GLASNIK MATEMATIČKI
Vol. 35(55)(2000), 339 – 354

EXTENSION DIMENSION OF INVERSE LIMITS

Sibe Mardešić
University of Zagreb, Croatia

Abstract. Recently L.R. Rubin and P.J. Schapiro have considered inverse sequences $\boldsymbol{X}$ of metrizable spaces X_i, whose extension dimension $\dim X_i \leq P$, i.e., $P \in \mathrm{AE}(X_i)$, where P is an arbitrary polyhedron (or CW-complex). They proved that $\dim X \leq P$, where $X = \lim \boldsymbol{X}$. The present paper generalizes their result to inverse sequences of stratifiable spaces, giving at the same time a more conceptual proof.

1. Introduction

By a polyhedron P we mean the geometric realization $|K|$ of a simplicial complex K endowed with the CW-topology. We say that the extension dimension of a space X does not exceed P, and we write $\dim X \leq P$, provided every mapping $f\colon A \to P$ from a closed subset $A \subseteq X$ to P admits an extension to all of X, i.e., P is an absolute extensor for X, $P \in \mathrm{AE}(X)$. Formally, extension dimension (for compacta) was first introduced in a 1994 paper by A. Dranishnikov [9]. It was further studied by A. Dranishnikov and J. Dydak [10] and other authors.

A classical theorem of dimension theory asserts that, for normal spaces X, the covering dimension $\dim X \leq n$ if and only if $\dim X \leq S^n$ (see e.g., Theorem 3.2.10 of [12]). If G is an abelian group and $K = K(G, n)$ is an Eilenberg-MacLane complex, then for paracompact spaces X, the cohomological dimension $\dim_G X \leq n$ if and only if $\dim X \leq |K|$. This follows from the work of H. Cohen [7], P.J. Huber [13], E.G. Sklyarenko [19] and Y. Kodama [14].

2000 *Mathematics Subject Classification.* 54B35, 54C55, 54F45.
Key words and phrases. inverse limit, covering dimension, cohomological dimension, extension theory, extension dimension, metrizable space, stratifiable space.

It is well known that for inverse systems $\boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ of compact Hausdorff spaces with $\dim X_\lambda \leq n$ the inverse limit X has dimension $\dim X \leq n$ (see e.g., Theorem 3.3.6 of [12]). The following general proposition is also easily proved (see Theorem 2.2 of [8]).

Proposition 1. *Let $\boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ be an inverse system of compact Hausdorff spaces with inverse limit X and let P be a polyhedron. If* $\dim X_\lambda \leq P$*, for every $\lambda \in \Lambda$, then also* $\dim X \leq P$.

Much deeper is a 1959 result of K. Nagami [16] (also see Theorem 4.1.22 of [12]), which asserts that the limit X of an inverse sequence $\boldsymbol{X} = (X_i, p_{ii+1})$ of metrizable spaces X_i with dimension $\dim X_i \leq n$ has dimension $\dim X \leq n$. Its generalization to an arbitrary polyhedron P and inverse sequences of metrizable spaces such that $\dim X_i \leq P$, for every $i \in \mathbb{N}$, was recently proved by L.R. Rubin and P.J. Schapiro [18]. The case of separable metric spaces was obtained earlier by A. Chigogidze [6]. In this theorem (as well as in Nagami's theorem) the assumption that the spaces X_i are metrizable cannot be replaced by the weaker condition that the spaces X_i be paracompact. Indeed, in 1980 M.G. Charalambous exhibited an inverse sequence of paracompact 0-dimensional spaces X_i, whose limit X is a normal space and $\dim X > 0$ [5].

The purpose of the present paper is to generalize the Rubin – Schapiro theorem to inverse sequences of stratifiable spaces and, more important, to give a more conceptual proof. The following is our main result.

Theorem 1. *Let P be a polyhedron and let $\boldsymbol{X} = (X_i, p_{ii+1})$ be an inverse sequence of stratifiable spaces with limit X. If* $\dim X_i \leq P$*, for all i, then also* $\dim X \leq P$.

The proof of Theorem 1 is a modification of the "natural" proof of Proposition 1. Therefore, we first outline that proof.

Proof of Proposition 1. Let $A \subseteq X$ be a closed set and let $f\colon A \to P$ be a mapping. Choose an open covering $\mathcal{V}$ of P such that any two $\mathcal{V}$-near mappings into P are homotopic. Consider the inverse system $\boldsymbol{A} = (A_\lambda, p_{\lambda\lambda'}, \Lambda)$, where $A_\lambda = p_\lambda(A)$ ($p_\lambda\colon X \to X_\lambda$ are the natural projections) and $p_{\lambda\lambda'}\colon A_{\lambda'} \to A_\lambda$ are the restrictions to $A_{\lambda'}$ of $p_{\lambda\lambda'}\colon X_{\lambda'} \to X_\lambda$. It is readily seen that A is the limit of $\boldsymbol{A}$ with projections $p_\lambda\colon A \to A_\lambda$, which are the restrictions to A of the projections $p_\lambda\colon X \to X_\lambda$. Since A is compact and $\boldsymbol{A}$ is an inverse system of compact spaces, $\boldsymbol{p} = (p_\lambda)\colon A \to \boldsymbol{A}$ is a resolution (see Theorem 1 of I.6.1 in [15]). Therefore, by a characterization of resolutions (see I.6.2 of [15]), there exist a $\lambda \in \Lambda$ and a mapping $g_\lambda\colon A_\lambda \to P$ such that the mapping $g = g_\lambda p_\lambda\colon A \to P$ is $\mathcal{V}$-near to f. Clearly, g factors through A_λ and, by the choice of $\mathcal{V}$, it is homotopic to f. Since $\dim X_\lambda \leq P$, g_λ extends to a mapping $h_\lambda\colon X_\lambda \to P$. Therefore, $h_\lambda p_\lambda\colon X \to P$ is an extension of g to all of X. Now the homotopy extension theorem implies that also f admits an extension to all of X. $\square$

