we say that a function $f : T \to X$ is *m-integrable* if there exists a Cauchy sequence $(f_n) \subseteq \mathcal{E}_X(v(m))$ which converges to f $v(m)$-almost everywhere on T. The integral of f with respect to m is the element $\int f(t)\, m(dt) \in Z$ defined by

$$\int f(t)\, m(dt) = \lim_{n \to \infty} \int f_n(t)\, m(dt).$$

We denote by $\mathcal{L}^1_X(m)$ the space of all m-integrable functions $f : T \to X$. Furthermore, if $f \in \mathcal{L}^1_X(m)$ and $A \in \mathcal{M}(v(m))$, then $f\chi_A \in \mathcal{L}^1_X(m)$ and we define

$$\int_A f(t)\, m(dt) = \int f(t) \chi_A m(dt).$$

Let now $F : T \to \mathcal{P}_f(X)$ be a multifunction. Then we say that F is *$v(m)$-measurable* if the set $F^-(C) = \{t \in T \mid \ F(t) \cap C \neq \emptyset\}$ belongs to $\mathcal{M}(v(m))$ for every set $C \in \mathcal{P}_f(X)$ and if for every $v(m)$-integrable set A there exists a $v(m)$-negligible set $N \subseteq A$ and a countable set $H \subseteq X$ such that $F(A \backslash N) \subseteq \overline{H}$. Furthermore, F is said to be *integrably bounded* if there exists a $k \in \mathcal{L}^1_{\mathbb{R}}(m)$ such that

$$\|F(t)\| \ \leq \ k \ \ v(m) - \text{a.e on } T.$$

A function $f : T \to X$ is called a *selector* of F if $f(t) \in F(t)$ $v(m)$-almost everywhere on T. The set of all $v(m)$ - measurable selectors of F will be denoted by S_F. In addition, we put

$$S_F^1(m) = \{f \in \mathcal{L}_X^1(m) \mid f \in S_F\}.$$

Hence $S_F^1(m)$ denotes the class of all integrable selectors of F. It follows clearly that $S_F^1(m)$ is a closed subset of $\mathcal{L}_X^1(m)$.

3. Integration of multifunctions with respect to a multimeasure

Throughout this section we will assume that X, Y and Z are Banach spaces, T will denote a non-empty point set on which no topological structure is required and $\mathcal{R}$ is a ring of subsets of T.

Definition 3.1. *If $M : \mathcal{R} \to \mathcal{P}_f(Y)$ is a multimeasure and $F : T \to \mathcal{P}(X)$ is a multifunction, then for every set $A \in \mathcal{M}(v(M))$ we define the* **integral of F with respect to M**, *denoted by $\int_A F(t)\,M(dt)$, by the equality*

$$\int_A F(t)M(dt) = \left\{ \int_A f(t)\,m(dt) \mid f \in S_F^1(m), \;\; m \in S_M \right\}.$$

We note that the integral of F with respect to M will always exist, even if F is not $v(M)$-measurable. Moreover, if $S_F^1(m) = \emptyset$ for all $m \in S_M$, then $\int_A F(t)M(dt) = \emptyset$. Also, if $v(M, A) = 0$ for $A \in \mathcal{M}(v(M))$ and $S_F^1(m) \neq \emptyset$ for $m \in S_M$, then $\int_A F(t)M(dt) = \{0\}$.

Example 3.2.

Let $T = [0, 1]$, Σ is the Lebesgue σ-algebra of subsets of T and λ is the Lebesgue measure on Σ. If we define $F : T \to I\!R$ by $F(t) = [0, 1]$ and $M : \Sigma \to I\!R$ by $M(A) = [0, \infty)$, then $\int_A F(t)\,M(dt) = [0, \infty)$.

Theorem 3.3. *If X is a separable Banach space, $M : \mathcal{R} \to \mathcal{P}_f(Y)$ is a multimeasure of bounded variation $v(M)$ and if $F : T \to \mathcal{P}_f(X)$ is an integrably bounded $v(M)$-measurable multifunction, then $\int_A F(t)M(dt) \neq \emptyset$ for every $A \in \mathcal{M}(v(M))$.*

Proof. From Theorem 2.5 of [16] we obtain a selector $m : \mathcal{R} \to Y$ of M. By the integrably boundedness of F, there exists a $k \in \mathcal{L}_{I\!R}^1(m)$ such that $\|F(t)\| \leq k$ $v(m)$-almost everywhere on T. Corollary 7.5 of [20] then provides F with a $v(m)$-measurable selector $f : T \to X$. Since $\|f(t)\| \leq k(t)$ $v(m)$-almost everywhere on T, Proposition 19 on page 136 of [11] implies that $f \in \mathcal{L}_X^1(m)$ so that $S_F^1(m) \neq \emptyset$ for every $m \in S_M$; therefore $\int_A F(t)M(dt) \neq \emptyset$ for every $A \in \mathcal{M}(v(M))$. ∎

Theorem 3.4 ([20], p99, Theorem 10.5). *Let X be a separable Banach space, $M : \mathcal{R} \to \mathcal{P}_f(Y)$ a multimeasure of bounded variation and let $F : T \to \mathcal{P}(X)$ be an integrably bounded multifunction such that $Gr_F \in \mathcal{S}(\mathcal{M}(v(M)) \times \mathcal{S}(\mathcal{B}_X))$.*

(a) *If T is a countable union of sets of $\mathcal{R}$ and if the bounding function k belongs to $\mathcal{L}_{I\!R}^1(v(M))$, then $\int_A F(t)M(dt) \neq \emptyset$ for every $A \in \mathcal{M}(v(M))$.*

(b) *If $T \in \mathcal{R}$ and if the bounding function k belongs to $\mathcal{L}^1_{\mathbb{R}}(M)$, then $\int_A F(t)M(dt) \neq \emptyset$ for every $A \in \mathcal{M}(v(M))$.*

In our next two results we list some useful properties of the bilinear integral of a multifunction F with respect to a multimeasure M. The first theorem is the set-valued version of the results of [11] on page 109. The second result shows that if F and M are both positive, then the integral of F with respect to M will also be positive, and vice versa. If X, Y and Z are Banach lattices, then we denote by X_+, Y_+ and Z_+ the positive cones of X, Y and Z respectively.

THEOREM 3.5. *Suppose that X, Y and Z are Banach lattices, let $M : \Sigma(v(M)) \to \mathcal{P}(Y)$ be a multimeasure of bounded variation $v(M)$ and let $F : T \to \mathcal{P}(X)$ be an integrably bounded $v(M)$-measurable multifunction.*

(a) *If $Y = \mathcal{L}(X, Z)$ and if $M(A) \subseteq Y_+$ for all $A \in \Sigma(v(M))$, then for all $A \in \Sigma(v(M))$ the mapping $F \mapsto \int_A F(t)\, M(dt)$ of T into Z is increasing.*

(b) *If $M(A) \subseteq Y_+$ for all $A \in \Sigma(v(M))$ and if $F(t) \subseteq X_+$ $v(M)$-almost everywhere on T, then*

$$\int_E F(t)\, M(dt) \subseteq \int_F F(t)\, M(dt),$$

for all $E, F \in \Sigma(v(M))$ with $E \subseteq F$.

(c) *If $N : \Sigma(v(N)) \to \mathcal{P}(Y)$ is a multimeasure of bounded variation $v(N)$ such that $M(A) \subseteq N(A)$ for all $A \in \Sigma(v(N))$ and if $F(t) \subseteq X_+$ $v(N)$-almost everywhere on T, then*

$$\int_A F(t)\, M(dt) \subseteq \int_A F(t)\, N(dt).$$

(d) *For all $A \in \Sigma(v(M))$ we have that*

$$\| \int_A F(t)\, M(dt) \| \leq \int_A \|F(t)\| v(M, dt).$$

THEOREM 3.6. *Suppose that X, Y and Z are Banach lattices, let $M : \Sigma(v(M)) \to \mathcal{P}(Y)$ be a multimeasure of bounded variation $v(M)$ and let $F : T \to \mathcal{P}(X)$ be an integrably bounded $v(M)$-measurable multifunction. If $Y = \mathcal{L}(X, Z)$, $M(A) \subseteq Y_+$ for all $A \in \Sigma(v(M))$ and if $F(t) \subseteq X_+$ $v(M)$-almost everywhere on T, then $\int_A F(t)\, M(dt) \subseteq Z_+$. Conversely, if $X = \mathcal{L}(Y, Z)$, $M(A) \subseteq Y_+$ for all $A \in \Sigma(v(M))$ and if $\int_A F(t)\, M(dt) \subseteq Z_+$, then $F(t) \subseteq X_+$ $v(M)$-almost everywhere on T.*

PROOF. Let $M(A) \subseteq Y_+$ for all $A \in \Sigma(v(M))$ and let $F(t) \subseteq X_+$ $v(M)$-almost everywhere on T. From

$$M(A) = \{m(A) \mid m \in S_M\},$$

follows that $(y', m(A)) \geq 0$ for every $y' \in Y'_+$ and $m \in S_M$. Consequently, for $y' \in Y'_+$, $m \in S_M$ and $f \in S^1_F(m)$ we have that

$$(y', \int_A f(t)\, m(dt)) = \int_A f(t)(y', m(dt)) \geq 0$$

so that $\int_A F(t)\, M(dt) \subseteq Z_+$.

Conversely, by Lemma 1.1 of [17] we obtain a sequence $(f_k) \subseteq S^1_F(v(M))$ such that

$$F(t) = \overline{\{f_k(t) \mid k \in I\!N\}}$$

$v(M)$-almost everywhere on T. Since $\int_A F(t)\, M(dt) \subseteq Z_+$, it then follows that $\int_A f_k(t)\, m(dt) \in Z_+$ for all $m \in S_M$ and $k \in I\!N$. Consequently, for all $z' \in Z'_+$ and all $A \in \Sigma(v(M))$,

$$0 \leq \left(z', \int_A f_k(t)\, m(dt)\right) = \int_A (z', f_k(t))\, m(dt).$$

Since $m(A) \in M(A) \subseteq Y_+$ it then follows that $0 \leq (z', f_k(t))$ and hence $f_k(t) \in X_+$ for each $k \in I\!N$. We then conclude that $F(t) \subseteq X_+$ $v(M)$-almost everywhere on T. ∎

The next theorem shows that the bilinear integral of a multifunction with respect to a multimeasure is in fact a multimeasure.

THEOREM 3.7. *Let $M : \mathcal{R} \to \mathcal{P}_f(Y)$ be a multimeasure of bounded variation $v(M)$ and let $F : T \to \mathcal{P}_f(X)$ be an integrably bounded $v(M)$-measurable multifunction. If for each $A \in \Sigma(v(M))$ we define $N(A) = \int_A F(t)M(dt)$, then $N : \Sigma(v(M)) \to \mathcal{P}(Z)$ is a multimeasure of bounded variation.*

PROOF. We first show that N is of bounded variation. So let $(A_k) \subseteq T$ be a sequence of mutually disjoint sets of $\Sigma(v(M))$. From

$$\|N(A_k)\| \leq \| \int_{A_k} F(t)\, M(dt) \| \leq \int_T \|F(t)\|\ v(M, dt),$$

follows immediately that N is indeed of bounded variation.

To show that N is a multimeasure, let (A_k) be a sequence of mutually disjoint sets in $\Sigma(v(M))$ and let $A = \cup_{k=1}^{\infty} A_k$. Then we need to prove that

$$N(A) = \sum_{k=1}^{\infty} N(A_k).$$

For this purpose, let $z_k \in N(A_k)$ for $k \in I\!N$. Then there exist sequences $(m_k) \subseteq S_M$ and $(f_k) \subseteq S^1_F(m_k)$ such that $z_k = \int_{A_k} f_k(t)\, m_k(dt)$ for $k \in I\!N$. Define $f : T \to X$ by

$$f(t) = \begin{cases} f_k(t) & \text{if } t \in A_k \\ f_1(t) & \text{if } t \in T \backslash A \end{cases}$$

and $m : \Sigma(v(m)) \to Y$ by

$$m = \chi_{A_1} m_1 + \chi_{A_2} m_2 + \ldots + \chi_{T \setminus \cup_{i=1}^{n-1} A_i} m_n,$$

where $\chi_A m_k(B) = m_k(A \cap B)$ for $k = 1, 2, \ldots, n$. By the decomposability of S_F and S_M we then have that $f \in S_F^1(m)$ and $m \in S_M$, respectively. Consequently, for $z' \in Z'$, we have that

$$\begin{aligned} \left(z', \sum_{k=1}^{n} z_k \right) &= \left(z', \sum_{k=1}^{n} \int_{A_k} f(t)\, m_k(dt) \right) \\ &= \left(z', \int_{\cup_{k=1}^{n} A_k} f(t)\, m(dt) \right) \to \left(z', \int_A f(t)\, m(dt) \right) \end{aligned}$$

as $n \to \infty$. This means that the series $\sum_{k=1}^{\infty} z_k$ converges weakly to $z = \int_A f(t)\, m(dt)$ and a similar property holds for every subseries of $\sum_{k=1}^{\infty} z_k$. By the Orlicz-Pettis theorem follows that the series $\sum_{k=1}^{\infty} z_k$ converges unconditionally to $z \in N(A)$. This means that the series $\sum_{k=1}^{\infty} N(A_k)$ is unconditionally convergent and is contained in $N(A)$.

To prove the inverse inclusion, let $z \in N(A)$ with $A \in \Sigma(v(M))$. Then $z = \int_A f(t)\, m(dt)$ for some $m \in S_M$ and $f \in S_F^1(m)$. Then, as before, the series $\sum_{k=1}^{\infty} \int_{A_k} f(t)\, m(dt)$ converges to z. This shows that $z \in \Sigma_{k=1}^{\infty} N(A_k)$, which concludes the proof. $\square$

We have seen from the previous theorem that if $N(A) = \int_A F(t)\, M(dt)$, where M is a closed-valued multimeasure of bounded variation $v(M)$ and F is an integrably bounded $v(M)$-measurable multifunction with closed values, then N is a multimeasure. We now investigate the relationship between S_M, the selectors of M, and S_N, the selectors of N.