REMARK 1. *Note that whenever a mapping $g\colon A \to P$ factors through some A_λ, then it also factors through all $A_{\lambda'}$, where λ' ranges through a cofinal subset of Λ. Indeed, it suffices to consider indices $\lambda' \geq \lambda$ and put $g_{\lambda'} = g_\lambda p_{\lambda\lambda'}$.*

In the case of an inverse system $\boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ of non-compact spaces with limit X and projections $p_\lambda\colon X \to X_\lambda$ we must replace factorization of mappings by the more general notion of a filtered factorization of mappings, defined as follows. Let $\boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ be an inverse system of spaces with limit X and projections $p_\lambda\colon X \to X_\lambda$. Let $U \subseteq X$ be a non-empty open set and let $g\colon U \to P$ be a mapping. By a *filtered factorization* of g through $\boldsymbol{X}$ we mean a family of open sets $G_\lambda \subseteq X_\lambda$, $\lambda \in \Lambda$, and a family of mappings $g_\lambda\colon \overline{G_\lambda} \to P$ which satisfy the following conditions.

$$p_{\lambda\lambda'}^{-1}(\overline{G_\lambda}) \subseteq G_{\lambda'},\ \lambda < \lambda', \tag{1.1}$$

$$\bigcup_{\lambda\in\Lambda} p_\lambda^{-1}(G_\lambda) = U, \tag{1.2}$$

$$g|p_\lambda^{-1}(\overline{G_\lambda}) = g_\lambda p_\lambda|p_\lambda^{-1}(\overline{G_\lambda}). \tag{1.3}$$

Condition (1.1) implies

$$p_\lambda^{-1}(G_\lambda) \subseteq p_{\lambda'}^{-1}(G_{\lambda'}),\ \lambda \leq \lambda', \tag{1.4}$$

which together with (1.2) shows that the sets $p_\lambda^{-1}(G_\lambda)$ form an increasing filtration of U. On the other hand, (1.3) gives factorizations of g restricted to members of that filtration. Some of the sets G_λ can be empty, but not all because of (1.2).

REMARK 2. *If the projections p_λ are surjective, condition* (1.3) *implies*

$$g_{\lambda'}|p_{\lambda\lambda'}^{-1}(\overline{G_\lambda}) = g_\lambda p_{\lambda\lambda'}|p_{\lambda\lambda'}^{-1}(\overline{G_\lambda}),\ \lambda \leq \lambda'. \tag{1.5}$$

The key step in the proof of Theorem 1 is the following theorem which could prove useful in other situations as well.

THEOREM 2. *Let $\boldsymbol{X} = (X_i, p_{ii+1})$ be an inverse sequence of paracompact perfectly normal spaces with limit X and surjective projections $p_i\colon X \to X_i$. Let P be a polyhedron, $\mathcal{V}$ an open covering of P and $U \subseteq X$ an open set. Then every mapping $f\colon U \to P$ admits a mapping $g\colon U \to P$, which is $\mathcal{V}$-near to f and admits a filtered factorization through $\boldsymbol{X}$.*

In [17] Rubin has established a more general version of his result with Schapiro by replacing the condition $\dim X_i \leq P$, $i \in \mathbb{N}$, by the weaker condition $\dim \boldsymbol{X} \leq P$. For an inverse system $\boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ the latter condition means that, for every $\lambda \in \Lambda$, closed set $A_\lambda \subseteq X_\lambda$ and mapping $f_\lambda\colon A_\lambda \to P$, there exists a $\lambda' \geq \lambda$ such that the mapping $f_\lambda p_{\lambda\lambda'}|p_{\lambda\lambda'}^{-1}(A_\lambda)$ extends to all of $X_{\lambda'}$. The analogous result for inverse sequences of stratifiable spaces is established in Theorem 4. The proof also uses Theorem 2.

2. Preliminaries

We now recall some well-known notions and facts from geometric and general topology needed in our proofs.

If $\mathcal{U}$ is an open covering of a space X, let $N(\mathcal{U})$ denote the corresponding nerve. A mapping $f\colon X \to |N(\mathcal{U}|)$ is called canonical if $f^{-1}(\mathrm{St}\,(U, N(\mathcal{U}))) \subseteq U$, for every $U \in \mathcal{U}$. An open covering is normal if it admits a canonical mapping. In a paracompact space every open covering is normal. Every polyhedron P admits an open covering $\mathcal{V}$ such that any two $\mathcal{V}$-near mappings into P are homotopic (see e.g., Theorem 2.6 of [3]). Every open covering $\mathcal{V}$ of a polyhedron P admits a triangulation K such that the closed stars of K refine $\mathcal{V}$ (see e.g., Theorem 4, Appendix 1 of [15]). Two mappings f, g into the geometric realization $|K|$ of a simplicial complex K are said to be contiguous, denoted by $f \equiv g$, provided every point $x \in X$ admits a simplex $\sigma \in K$ such that $f(x), g(x) \in |\sigma|$, where $|\sigma|$ denotes the closure of σ in $|K|$.