Proposition 3.8. *Let $M : \Sigma(v(M)) \to \mathcal{P}_k(Y)$ be a multimeasure of bounded variation $v(M)$, let $F : T \to \mathcal{P}_k(X)$ be an integrably bounded $v(M)$-measurable multifunction and for each $A \in \Sigma(v(M))$ let $N(A) = \int_A F(t) M(dt)$.*

(a) *If $m \in S_M$ and $f \in S_F^1(m)$, then the measure defined by $n(A) = \int_A f(t)\, m(dt)$ is a selector of N.*
(b) *If $n \in S_N$, then there exist an $m \in S_M$ and an $f \in S_F^1(m)$ such that $n(A) = \int_A f(t)\, m(dt)$, $A \in \Sigma(v(M))$.*

Proof. (a) Let $m \in S_M$ (which exists by Theorem 2.5 of [16]) and let $f \in S_F^1(m)$ (whose existence is guaranteed by Proposition 3.1 of [21]). Then the measure $n : \Sigma(v(m)) \to Y$ defined by $n(A) = \int_A f(t)\, m(dt)$ is clearly a selector of N.

(b) Since N is a compact-valued multimeasure of bounded variation (by Theorem 3.6), it follows from Theorem 2.5 of [16] that $S_N \neq \emptyset$. Let $n \in$

S_N. From Theorem 1 of [13] follows that $N(A) = \{n(A) \mid n \in S_N\}$. But Proposition 3.1 of [21] implies that $\int_A F(t)\, M(dt) \neq \emptyset$, that is, there exist an $m \in S_M$ and an $f \in S_F^1(m)$ such that $\int_A f(t)\, m(dt) \in \int_A F(t)\, M(dt) = N(A)$. Consequently, if $n \in S_N$, then $n(A) = \int_A f(t)\, m(dt)$, $A \in \Sigma(v(M))$. $\square$

Let $ca(Y)$ denote the space of all Y-valued measures on $\Sigma(v(m))$. We now discuss the topology of pointwise weak convergence on $ca(Y)$. If we consider $\mathcal{E}_{I\!R}(v(m)) \otimes Y'$, then $\mathcal{E}_{I\!R}(v(m)) \otimes Y'$ and $ca(Y)$ can be put into duality as follows:

$$(m, y) = \left(m, \sum_{k=1}^{n} \chi_{A_k} \otimes y'_k\right) = \sum_{k=1}^{n} (y'_k, m(A_k)),$$

where $\{A_1, \ldots, A_n\}$ is a finite $v(m)$-measurable partition of T and $y'_k \in Y'$, $k = 1, 2, \ldots, n$. Then it follows that the topology of pointwise weak convergence on $ca(Y)$ is in fact the $w(ca(Y), \mathcal{E}_{I\!R}(v(m)) \otimes Y')$-topology.

THEOREM 3.9. *Suppose that X is a separable Banach space and Z is finite-dimensional. Let $M : \Sigma(v(M)) \to \mathcal{P}_{wk}(Y)$ be a multimeasure of bounded variation $v(M)$ and suppose that $F : T \to \mathcal{P}_{wk}(X)$ is an integrably bounded $v(M)$-measurable multifunction. If for each $A \in \Sigma(v(M))$ we define $N(A) = \int_A F(t)M(dt)$, then $N : \Sigma(v(M)) \to \mathcal{P}_{wk}(Z)$ is a multimeasure of bounded variation.*

PROOF. The fact that N is of bounded variation follows just like before. By making use of the facts that S_M is $w(ca(Y), \mathcal{E}_{I\!R}(v(m)) \otimes Y')$-compact and $S_F^1(m)$ is weakly compact in $\mathcal{L}_X^1(m)$, it follows that N is closed-valued.

We will now make use of Theorem 1 of [29] in order to show that N is a multimeasure. Let $A, B \in \Sigma(v(M))$ with $A \cap B = \emptyset$. To prove that $N(A \cup B) = N(A) + N(B)$, we only need to show that $N(A) + N(B) \subseteq N(A \cup B)$ because the inverse inclusion follows from the definition of N. So, if $z \in N(A) + N(B)$, then $z = \int_A f_1 m_1(dt) + \int_B f_2 m_2(dt)$, where $f_i \in S_F^1(m_i)$ and $m_i \in S_M$, for $i = 1, 2$. Put $f = \chi_A f_1 + \chi_B f_2$ and $m = \chi_A m_1 + \chi_B m_2$. Then $f \in S_F^1(m)$ and $m \in S_M$ because both S_F and S_M are decomposable, and therefore $z = \int_{A \cup B} f\, dm \in N(A \cup B)$.

Finally, let (A_k) be an increasing sequence in $\Sigma(v(M))$ and put $A = \cup_{k=1}^{\infty} A_k$. Then

$$\begin{aligned} H(N(A), N(A_k)) &= H(N(A_k) + N(A \backslash A_k), N(A_k)) \\ &\leq \|N(A \backslash A_k)\| \\ &\leq \int_{A \backslash A_k} \|\, F(t) \,\|\; v(M, dt) \longrightarrow 0 \end{aligned}$$

as $k \to \infty$. This shows that N is indeed a strong multimeasure. $\square$

We now investigate the convexity of $\int_A F(t)\, M(dt)$. In particular, we will see that if Z is finite dimensional, then $\int_A F(t)\, M(dt)$ is convex. The convexity fails in the infinite dimensional case; in fact, as it turns out, the closure of the integral will be convex (see Example 3.2 below). For results on the convexity of the integral of a multifunction with respect to a vector measure, see [6], [17] and [4]. Central to our proofs is the Lyapunov convexity theorem.

THEOREM 3.10. *Suppose that X is a separable Banach space and Z is finite-dimensional. If $M : \Sigma(v(M)) \to \mathcal{P}_f(Y)$ is a non-atomic multimeasure of bounded variation $v(M)$ and $F : T \to \mathcal{P}_{wf}(X)$ is an integrably bounded $v(M)$-measurable multifunction, then $\int_A F(t)\, M(dt)$ is a convex set for each $A \in \Sigma(v(M))$.*

PROOF. If for $A \in \Sigma(v(M))$ we put $N(A) = \int_A F(t)\, M(dt)$, then from Theorem 4.2 of [3] follows that we only need to show that N is a bounded non-atomic multimeasure. The fact that N is a strong multimeasure of bounded variation follows from Theorem 3.6. Therefore it only remains to show that N is non-atomic. For this purpose, if $m \in S_M$, let $f \in S^1_F(m)$ and define the set functions $n : \Sigma(v(m)) \to Z$ and $\nu : \Sigma(v(M)) \to I\!R_+$ by

$$n(A) = \int_A f(t)\, m(dt) \ \text{ and } \ \nu(A) = \int_A \|f(t)\|\, v(M, dt)$$

for each $A \in \Sigma(v(M))$, respectively. By Proposition 3.7(a) we have that $n \in S_N$, and ν is $v(M)$-continuous. We now proceed by showing that n is ν-continuous, because then N will be $v(M)$-continuous, and hence non-atomic). Indeed, let $A \in \Sigma(v(M))$ and let $\{A_j \mid j \in J\}$ be an arbitrary finite partition of A into mutually disjoint sets $A_j \in \Sigma(v(M))$. Then

$$\begin{aligned}
\sum_{j\in J} \|n(A_j)\| &= \sum_{j\in J} \| \int_{A_j} f(t)\, m(dt))\| \\
&\leq \sum_{j\in J} \int_{A_j} \|f(t)\|\, v(M, dt) \\
&= \sum_{j\in J} \nu(A_j) \\
&= \nu(A).
\end{aligned}$$

Then, since $v(n, A) = \sup_J \sum_{j\in J} \|n(A_j)\|$, it follows that $v(n) \leq \nu$ and consequently n is ν-continuous; therefore n is $v(M)$-continuous. But from $N(A) = \{n(A) \mid \ n \in S_N\}$ follows immediately that N is $v(M)$-continuous. $\square$

EXAMPLE 3.11.