Every paracompact space is normal. Paracompact perfectly normal spaces (open sets are F_σ-sets) are hereditarily paracompact, i.e., all of their subsets are paracompact (see e.g., Exercise 5. 139 in [1]). A T_1-space X is stratifiable provided with every open set $U \subseteq X$ one can associate a sequence of open sets $U_n \subseteq X$ in such a way that the following conditions be fulfilled.

(S1) $\overline{U_n} \subseteq U$,

(S2) $\bigcup_{n=1}^{\infty} U_n = U$,

(S3) $U \subseteq V \Rightarrow U_n \subseteq V_n$.

Stratifiable spaces were introduced in 1961 by J. Ceder [4] as a generalization of metrizable spaces. Every stratifiable space is paracompact and perfectly normal. Every subset of a stratifiable space is stratifiable. Hence, stratifiable spaces are hereditarily paracompact. The direct sum of an arbitrary collection of stratifiable spaces is stratifiable. The direct product of a countable collection of stratifiable spaces is stratifiable. Consequently, the limit of an inverse sequence of stratifiable spaces is a stratifiable space. All polyhedra are stratifiable spaces. Polyhedra are absolute neighborhood extensors (ANE's) for stratifiable spaces [3]. It easily follows that polyhedra have the homotopy extension property for stratifiable spaces, i.e., if P is a polyhedron, X is a stratifiable space and $A \subseteq X$ is a closed set, then every mapping $f\colon (X \times 0) \cup (A \times I) \to P$ extends to all of $X \times I$.

3. Filtered factorizations of mappings

This section is devoted to the proof of Theorem 2. For this we need some lemmas.

Lemma 1. *Let X, X' be topological spaces and let $p\colon X \to X'$ be a surjective mapping. Let $\mathcal{U} = (U_\gamma, \gamma \in \Gamma)$ be a collection of non-empty open sets in X and let $U \subseteq X$ be their union. Similarly, let $\mathcal{U}' = ({U'}_\gamma, \gamma \in \Gamma')$ be a collection of non-empty open sets in X' and let $U' \subseteq X'$ be their union. If $\Gamma' \subseteq \Gamma$ and*

$$p^{-1}(U'_\gamma) \subseteq U_\gamma, \text{ for } \gamma \in \Gamma', \tag{3.6}$$

then the inclusion $\Gamma' \hookrightarrow \Gamma$ induces a simplicial injection $s\colon N(\mathcal{U}') \to N(\mathcal{U})$ between the corresponding nerves. Furthermore, if $f\colon U \to |N(\mathcal{U})|$ and $f'\colon U' \to |N(\mathcal{U}')|$ are canonical mappings, then $f|p^{-1}(U')$ and $sf'p|p^{-1}(U')$ are contiguous mappings into $N(\mathcal{U})$.

Proof. Let the vertices $U'_{\gamma_0}, \ldots, U'_{\gamma_n}$, $\gamma_0, \ldots, \gamma_n \in \Gamma'$, span a simplex of $N(\mathcal{U}')$. Then $U'_{\gamma_0} \cap \ldots \cap U'_{\gamma_n} \neq \emptyset$. Since $p\colon X \to X'$ is a surjection, it follows that

$$\emptyset \neq p^{-1}(U'_{\gamma_0} \cap \ldots \cap U'_{\gamma_n}) = p^{-1}(U'_{\gamma_0}) \cap \ldots \cap p^{-1}(U'_{\gamma_n}) \subseteq U_{\gamma_0} \cap \ldots \cap U_{\gamma_n} \tag{3.7}$$

and thus, the vertices $U_{\gamma_0} = s(U_{\gamma'_0}), \ldots, U_{\gamma_n} = s(U_{\gamma'_n})$ span a simplex of $N(\mathcal{U})$.

We will now prove that the mappings $f|p^{-1}(U')$ and $sf'p|p^{-1}(U')$ are contiguous. Let $x \in p^{-1}(U')$ and let $U_{\gamma_0}, \ldots, U_{\gamma_n}$ be the vertices of the simplex $\sigma \in N(\mathcal{U})$ which contains $f(x)$ in its interior. Similarly, let $U'_{\gamma'_0}, \ldots, U'_{\gamma'_m}$ be the vertices of the simplex $\sigma' \in N(\mathcal{U}')$ which contains $f'p(x)$ in its interior. Since s is a simplicial injection, the point $sf'p(x)$ lies in the interior of the simplex $\sigma'' \in N(\mathcal{U})$, whose vertices are $U_{\gamma'_0}, \ldots, U_{\gamma'_m}$. It thus suffices to show that the vertices $U_{\gamma_0}, \ldots, U_{\gamma_n}, U_{\gamma'_0}, \ldots, U_{\gamma'_m}$ span a simplex of $N(\mathcal{U})$, i.e., that

$$U_{\gamma_0} \cap \ldots \cap U_{\gamma_n} \cap U_{\gamma'_0} \cap \ldots \cap U_{\gamma'_m} \neq \emptyset. \tag{3.8}$$

Note that

$$f(x) \in \operatorname{St}(U_{\gamma_0}, N(\mathcal{U})) \cap \ldots \cap \operatorname{St}(U_{\gamma_n}, N(\mathcal{U})). \tag{3.9}$$

Since f is a canonical mapping, $f^{-1}(\operatorname{St}(U_\gamma, N(\mathcal{U})) \subseteq U_\gamma$ and thus, (3.9) implies

$$x \in U_{\gamma_0} \cap \ldots \cap U_{\gamma_n}. \tag{3.10}$$

Similarly,

$$f'p(x) \in \operatorname{St}(U'_{\gamma'_0}, N(\mathcal{U}')) \cap \ldots \cap \operatorname{St}(U'_{\gamma'_m}, N(\mathcal{U}')) \tag{3.11}$$

and (3.6) implies

$$x \in p^{-1}(U'_{\gamma'_0}) \cap \ldots \cap p^{-1}(U'_{\gamma'_m}) \subseteq U_{\gamma'_0} \cap \ldots \cap U_{\gamma'_m}. \tag{3.12}$$

Clearly, (3.10) and (3.12) yield (3.8). □

LEMMA 2. *Let* $\boldsymbol{X} = (X_i, p_{ii+1})$ *be an inverse sequence of hereditarily paracompact spaces with limit* X *and surjective projections* $p_i\colon X \to X_i$, $i \in \mathbb{N}$. *Let* $U \subseteq X$ *be an open set and let* $f\colon U \to P$ *be a mapping into a polyhedron* $P = |K|$. *Then there exist open sets* $U_i \subseteq X_i$ *such that*

$$p_{ii+1}^{-1}(U_i) \subseteq U_{i+1}, \tag{3.13}$$

$$\bigcup_{i\in\mathbb{N}} p_i^{-1}(U_i) = U. \tag{3.14}$$

Moreover, there exist maps $h_i\colon U_i \to |K|$, $i \in \mathbb{N}$, *such that*

$$h_i p_{ii+1} | p_{ii+1}^{-1}(U_i) \equiv h_{i+1} | p_{ii+1}^{-1}(U_i), \tag{3.15}$$

$$h_i p_i | p_i^{-1}(U_i) \equiv f | p_i^{-1}(U_i). \tag{3.16}$$

PROOF. Let Γ be the set of all vertices of K for which

$$U_\gamma = f^{-1}(\mathrm{St}\,(\gamma, K)) \neq \emptyset. \tag{3.17}$$

Consider the open covering $\mathcal{U} = (U_\gamma, \gamma \in \Gamma)$ of U and note that Γ is a subset of the set K^0 of all the vertices of K. For $i \in \mathbb{N}$ and $\gamma \in \Gamma$ consider the open subset $U_{i\gamma}$ of X_i, defined as the union of all open sets $V \subseteq X_i$, for which $p_i^{-1}(V) \subseteq U_\gamma$. Let $\Gamma_i \subseteq \Gamma$ be the set of all $\gamma \in \Gamma$, for which $U_{i\gamma} \neq \emptyset$. Clearly, $\Gamma_i \subseteq \Gamma$ and

$$p_i^{-1}(U_{i\gamma}) \subseteq U_\gamma \subseteq U, \text{ for } \gamma \in \Gamma_i. \tag{3.18}$$

Put $\mathcal{U}_i = (U_{i\gamma}, \gamma \in \Gamma_i)$ and

$$U_i = \bigcup_{\gamma\in\Gamma_i} U_{i\gamma}. \tag{3.19}$$

To prove (3.13) note that $p_i = p_{ii+1} p_{i+1}$ implies

$$p_{i+1}^{-1}(p_{ii+1}^{-1}(U_{i\gamma})) = p_i^{-1}(U_{i\gamma}) \subseteq U_\gamma. \tag{3.20}$$

Since $U_{i+1\gamma}$ contains all open subsets of $V \subseteq X_{i+1}$ satisfying $p_{i+1}^{-1}(V) \subseteq U_\gamma$, formula (3.20) shows that

$$p_{ii+1}^{-1}(U_{i\gamma}) \subseteq U_{i+1\gamma}. \tag{3.21}$$

Note that (3.21) implies $\Gamma_i \subseteq \Gamma_{i+1}$. Indeed, if $\gamma \in \Gamma_i$, i.e., $U_{i\gamma} \neq \emptyset$, then $p_{ii+1}^{-1}(U_{i\gamma}) \neq \emptyset$, because the surjectivity of p_i implies the surjectivity of p_{ii+1}. Consequently, $U_{i+1\gamma} \neq \emptyset$, i.e., $\gamma \in \Gamma_{i+1}$. Finally, (3.21), (3.19) for i and $i+1$ and the inclusion $\Gamma_i \subseteq \Gamma_{i+1}$ yield (3.13).