(i) The following example shows that Theorem 3.9 fails if Z is infinite-dimensional: Let $T = [0,1]$, suppose that Σ is the Lebesgue σ-algebra of subsets of T and λ is the Lebesgue measure on Σ. Put $Z = L^1_{\mathbb{R}}(T,\Sigma,\lambda)$ and define $F : T \to \mathbb{R}$ by $F(t) = \{0,1\}$ and $M : \Sigma \to L^1_{\mathbb{R}}(T,\Sigma,\lambda)$ by $M(A) = \{\chi_A\}$. Then

$$\int_A F(t)\, M(dt) = \left\{ \int_A F(t)\, m(dt) \mid m \in S_M \right\} = \{R(m) \mid m \in S_M\} = \{0,1\},$$

which is not convex.

(ii) The following example shows that Theorem 3.9 fails if the multimeasure M is atomic: Let $T = \{t_0\}$ and $\Sigma = \{\emptyset, T\}$. Define $F : T \to \mathbb{R}$ by $F(t) = \{0,1\}$ and $M : \Sigma \to \mathbb{R}$ by $M(\emptyset) = \{0\}$ and $M(T) = \{1\}$. Then M is atomic and $\int_A F(t)\, M(dt) = \{0,1\}$, which is not convex.

Theorem 3.12. *Suppose that X and Z are separable Banach spaces. If $M : \Sigma(v(M)) \to \mathcal{P}_f(Y)$ is a non-atomic multimeasure of bounded variation $v(M)$ and $F : T \to \mathcal{P}_f(X)$ is an integrably bounded $v(M)$-measurable multifunction, then $\overline{\int_A F(t)\, M(dt)}$ is a convex subset of Z for each $A \in \Sigma(v(M))$.*

Proof. Let $z_1, z_2 \in \overline{\int_A F(t)\, M(dt)}$ for $A \in \Sigma(v(M))$, let $\epsilon > 0$ and let $\alpha \in [0,1]$. Then we need to establish the existence of an $m \in S_M$ and an $f \in S^1_F(m)$ such that

$$\|\alpha z_1 + (1-\alpha) z_2 - \int_A f(t)\, m(dt)\| < \epsilon.$$

Since $z_1, z_2 \in \overline{\int_A F(t)\, M(dt)}$, there exist $m_i \in S_M$ and $f_i \in S^1_F(m_i)$, $i = 1,2$, such that

$$\|z_1 - \int_A f_1(t)\, m_1(dt)\| < \frac{\epsilon}{2} \ \text{ and } \ \|z_2 - \int_A f_2(t)\, m_2(dt)\| < \frac{\epsilon}{2}.$$

Define the set functions $n_i : \Sigma(v(M)) \to Z$ $(i = 1,2)$ by

$$n_i(A) = \int_A f_i(t)\, m_i(dt), i = 1,2.$$

In the same way as in the proof of the previous theorem we can prove that each n_i is a non-atomic measure. Consider now the Banach space $Z \times Z$ with norm defined by $\|(z_1,z_2)\| = \sqrt{\|x_1\|^2 + \|x_2\|^2}$ and define the set function $n : \Sigma(v(M)) \to Z \times Z$ by

$$n(A) = (n_1(A), n_2(A)) = \left(\int_A f_1(t)\, m_1(dt), \int_A f_2(t)\, m_2(dt) \right).$$

Then n is a non-atomic measure with finite variation. Indeed, suppose, on the contrary that E is a atom for n. Then, for all $E' \subset E$, $E' \in \Sigma(A, v(m))$, we have that either $n(E') = 0$ or $n(E \backslash E') = 0$. This in turn implies that either $n_1(E') = 0 = n_2(E')$ or $n_1(E \backslash E') = 0 = n_2(E \backslash E')$, contradicting the fact that both n_1 and n_2 is non-atomic. From the Lyapunov convexity theorem, it follows that the closure of the range $R(n)$ of n is convex in $Z \times Z$. Consequently, if $A \in \Sigma(v(M))$, then

$$\alpha n(A) + (1-\alpha)n(\emptyset) = \alpha n(A) \in \overline{R(n)}.$$

This means that there exists a set $A_\alpha \subseteq A$ such that

$$\|\alpha n(A_\alpha) - n(A_\alpha)\| < \frac{\epsilon}{4} \text{ and } \|(1-\alpha)n(A) - n(A \backslash A_\alpha)\| < \frac{\epsilon}{4},$$

that is

$$\|\alpha \int_A f_i(t)\, m_i(dt) - \int_{A_\alpha} f_i(t)\, m_i(dt)\| < \frac{\epsilon}{4}$$

and

$$\|(1-\alpha) \int_A f_i(t)\, m_i(dt) - \int_{A \backslash A_\alpha} f_i(t)\, m_i(dt)\| < \frac{\epsilon}{4}$$

for $i = 1, 2$. If we put

$$f = f_1 \chi_{A_\alpha} + f_2 \chi_{T \backslash A_\alpha} \text{ and } m = \chi_{A_\alpha} m_1 + \chi_{T \backslash A_\alpha} m_2,$$

then $m \in S_M$, $f \in S_F^1(m)$ and $m(A) = m_1(A_\alpha) + m_2(A \backslash A_\alpha)$ for all $A \in \Sigma(v(M))$. The result then follows from the fact that

$$\begin{aligned}
& \|\alpha z_1 + (1-\alpha)z_2 - \int_A f(t)\, m(dt)\| \\
\leq\ & \|\alpha z_1 - \alpha \int_A f_1(t)\, m_1(dt)\| + \|\alpha \int_A f_1(t)\, m_1(dt) - \int_{A_\alpha} f_1(t)\, m_1(dt)\| + \\
& \|(1-\alpha)z_2 - (1-\alpha) \int_A f_2(t)\, m_2(dt)\| + \\
& \qquad \|(1-\alpha) \int_A f_2(t)\, m_2(dt) - \int_{A \backslash A_\alpha} f_2(t)\, m_2(dt)\| \\
<\ & \alpha \frac{\epsilon}{2} + \frac{\epsilon}{4} + (1-\alpha)\frac{\epsilon}{2} + \frac{\epsilon}{4} = \epsilon
\end{aligned}$$

∎

By making use of the facts that that S_M is $w(ca(Y), \mathcal{E}_{I\!R}(v(m)) \otimes Y')$-compact and $S_F^1(m)$ is weakly compact in $\mathcal{L}_X^1(m)$, we have

THEOREM 3.13. *Let T be a countable union of sets of the ring $\mathcal{R}$ and suppose that $Y = Z$ is a separable reflexive Banach space. If $F : T \to \mathcal{P}_{kc}(I\!R^n)$ is an integrably bounded $v(M)$-measurable multifunction and $M : \Sigma(v(M)) \to \mathcal{P}_{wkc}(Y)$ is a multimeasure of bounded variation $v(M)$, then for each $A \in \Sigma(v(M))$ the set $\int_A F(t)\, M(dt)$ is a convex and $w(Y, Y')$-compact subset of Y.*

In [3] Artstein discussed Radon-Nikodým derivatives of multimeasures whose values are convex sets in $I\!R^n$ while Castaing [5] and Godet-Thobie [14] gave Radon-Nikodým theorems for multimeasures with compact and convex values in a locally convex topological space. Note that Theorem 9.1 on page 120 in [3] has been shown in [9, pp. 305, 308] to be false. Costé [7] and Hiai [16] discussed Radon-Nikodým theorems for multimeasures whose values are closed, bounded and convex sets in a separable Banach space. Papageorgiou [24] proved two set-valued Radon-Nikodým theorems for transition multimeasures, and the results were recently ([25]) extended to the case where the dominating control measure is a transition measure. We now continue by establishing Radon-Nikodým-type theorems for our bilinear set-valued integral. In our first result the range spaces of the multimeasure and multifunction are finite-dimensional while in the results thereafter we take the range spaces to be arbitrary Banach spaces.