To establish (3.14) first note that (3.18) and (3.19) imply $\cup_i\, p_i^{-1}(U_i) \subseteq U$. To prove the converse inclusion consider a point $x \in U$ and choose $\gamma \in \Gamma$ so that $x \in U_\gamma$. Since $X = \lim \boldsymbol{X}$ and U_γ is open, there is an $i \in \mathbb{N}$ and there is an open set $V \subseteq X_i$ such that $p_i(x) \in V$ and $p_i^{-1}(V) \subseteq U_\gamma$. Therefore, $\emptyset \neq V \subseteq U_{i\gamma}$ and $\gamma \in \Gamma_i$. Moreover,

$$x \in p_i^{-1}(V) \subseteq p_i^{-1}(U_{i\gamma}) \subseteq p_i^{-1}(U_i). \tag{3.22}$$

Let us now see that $N(\mathcal{U})$ can be viewed as a subcomplex of K and f can be viewed as a canonical mapping $f\colon U \to |N(\mathcal{U})| \subseteq |K|$. To verify the first assertion it suffices to see that the inclusion $\Gamma \hookrightarrow K^0$ induces a simplicial injection $s\colon N(\mathcal{U}) \to K$. Indeed, if $U_{\gamma_0}, \dots, U_{\gamma_n}$ span a simplex of $N(\mathcal{U})$, then $U_{\gamma_0} \cap \ldots \cap U_{\gamma_n} \neq \emptyset$ and thus,

$$f^{-1}(\mathrm{St}\,(\gamma_0, K) \cap \ldots \cap \mathrm{St}\,(\gamma_n, K)) \neq \emptyset, \tag{3.23}$$

which implies $\mathrm{St}\,(\gamma_0, K) \cap \ldots \cap \mathrm{St}\,(\gamma_n, K) \neq \emptyset$. Consequently, the vertices $\gamma_0 = s(U_{\gamma_0}), \dots, \gamma_n = s(U_{\gamma_n})$ span a simplex of K. Also note that $f(U) \subseteq |N(\mathcal{U})| \subseteq |K|$. Indeed, if $x \in U$ and the vertices $\gamma_0, \dots, \gamma_n$ span a simplex $\sigma \in K$ which contains $f(x)$ in its interior, then $f(x)$ lies in $\mathrm{St}\,(\gamma_0, K) \cap \ldots \cap \mathrm{St}\,(\gamma_n, K)$ and therefore, $x \in U_{\gamma_0} \cap \ldots \cap U_{\gamma_n}$. Consequently, $U_{\gamma_0} \cap \ldots \cap U_{\gamma_n} \neq \emptyset$. However, this implies that $\gamma_0, \dots, \gamma_n \in \Gamma$ and $U_{\gamma_0}, \dots, U_{\gamma_n}$ are vertices which span a simplex of $N(\mathcal{U})$. Moreover, for $\gamma \in \Gamma$, (3.17) implies

$$f^{-1}(\mathrm{St}\,(U_\gamma, N(\mathcal{U})) \subseteq f^{-1}(\mathrm{St}\,(\gamma, K)) = U_\gamma, \tag{3.24}$$

which shows that $f\colon U \to |N(\mathcal{U})|$ is a canonical mapping.

In order to define the mappings $h_i\colon U_i \to |K|$ we first choose, for every $i \in \mathbb{N}$, a canonical mapping $g_i\colon U_i \to |N(\mathcal{U}_i)|$. Such mappings exist because X_i is hereditarily paracompact and thus, U_i is paracompact. We then apply Lemma 1 to the spaces X, X_i, the mapping p_i, the open coverings $\mathcal{U}$ of U and $\mathcal{U}_i$ of U_i and the canonical mappings $f\colon U \to |N(\mathcal{U})|$ and $g_i\colon U_i \to |N(\mathcal{U}_i)|$. We obtain a simplicial injection $s^i\colon N(\mathcal{U}_i) \to N(\mathcal{U})$ induced by the inclusion $\Gamma_i \hookrightarrow \Gamma$. Define the desired mapping $h_i\colon U_i \to |N(\mathcal{U})| \subseteq P$ by

$$h_i = s^i g_i. \tag{3.25}$$

One obtains (3.16) as an immediate consequence of the assertion of Lemma 1. To prove (3.15), apply Lemma 1 to the surjection p_{ii+1}, to the collections $\mathcal{U}_{i+1}$ and $\mathcal{U}_i$ and to the canonical mappings $g_{i+1}\colon U_{i+1} \to |N(\mathcal{U}_{i+1})|$ and $g_i\colon U_i \to |N(\mathcal{U}_i)|$. One obtains a simplicial injection $s^{ii+1}\colon N(\mathcal{U}_i) \to N(\mathcal{U}_{i+1})$, induced by the inclusion $\Gamma_i \hookrightarrow \Gamma_{i+1}$, such that

$$s^{ii+1} g_i p_{ii+1} | p_{ii+1}^{-1}(U_i) \equiv g_{i+1} | p_{ii+1}^{-1}(U_i). \tag{3.26}$$