Theorem 3.14 (Radon-Nikodým). *Let T be a countable union of sets of the ring $\mathcal{R}$, μ is a scalar measure on $\mathcal{R}$ and let $M : \Sigma(v(M)) \to \mathcal{P}_{kc}(I\!R^n)$ be a multimeasure of bounded variation $v(M)$. If M is μ-continuous on $\mathcal{R}$, then there exists an integrably bounded $v(M)$-measurable multifunction $F : T \to \mathcal{P}_{kc}(I\!R^n)$ such that*

$$M(A) = \int_A F(t)\,\mu(dt)$$

for each $A \in \Sigma(v(M))$.

Proof. From Theorem 6 on page 67 of [15] follows that

$$M(A) = \{m(A) \mid m \in S_M\}$$

for all $A \in \mathcal{R}$. This means that every $m \in S_M$ is μ-continuous on $\mathcal{R}$. Since μ has the direct sum property, from Theorem 5 on page 182 of [11] follows that for each $m \in S_M$ there exists a locally μ-integrable function $f_m : T \to I\!R^n$ such that

$$m(A) = \int_A f_m(t)\,\mu(dt), \quad A \in \mathcal{R}.$$

Put

$$K = \{f_m\chi_A \in \mathcal{L}^1_{I\!R^n}(\mu) \mid A \in \mathcal{R}, m \in S_M\}.$$

We first show that K is a closed subset of $\mathcal{L}^1_{I\!R^n}(\mu)$. So let $f_{m_k}\chi_A \in K$ and let f be such that $f_{m_k}\chi_A \to f$ in $\mathcal{L}^1_{I\!R^n}(\mu)$. Since S_M is compact for the topology of pointwise convergence, there exists an $m' \in S_M$ such that $m_k(A) \to m'(A)$

for all $A \in \mathcal{R}$. But

$$\begin{aligned}\lim_{k\to\infty}\int(f_{m_k}\chi_A)(t)\,\mu(dt) &= \lim_{k\to\infty}\int_A f_{m_k}(t)\,\mu(dt)\\ &= \int_A f_{m'}(t)\,\mu(dt)\\ &= \int(f_{m'}\chi_A)(t)\,\mu(dt)\end{aligned}$$

so that $f = f_{m'}\chi_A \in K$.

Evidently, K is convex, and by Theorem 3.1 of [17] we obtain an integrably bounded μ-measurable multifunction $F : T \to \mathcal{P}_{kc}(\mathbb{R}^n)$ such that $K = S_F^1(\mu)$. Then, for each $A \in \mathcal{R}$,

$$\begin{aligned}M(A) = \{m(A) \mid m \in S_M\} &= \left\{\int_A f_m(t)\,\mu(dt) \mid m \in S_M\right\}\\ &= \left\{\int(f_m\chi_A)(t)\,\mu(dt) \mid f_m\chi_A \in K\right\}\\ &= \left\{\int_A f_m(t)\,\mu(dt) \mid f_m \in S_F^1(\mu)\right\}\\ &= \int_A F(t)\,\mu(dt).\end{aligned}$$

□

Corollary 3.15. *Under the hypotheses of Theorem 3.12 follows that there exists a unique integrably bounded $v(M)$-measurable multifunction $F : T \to \mathcal{P}_{kc}(\mathbb{R}^n)$ such that*

$$M(A) = \int_A F(t)\,\mu(dt)$$

for each $A \in \Sigma(v(M))$.

Proof. Let $G : T \to \mathcal{P}_{kc}(\mathbb{R}^n)$ be a multifunction such that $M(A) = \int_A G(t)\,\mu(dt)$ for each $A \in \Sigma(v(M))$. Define $\phi : S_M \to S_F^1(\mu)$ by

$$\phi(m) = \frac{m(dt)}{\mu(dt)}.$$

Then it follows that ϕ is a linear isometric bijection and

$$S_F^1(\mu) = \phi(S_M) = S_G^1(\mu).$$

From Corollary 1.2 of [17] follows then that $F(t) = G(t)$ $v(M)$-almost everywhere on T. □

Corollary 3.16. *Let T be a countable union of sets of the ring $\mathcal{R}$ and let μ be a scalar measure on $\mathcal{R}$. If $M : \Sigma(v(M)) \to \mathcal{P}_{kc}(I\!\!R^n)$ is a μ-continuous multimeasure of bounded variation $v(M)$, then*

$$S_M = \left\{ \int_{(\cdot)} f(t)\,\mu(dt) \mid f \in S^1_F(\mu), M(dt) = F(t)\mu(dt) \right\}.$$

Proof. Let $m \in S_M$. Then for all $A \in \Sigma(v(M))$ we have that $m(A) \in M(A)$ so that m is also μ-continuous. From Theorem 5 on page 182 of [11] we obtain an $f \in \mathcal{L}^1_{I\!\!R}(\mu)$ such that $m(A) = \int_A f(t)\,\mu(dt)$ for every $A \in \Sigma(\mu)$. Then $\int_A f(t)\,\mu(dt) \in \int_A F(t)\,\mu(dt)$ for all $A \in \Sigma(\mu)$. This shows that $f \in S^1_F(\mu)$ and consequently

$$S_M \subseteq \left\{ \int_{(\cdot)} f(t)\,\mu(dt) \mid\ f \in S^1_F(\mu), M(dt) = F(t)\mu(dt) \right\}.$$

For the inverse inclusion, let $f \in S^1_F(\mu)$ and consider $m(A) = \int_A f(t)\,\mu(dt)$, $A \in \Sigma(\mu)$. Then Proposition 3.7(a) implies that $m \in S_M$, and

$$\left\{ \int_{(\cdot)} f(t)\,\mu(dt) \mid f \in S^1_F(\mu), M(dt) = F(t)\mu(dt) \right\} \subseteq S_M.$$

$\square$

Corollary 3.17. *Let T be a countable union of sets of the ring $\mathcal{R}$, μ is a scalar measure on $\mathcal{R}$ and for $i = 1, 2$ let $M_i : \Sigma(v(M_i)) \to \mathcal{P}_{kc}(I\!\!R^n)$ be a μ-continuous multimeasure of bounded variation $v(M_i)$. If $M_1(A) \subseteq M_2(A)$ for every $A \in \Sigma(v(M_2))$, then for $i = 1, 2$ there exists an integrably bounded $v(M_i)$-measurable multifunction $F_i : T \to \mathcal{P}_{kc}(I\!\!R^n)$ such that $F_1(t) \subseteq F_2(t)$.*