Also note that $s^{i+1} s^{ii+1}\colon N(\mathcal{U}_i) \to N(\mathcal{U})$ is a simplicial mapping induced by the inclusions $\Gamma_i \hookrightarrow \Gamma_{i+1} \hookrightarrow \Gamma$. Consequently, it coincides with $s^i\colon N(\mathcal{U}_i) \to N(\mathcal{U})$. Note that the compositions of two contiguous mappings with the same simplicial mapping are contiguous mappings. Therefore, (3.25) and (3.26) yield

$$h_i p_{ii+1}|p_{ii+1}^{-1}(U_i) = s^{i+1}s^{ii+1}g_i p_{ii+1}|p_{ii+1}^{-1}(U_i) \equiv \\ s^{i+1}g_{i+1}|p_{ii+1}^{-1}(U_i) = h_{i+1}|p_{ii+1}^{-1}(U_i). \; \square \tag{3.27}$$

REMARK 3. *Note that the set $U_{i\gamma}$ was defined as the maximal open set V having the property that $p_i^{-1}(V)$ is contained in U_γ. Rubin and Schapiro* [18] *defined such an open set as the U_γ-response to p_i and denoted it by* $\mathrm{resp}(U_\gamma, p_i)$. *We use this construction also in the proof of Theorem 3, where* $V_i = \mathrm{resp}(V, p_i)$.

LEMMA 3. *Let $\boldsymbol{X} = (X_i, p_{ii+1})$ be an inverse sequence of perfectly normal spaces with limit X and projections $p_i\colon X \to X_i$. Let $U \subseteq X$ and $U_i \subseteq X_i$ be open sets which satisfy* (3.13) *and* (3.14). *Then there exist open sets $G_i \subseteq X_i$ such that*

$$\overline{G_i} \subseteq U_i, \tag{3.28}$$

$$p_{ii+1}^{-1}(\overline{G_i}) \subseteq G_{i+1}, \tag{3.29}$$

$$\bigcup_{i\in\mathbb{N}} p_i^{-1}(G_i) = U. \tag{3.30}$$

PROOF. In a perfectly normal space Y each open set V is a cozero-set, i.e., it is of the form $\phi^{-1}(0,1]$, for some mapping $\phi : Y \to [0,1]$ (see e.g., Corollary 1.5.12 of [11]). It is therefore easy to see that, for every $i \in \mathbb{N}$, the open set U_i can be represented as the union of a sequence of open subsets $V_i^1, V_i^2, \dots$ of X_i such that

$$V_i^n \subseteq \overline{V_i^n} \subseteq V_i^{n+1}, \tag{3.31}$$

$$U_i = \bigcup_{j\in\mathbb{N}} V_i^j. \tag{3.32}$$

We define the open sets $G_i \subseteq X_i$ by induction on i. In addition to conditions (3.28) and (3.29) we also require that

$$p_{ik}^{-1}(\overline{V_i^k}) \subseteq G_k, \text{ for } 1 \le i \le k. \tag{3.33}$$

Since $\overline{V_1^1} \subseteq U_1$ and X_1 is normal, it is possible to choose an open set G_1 in X_1 for which $\overline{G_1} \subseteq U_1$ and $\overline{V_1^1} \subseteq G_1$. Now assume that we have already defined sets $G_1, \dots, G_i$ in accordance with (3.28), (3.29) and (3.33). We choose as G_{i+1} an open set in X_{i+1} such that $\overline{G_{i+1}} \subseteq U_{i+1}$ and G_{i+1} contains the closed sets $p_{ii+1}^{-1}(\overline{G_i})$ and $p_{ji+1}^{-1}(\overline{V_j^{i+1}})$, where $1 \le j \le i+1$. Since this is a finite collection of closed sets contained in U_{i+1}, the existence of G_{i+1} is a consequence of the normality of X_{i+1}. Clearly, the sets G_i constructed in this way satisfy (3.28), (3.29) and (3.33). It remains to prove that they also satisfy (3.30).

Given a point $x \in U$, (3.14) shows that there exists an index $i \in \mathbb{N}$ such that $x \in p_i^{-1}(U_i)$. Therefore, by (3.32), there exists an integer $j \geq 1$ such that $p_i(x) \in V_i^j$ and thus, $p_i(x) \in V_i^k$, for $k \geq j$. If also $k \geq i$, one has $p_i(x) = p_{ik}p_k(x)$ and thus,

$$p_k(x) \in p_{ik}^{-1}(p_i(x)) \subseteq p_{ik}^{-1}(V_i^k). \tag{3.34}$$

By (3.33), $p_k(x) \in G_k$ and thus, x is contained in $p_k^{-1}(G_k) \subseteq \cup_i p_i^{-1}(G_i)$. Consequently, U is contained in the left side of (3.30). The opposite inclusion is an immediate consequence of $G_i \subseteq U_i$ and of (3.14). $\square$

Remark 4. *The natural analogues of Lemmas* 2 *and* 3 *hold also for inverse systems indexed by cofinite directed sets* Λ.