Proof. From Theorem 3.12 we obtain an integrably bounded $v(M_i)$-measurable multifunction $F_i : T \to \mathcal{P}_{kc}(I\!\!R^n)$ such that

$$M(A_i) = \int_A F_i(t)\,\mu(dt), i = 1, 2.$$

Since $M_1(A) \subseteq M_2(A)$ for every $A \in \Sigma(v(M_2))$, it follows that $\sigma(p, M_1(A)) \leq \sigma(p, M_2(A))$ for $p \in I\!\!R^n$. Consequently, for $p \in I\!\!R^n$,

$$\begin{aligned}\int_A \sigma(p, F_1(t))\,\mu(dt) &= \sigma(p, \int_A F_1(t)\,\mu(dt)) \\ &\leq \sigma(p, \int_A F_2(t)\,\mu(dt)) = \int_A \sigma(p, F_2(t))\,\mu(dt)\end{aligned}$$

and we deduce that $\sigma(p, F_1(t)) \leq \sigma(p, F_2(t))$. Since both F_1 and F_2 are convex and compact-valued, it then follows that $F_1(t) \subseteq F_2(t)$. $\square$

COROLLARY 3.18. *Let T be a countable union of sets of the ring $\mathcal{R}$ and μ is a non-atomic scalar measure on $\mathcal{R}$. If $M : \Sigma(v(M)) \to \mathcal{P}_{kc}(\mathbb{R}^n)$ is a μ-continuous multimeasure of bounded variation $v(M)$, then there exists a multimeasure $N : \Sigma(v(M)) \to \mathcal{P}_{kc}(\mathbb{R}^n)$ of bounded variation such that $co\, M(A) = N(A)$ for each $A \in \Sigma(v(M))$.*

PROOF. From Theorem 3.12 follows that there exists an integrably bounded $v(M)$-measurable multifunction $F : T \to \mathcal{P}_{kc}(\mathbb{R}^n)$ such that $M(A) = \int_A F(t)\,\mu(dt)$ for each $A \in \Sigma(v(M))$. But then

$$co\, M(A) = co \int_A F(t)\,\mu(dt) = \int_A F(t)\,\mu(dt) = \int_A co\, F(t)\,\mu(dt),$$

where the last equality follows from the fact that $\int_A F(t)\,\mu(dt)$ is a convex set, and

$$\begin{aligned} \sigma(p, \int_A F(t)\mu(dt)) &= \int_A \sigma(p, F(t))\,\mu(dt) \\ &= \int_A \sigma(p, co\, F(t))\,\mu(dt) \\ &= \sigma(p, \int_A co\, F(t)\mu(dt)) \end{aligned}$$

for every $p \in \mathbb{R}^n$. If we put $N(A) = \int_A co\, F(t)\,\mu(dt)$, then N is the desired multimeasure. Indeed, by Lemma 8.3 of [20] follows that $co\, F$ is μ-measurable. According to Theorem 3.6 we then only need to show that $co\, F$ is integrably bounded. To start with, first note that from the integrably boundedness of F we obtain a $k \in \mathcal{L}^1_{\mathbb{R}}(\mu)$ such that $\|F(t)\| \leq k(t)$ for every $t \in T \backslash N$, where N is some μ-negligible subset of T. Let $x(t) \in co\, F(t)$ for $t \in T$. Then $x(t) = \sum_{j=1}^{n+1} \alpha_j(t) x_j(t)$, where $x_j(t) \geq 0$, $\sum_{j=1}^{n+1} \alpha_j(t) = 1$ and $x_j(t) \in F(t)$ for $j = 1, 2, \ldots, n+1$. If $t \in T \backslash N$, then it follows easily that $\|x(t)\| \leq k(t)$ so that $co\, F$ is indeed integrably bounded. □

THEOREM 3.19. *Let T be a countable union of sets of the ring $\mathcal{R}$ and let X and Z be separable Banach spaces such that $Y \subseteq \mathcal{L}(X, Z)$ and $Z = W'$, where W is a norming subspace of Z'. If $M : \Sigma(v(M)) \to \mathcal{P}_k(Y)$ is a multimeasure of bounded variation $v(M)$ and $F : T \to \mathcal{P}_f(X)$ is an integrably bounded $v(M)$-measurable multifunction, then there exists an integrably bounded $v(M)$-measurable multifunction $G : T \to \mathcal{P}_k(Z)$ such that*

$$\int_A F(t)\, M(dt) = \int_A G(t)\, v(M, dt)$$

for each $A \in \Sigma(v(M))$.

PROOF. From Theorem 3.2 we have that $\int_A F(t)\,M(dt) \neq \emptyset$ for $A \in \Sigma(v(M))$. Then

$$\begin{aligned}\int_A F(t)\,M(dt) &= \left\{\int_A f(t)\,m(dt) \mid f \in S_F^1(m), m \in S_M\right\}\\ &= \left\{\int_A U_m f(t)\,v(m,dt) \mid f \in S_F^1(m), m \in S_M\right\},\end{aligned}$$

where $U_m : T \to \mathcal{L}(Y,Z)$ is the function whose existence is guaranteed by Theorem 4 on page 263 of [11]. If, for each $t \in T$, we define

$$G(t) = \{(U_m f)(t) \mid f \in S_F^1(m), m \in S_M\},$$

then G is the desired multifunction. Indeed, first note that $G(t) \in \mathcal{P}_k(Z)$ because both S_M and S_F^1 are compact for the topology of pointwise convergence. To prove the integrably boundedness of G, let $z \in G(t)$ for all $t \in T$. Then there exist $m' \in S_M$ and $f' \in S_F^1(m')$ such that $z = U_{m'}(t)f'(t)$ for all $t \in T$. Therefore

$$\|z\| = \|U_{m'}(t)f'(t)\| \le \|U_{m'}(t)\|\,\|f'(t)\| = \|f'(t)\|,$$

which implies that G is indeed integrably bounded.

Furthermore, since the mapping $t \mapsto (U_m f)(t)$ is $v(m)$-measurable for each $m \in S_M$ and $f \in S_F^1(m)$, it follows immediately that G is also $v(M)$-measurable. Obviously, for each $A \in \Sigma(v(M))$ we have that

$$\int_A F(t)\,M(dt) = \int_A G(t)\,v(M,dt).$$

□

By making use of Theorem 5 on page 269 of [11] and the remarks on page 271 of [11] we now have the following corollary, the proof of which is similar to the previous theorem.

COROLLARY 3.20. *Let T be a countable union of sets of the ring $\mathcal{R}$ and let X and Z be separable Banach spaces such that $Y \subseteq \mathcal{L}(X,Z)$ and $Z = W'$, where W is a norming subspace of Z'. Suppose that $M : \Sigma(v(M)) \to \mathcal{P}_k(Y)$ is a multimeasure of bounded variation $v(M)$ and let $F : T \to \mathcal{P}_f(X)$ be an integrably bounded $v(M)$-measurable multifunction. If μ is a scalar measure on $\mathcal{R}$ with the direct sum property such that M is μ-continuous, then there exists an integrably bounded $v(M)$-measurable multifunction $G : T \to \mathcal{P}_k(Z)$ such that*

$$\int_A F(t)\,M(dt) = \int_A G(t)\,\mu(dt)$$

for each $A \in \Sigma(v(M))$.

THEOREM 3.21. *Let T be a countable union of sets of the ring $\mathcal{R}$ and let μ be a scalar measure on $\mathcal{R}$. If $M : \Sigma(v(M)) \to \mathcal{P}_{kc}(\mathbb{R}^n)$ is a multimeasure of bounded variation $v(M)$ such that M is μ-continuous and if $F : T \to \mathcal{P}_f(X)$ is an integrably bounded $v(M)$- measurable multifunction, then there exists an integrably bounded $v(M)$-measurable multifunction $G : T \to \mathcal{P}_{kc}(\mathbb{R}^n)$ such that*

$$\int_A F(t)\, M(dt) = \int_A F(t)G(t)\, \mu(dt)$$

for all $A \in \Sigma(v(M))$.

PROOF. By Theorem 3.12 we obtain an integrably bounded $v(M)$-measurable multifunction $G : T \to \mathcal{P}_{kc}(\mathbb{R}^n)$ such that $M(A) = \int_A G(t)\, \mu(dt)$ for all $A \in \Sigma(v(M))$. Therefore, for $m \in S_M$, there exists a $g \in S^1_G(\mu)$ such that

$$m(A) = \int_A g(t)\, \mu(dt).$$

Then, since $m(dt) = g(t)\mu(dt)$, we have that

$$\int_A f(t)\, m(dt) = \int_A f(t)g(t)\, \mu(dt) \in \int_A F(t)G(t)\, \mu(dt),$$

for every $f \in S^1_F(m)$ and therefore $\int_A F(t)\, M(dt) \subseteq \int_A F(t)G(t)\, \mu(dt)$ for all $A \in \Sigma(v(M))$. The inverse inclusion follows similarly. □

THEOREM 3.22. *Let T be a countable union of sets of the ring $\mathcal{R}$ and let X and Z be separable Banach spaces such that $Y \subseteq \mathcal{L}(X, Z)$ and $Z = W'$, where W is a norming subspace of Z'. Suppose that $M : \Sigma(v(M)) \to \mathcal{P}_{wkc}(Y)$ is a multimeasure of bounded variation $v(M)$ and let $F : T \to \mathcal{P}_{wkc}(X)$ be an integrably bounded $v(M)$-measurable multifunction. If μ is a scalar measure on $\mathcal{R}$ with the direct sum property such that M is μ-continuous, then $\int_A F(t)\, M(dt)$ is a convex and $w(Z, Z')$-compact subset of Z for every $A \in \Sigma(v(M))$.*

PROOF. From Corollary 3.18 we obtain an integrably bounded $v(M)$-measurable multifunction $G : T \to \mathcal{P}_{wkc}(Z)$ such that $\int_A F(t)\, M(dt) = \int_A G(t)\, \mu(dt)$ for each $A \in \Sigma(v(M))$. If we put $N(A) = \int_A G(t)\, \mu(dt)$, then we know that $N : \Sigma(v(M)) \to \mathcal{P}_{wkc}(Z)$ is a multimeasure of bounded variation and $S_N \neq \emptyset$. Let $n \in S_N$ and define $\psi : L^1_Z(\mu) \to Z$ and $\phi : S_N \to Z$ by

$$\psi([g]) = \int g(t)\, \mu(dt), g \in [g] \;\text{ and }\; \phi(n) = \frac{n(dt)}{\mu(dt)}.$$

Then ψ is continuous with respect to the norm topologies of the spaces $L^1_Z(\mu)$ and Z. Also, since ϕ is a linear isometric bijection, it is continuous with respect to the norm topologies of the spaces $ca(Z)$ and $L^1_Z(\mu)$. Theorem 15 on page 422 of [12] asserts that ψ is continuous with respect to the topologies

$w(L^1_Z(\mu), L^\infty_{Z'}(\mu))$ and $w(Z, Z')$ of the spaces $L^1_Z(\mu)$ and Z, while ϕ is continuous with respect to the topologies $w(ca(Z), \mathcal{E}_{I\!R}(\mu) \otimes Z')$ and $w(L^1_Z(\mu), L^\infty_{Z'}(\mu))$ of the spaces $ca(Z)$ and $L^1_Z(\mu)$. We then have that

$$(\psi \circ \phi)(S_N) = \psi(\phi(S_N)) = \psi(S^1_G(\mu)) = \int_A G(t)\,\mu(dt) = \int_A F(t)\,M(dt).$$

Since $\psi \circ \phi$ is continuous with respect the topologies $w(ca(Z), \mathcal{E}_{I\!R}(\mu) \otimes Z')$ and $w(Z, Z')$ of $ca(Z)$ and Z, and since S_N is a convex and $w(ca(Z), \mathcal{E}_{I\!R}(\mu) \otimes Z')$-compact subset of $ca(Z)$, we then have that $\int_A F(t)\,M(dt)$ is a convex and $w(Z, Z')$-compact subset of Z. □

Corollary 3.23. *Under the conditions of the previous theorem we have that $\int_A F(t)\,M(dt)$ is a convex and closed subset of Z for every $A \in \Sigma(v(M))$.*

Proof. By the previous theorem, $\int_A F(t)\,M(dt)$ is a convex and $w(Z, Z')$-compact subset of Z; therefore $\int_A F(t)\,M(dt)$ is also $w(Z, Z')$-closed. The result then follows from page 422 of [12]. □

The following corollary is a result of the fact that the weak and strong topologies coincide on finite-dimensional spaces.

Corollary 3.24. *Let T be a countable union of sets of the ring $\mathcal{R}$ and let μ be a scalar measure on $\mathcal{R}$ with the direct sum property. If $M : \Sigma(v(M)) \to \mathcal{P}_{kc}(I\!R^n)$ is a μ-continuous multimeasure of bounded variation $v(M)$ and $F : T \to \mathcal{P}_{kc}(I\!R^m)$ is an integrably bounded $v(M)$-measurable multifunction, then $\int_A F(t)\,M(dt)$ is a convex and compact subset of $I\!R^{nm}$ for every $A \in \Sigma(v(M))$.*

Proof. If we put $Z = I\!R^{nm}$, $W = I\!R^{nm}$ and consider $I\!R^n \subseteq \mathcal{L}(I\!R^m, I\!R^{nm})$, then it follows immediately that $Z = W'$ and W is a norming subspace of Z'. By Theorem 3.20 follows then that $\int_A F(t)\,M(dt)$ is convex and $w(I\!R^{nm}, (I\!R^{nm})')$-compact, and therefore convex and compact in $I\!R^{nm}$. □

Theorem 3.25. *Let T be a countable union of sets of the ring $\mathcal{R}$ and let μ be a non-atomic scalar measure on $\mathcal{R}$. If $M : \Sigma(v(M)) \to \mathcal{P}_{kc}(I\!R^n)$ is a multimeasure of bounded variation $v(M)$ such that M is μ-continuous, and if $F : T \to \mathcal{P}_{kc}(I\!R^n)$ is an integrably bounded $v(M)$-measurable multifunction, then*

$$\int_A F(t)\,M(dt) = \int_A F(t)\,ext\,M(dt)$$

for all $A \in \Sigma(v(M))$.