Lemma 4. *Let X be a normal space and K a simplicial complex. Let $A \subseteq X$ be a closed set and let $V, U \subseteq X$ be open sets such that $A \subseteq V \subseteq \overline{V} \subseteq U$. If $h\colon U \to |K|$ and $g\colon V \to |K|$ are mappings such that $h|V$ and g are contiguous mappings, then there exists a mapping $k\colon U \to |K|$, which is contiguous to h and is such that*

$$k|A = g|A, \tag{3.35}$$

$$k|U\backslash V = h|U\backslash V. \tag{3.36}$$

Proof. By normality of X choose an open set $H \subseteq X$ such that $A \subseteq H \subseteq \overline{H} \subseteq V$. Choose a mapping $\phi\colon X \to [0,1]$ such that $\phi(A) = 1$ and $\phi(X\backslash H) = 0$. Then define $k\colon U \to |K|$ by

$$k(x) = \begin{cases} \phi(x)g(x) + (1-\phi(x))h(x), & x \in V, \\ h(x), & x \in U\backslash\overline{H}. \end{cases} \tag{3.37}$$

Note that, for every point $x \in V$, the points $g(x)$ and $h(x)$ belong to a closed simplex $|\sigma|$ from K. Therefore, $\phi(x)g(x) + (1-\phi(x))h(x)$ is a well-defined point of $|\sigma|$. Moreover, the two expressions in (3.37) assume the same values on $V \cap (U\backslash\overline{H})$, which shows that k is a well-defined mapping. Finally, for $x \in A$, $\phi(x) = 1$ and thus, $k(x) = g(x)$. Similarly, for $x \in U\backslash V$, $\phi(x) = 0$ and thus, $k(x) = h(x)$. $\square$

Proof of Theorem 2. Choose a triangulation K of P such that its closed stars form a closed covering which refines $\mathcal{V}$. Since paracompact perfectly normal spaces are hereditarily paracompact, we can apply Lemma 2 to $\boldsymbol{X} = (X_i, p_{ii+1})$, X, p_i, U and $f\colon U \to |K|$. We thus obtain open sets $U_i \subseteq X_i$ and mappings $h_i\colon U_i \to |K|$ such that (3.13)–(3.16) hold. The first two of these relations enable us to apply Lemma 3 and obtain open sets $G_i \subseteq X_i$ such that (3.28), (3.29), and (3.30) are fulfilled.

We will now define, by induction on i, a sequence of mappings $g_i\colon U_i \to |K|$. Consider the open sets $V_1 = \emptyset$,

$$V_{j+1} = G_{j+1} \cap p_{jj+1}^{-1}(U_j),\ j \in \mathbb{N}, \tag{3.38}$$

and note that

$$p_{jj+1}^{-1}(\overline{G_j}) \subseteq V_{j+1} \subseteq \overline{V_{j+1}} \subseteq U_{j+1},\ j \in \mathbb{N}, \tag{3.39}$$

For $i = 1$ put $g_1 = h_1$. Assume that we have already defined mappings $g_1, \dots, g_i$ and that they satisfy the following conditions.

$$g_{j+1}|p_{jj+1}^{-1}(\overline{G_j}) = g_j p_{jj+1}|p_{jj+1}^{-1}(\overline{G_j}),\ 1 \le j < i, \tag{3.40}$$

$$g_j|(U_j \backslash V_j) = h_j|(U_j \backslash V_j),\ 1 \le j \le i, \tag{3.41}$$

$$g_{j+1}|(U_{j+1} \backslash p_{jj+1}^{-1}(V_j)) \equiv h_{j+1}|(U_{j+1} \backslash p_{jj+1}^{-1}(V_j)),\ 1 \le j < i. \tag{3.42}$$

To define g_{i+1} apply Lemma 4 to the sets from (3.39) (for $j = i$) and to the mappings $h_{i+1}\colon U_{i+1} \to |K|$ and $h_i p_{ii+1}|V_{i+1}$. Note that the latter mapping is defined because $V_{i+1} \subseteq p_{ii+1}^{-1}(U_i)$. Moreover, by (3.15), $h_{i+1}|V_{i+1} \equiv h_i p_{ii+1}|V_{i+1}$. One obtains a mapping $k_{i+1}\colon U_{i+1} \to |K|$ such that

$$k_{i+1}|p_{ii+1}^{-1}(\overline{G_i}) = h_i p_{ii+1}|p_{ii+1}^{-1}(\overline{G_i}), \tag{3.43}$$

$$k_{i+1}|(U_{i+1} \backslash V_{i+1}) = h_{i+1}|(U_{i+1} \backslash V_{i+1}). \tag{3.44}$$

$$k_{i+1} \equiv h_{i+1}. \tag{3.45}$$

Now note that $V_i \subseteq G_i \subseteq U_i$. Furthermore, by (3.41), $g_i|(U_i \backslash V_i) = h_i|(U_i \backslash V_i)$ and therefore,

$$g_i p_{ii+1}|(p_{ii+1}^{-1}(U_i) \backslash p_{ii+1}^{-1}(V_i)) = h_i p_{ii+1}|(p_{ii+1}^{-1}(U_i) \backslash p_{ii+1}^{-1}(V_i)). \tag{3.46}$$

We define $g_{i+1}\colon U_{i+1} \to |K|$ by the formula

$$g_{i+1}(x) = \begin{cases} g_i p_{ii+1}(x), & x \in p_{ii+1}^{-1}(\overline{G_i}), \\ k_{i+1}(x), & x \in U_{i+1} \backslash p_{ii+1}^{-1}(V_i). \end{cases} \tag{3.47}$$