Proof. We only need to prove that $\int_A F(t)\,M(dt) \subseteq \int_A F(t)\,ext\,M(dt)$ because the inverse inclusion follows obviously. By Theorem 3.19 we have that $\int_A F(t)\,M(dt) = \int_A F(t)G(t)\,\mu(dt)$, where $G : T \to \mathcal{P}_{kc}(I\!R^n)$ is an integrably bounded $v(M)$-measurable multifunction. But since $F(t)G(t) = co\, ext\, F(t)G(t)$, we then have that

$$\int_A F(t)G(t)\,\mu(dt) = \int_A co\, ext\, F(t)G(t)\,\mu(dt) = \int_A ext\, F(t)G(t)\,\mu(dt).$$

Then we only need to show that $\int_A ext\, F(t)G(t)\,\mu(dt) \subseteq \int_A F(t) ext\, M(dt)$. For this purpose, let $h \in S^1_{FG}(\mu)$ and $m \in S_M$. Then the proof will be complete if we can show that $\int_A h(t)\,\mu(dt) = \int_A f(t)\,m(dt)$ for $f \in S^1_F(m)$ because then

$$\int_A h(t)\,\mu(dt) = \int_A f(t)\,m(dt) \in co \int_A f(t)\, ext\, M(dt) \in \int_A F(t)\, ext\, M(dt).$$

But if $g \in S^1_G(\mu)$ and $f \in S^1_F(m)$, then $\int_A h(t)\,\mu(dt) = \int_A f(t)g(t)\,\mu(dt) = \int_A f(t)\,m(dt)$ and the proof is complete. ∎

THEOREM 3.26. *Let T be a countable union of sets of the ring $\mathcal{R}$, and suppose that $M : \Sigma(v(M)) \to \mathcal{P}_{kc}(\mathbb{R}^p)$ is a multimeasure of bounded variation $v(M)$ and $F : T \to \mathcal{P}_f(\mathbb{R}^n)$ is an integrably bounded $v(M)$-measurable multifunction. If μ is a non-atomic scalar measure on $\mathcal{R}$ such that M is μ-continuous and if $Z = \mathbb{R}^{np}$, then*

$$\int_A F(t)\,M(dt) = \int_A F(t)\,co\,M(dt)$$

for all $A \in \Sigma(v(M))$.

PROOF. By Theorem 3.19 follows that there is an integrably bounded $v(M)$-measurable multifunction $G : T \to \mathcal{P}_{kc}(\mathbb{R}^p)$ such that $\int_A F(t)\,M(dt) = \int_A F(t)G(t)\,\mu(dt)$ for all $A \in \Sigma(v(M))$. Since $\int_A F(t)G(t)\,\mu(dt) = \int_A co\,F(t)G(t)\,\mu(dt)$ for all $A \in \Sigma(v(M))$, we only need to prove that

$$\int_A co\,F(t)G(t)\,\mu(dt) = \int_A F(t) co\,M(dt)$$

for all $A \in \Sigma(v(M))$. So let $m \in S_{co\,M}$. Then $m(A) \in co\,M(A)$ for all $A \in \Sigma(v(M))$. But

$$co\,M(A) = \int_A F(t)\,\mu(dt) = \int_A co\,F(t)\,\mu(dt),$$

that is, there is a $g \in S^1_{co\,F}(\mu)$ such that $m(A) = \int_A g(t)\,\mu(dt)$. Then since $m(dt) = g(t)\mu(dt)$, it follows immediately that $\int_A f(t)\,m(dt) = \int_A f(t)g(t)\,\mu(dt)$ and the proof is complete. ∎

THEOREM 3.27. *Let T be a countable union of sets of the ring $\mathcal{R}$ and let X and Y be separable Banach spaces. Suppose that $M : \Sigma(v(M)) \to \mathcal{P}_f(Y)$ is a multimeasure of bounded variation $v(M)$ and $F : T \to \mathcal{P}_f(X)$ is an integrably bounded $v(M)$-measurable multifunction. Then*

(a) $\overline{\int_A F(t)\,\overline{co}\,M(dt)} = \overline{co} \int_A F(t)\,M(dt)$ *for all $A \in \Sigma(v(M))$.*

(b) *If M is in addition non-atomic, then* $\overline{\int_A F(t)\,\overline{co}\,M(dt)} = \overline{\int_A F(t)\,M(dt)}$ *for all* $A \in \Sigma(v(M))$.

Proof. Since $S_{\overline{co}\,M} = \overline{co}S_M$ (by Theorem 4.2 of [23]), statement (a) then follows from the fact that

$$\begin{aligned}
\overline{\int_A F(t)\,\overline{co}\,M(dt)} &= \overline{\left\{\int_A f(t)\,m(dt) \mid f \in S^1_F(m), m \in S_{\overline{co}\,M}\right\}} \\
&= \overline{co}\left\{\int_A f(t)\,m(dt) \mid f \in S^1_F(m), m \in S_M\right\} \\
&= \overline{co}\int_A F(t)\,M(dt)
\end{aligned}$$

for all $A \in \Sigma(v(M))$. To prove the second statement, assume that M is non-atomic. Since $\overline{\int_A F(t)\,M(dt)}$ is convex, we have that $\overline{\int_A F(t)\,\overline{co}\,M(dt)} = \overline{co}\int_A F(t)\,M(dt) = \overline{\int_A F(t)\,M(dt)}$ for all $A \in \Sigma(v(M))$. $\square$

Note that instead of assuming that M is non-atomic in the second statement of the above theorem, we may let M be convex-valued. Indeed, if this is the case, then $\overline{S}_M = \overline{co}S_M = S_{\overline{co}\,M}$ so that $\overline{\int_A F(t)\,\overline{co}\,M(dt)} = \overline{\int_A F(t)\,M(dt)}$ for all $A \in \Sigma(v(M))$.

Theorem 3.28. *Let T be a countable union of sets of the ring $\mathcal{R}$ and let X and Y be separable Banach spaces. Suppose that $M : \Sigma(v(M)) \to \mathcal{P}_k(Y)$ is a multimeasure of bounded variation $v(M)$ and $F : T \to \mathcal{P}_k(X)$ is an integrably bounded $v(M)$-measurable multifunction. Then*

(a) $\overline{\int_A F(t)\,M(dt)} = \overline{\int_A F(t)\,ext\,M(dt)}$ *for all* $A \in \Sigma(v(M))$.

(b) *If M is in addition convex, then* $\int_A F(t)\,M(dt) = \overline{\int_A F(t)\,ext\,M(dt)}$ *for all* $A \in \Sigma(v(M))$.

Proof. (a) By the Krein-Milman theorem follows that $\overline{co}\,M = \overline{co}\,ext\,M$. Consequently, by applying the previous theorem twice, we have

$$\begin{aligned}
\overline{\int_A F(t)\,M(dt)} &= \overline{\int_A F(t)\,\overline{co}\,M(dt)} \\
&= \overline{\int_A F(t)\,\overline{co}\,ext\,M(dt)} \\
&= \overline{\int_A F(t)\,ext\,M(dt)}
\end{aligned}$$

for all $A \in \Sigma(v(M))$. For statement (b), note that since M is convex-valued, we have that $S_M = \overline{co}\,ext\,S_M$ and the result follows immediately. $\square$

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Department of Mechanical Engineering
Technikon Free State
Private Bag X20539
Bloemfontein 9300
South Africa
E-mail: hbrink@eng.tofs.ac.za

Department of Mathematics
University of Stellenbosch
Private Bag X1
Matieland 7602
South Africa
E-mail: pm@land.sun.ac.za

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