Note that the sets $p_{ii+1}^{-1}(\overline{G_i})$ and $U_{i+1} \backslash p_{ii+1}^{-1}(V_i)$ are closed subsets of U_{i+1} and their intersection S is contained in the set $p_{ii+1}^{-1}(U_i) \backslash p_{ii+1}^{-1}(V_i)$. Therefore, by (3.46), $g_i p_{ii+1}|S = h_i p_{ii+1}|S$. On the other hand, $S \subseteq p_{ii+1}^{-1}(\overline{G_i})$. Therefore, (3.43) shows that also $k_{i+1}|S = h_i p_{ii+1}|S$. Consequently, the mapping g_{i+1} is well defined. Clearly, (3.40) holds because of the first line in (3.47). In order to verify (3.41), note that $p_{ii+1}^{-1}(V_i) \subseteq p_{ii+1}^{-1}(\overline{G_i}) \subseteq V_{i+1}$ and thus, $U_{i+1} \backslash V_{i+1} \subseteq U_{i+1} \backslash p_{ii+1}^{-1}(V_i)$. Therefore, (3.44) and (3.47) show that (3.41) holds also for $j = i + 1$. Finally, (3.42) holds for $j = i$ because of (3.45) and (3.47).

We now define the mapping $g\colon U \to |K|$ by putting

$$g|p_i^{-1}(\overline{G_i}) = g_i p_i|p_i^{-1}(\overline{G_i}),\ i \in \mathbb{N}. \tag{3.48}$$

Since $p_i = p_{ii+1}p_{i+1}$, it follows that

$$p_i^{-1}(\overline{G_i}) = p_{i+1}^{-1}(p_{ii+1}^{-1}(\overline{G_i})) \subseteq p_{i+1}^{-1}(G_{i+1}) \tag{3.49}$$

However, by (3.40), $g_{i+1}p_{i+1}|p_i^{-1}(\overline{G_i}) = g_i p_i|p_i^{-1}(\overline{G_i})$ and thus,

$$g_{i+1}p_{i+1}|p_i^{-1}(G_i) = g_i p_i|p_i^{-1}(G_i), \tag{3.50}$$

which shows that g is well defined on $U = \cup_i p_i^{-1}(G_i)$. Continuity of g is a consequence of the fact that g on $p_i^{-1}(G_i)$ is given by the continuous mapping $g_i p_i$ and the sets $p_i^{-1}(G_i)$, $i \in \mathbb{N}$, form an open covering of U.

It remains to prove that, for every $x \in U$, the points $f(x)$ and $g(x)$ belong to a closed star of K. By (1.2) and (3.48), it suffices to prove that, for $x \in p_i^{-1}(G_i)$, the points $f(x)$ and $g_i p_i(x)$ belong to a closed star of K. We will prove this assertion by induction on i. The assertion is true for $i = 1$, because $g_1(p_1(x)) = h_1(p_1(x))$ and, by (3.16), $f|p_i^{-1}(U_1) \equiv h_1 p_1|p_i^{-1}(U_1)$. Let us now prove the assertion for $i+1$ assuming that it holds for i. By (3.47), $g_{i+1}p_{i+1}(x)$ equals $g_i p_i(x)$ or $k_{i+1}p_{i+1}(x)$. In the first case, the induction hypothesis implies that $f(x)$ and $g_i p_i(x)$ belong to a closed star of K. In the second case, (3.45), the points $k_{i+1}p_{i+1}(x)$ and $h_{i+1}p_{i+1}(x)$ belong to a closed simplex of K. However, by (3.16), $f(x)$ and $h_{i+1}p_{i+1}(x)$ also belong to a closed simplex of K. Consequently, $k_{i+1}p_{i+1}(x)$ and $f(x)$ belong to a closed star of K. $\square$

4. Proof of the main theorem

We shall see that Theorem 2 essentially reduces the proof of Theorem 1 to the following theorem.

Theorem 3. *Let $\boldsymbol{X} = (X_i, p_{ii+1})$ be an inverse sequence of paracompact perfectly normal spaces with limit X and surjective projections $p_i\colon X \to X_i$. Let P be a polyhedron, let $A \subseteq X$ be a closed set and $U \subseteq X$ an open set, $A \subseteq U$, and let $g\colon U \to P$ be a mapping which admits a filtered factorization through $\boldsymbol{X}$. If $\dim X_i \leq P$, for every $i \in \mathbb{N}$, then there exists a mapping $h\colon X \to P$, which extends $g|A$.*

Proof of Theorem 3. Let a filtered factorization of $g\colon U \to P$ be given by open sets $G_i \subseteq X_i$ and by mappings $g_i\colon \overline{G_i} \to P$, which satisfy the analogues of (3.6)–(3.8). Consider the open set $V = X \setminus A$ and let $V_i \subseteq X_i$ be the maximal open set for which $p_i^{-1}(V_i) \subseteq V$. Note that

$$p_{ii+1}^{-1}(V_i) \subseteq V_{i+1}, \tag{4.51}$$

$$\bigcup_i p_i^{-1}(V_i) = V. \tag{4.52}$$

An application of Lemma 3 to V and V_i yields open sets $H_i \subseteq X_i$ such that

$$\overline{H_i} \subseteq V_i, \tag{4.53}$$

$$p_{ii+1}^{-1}(\overline{H_i}) \subseteq H_{i+1}, \tag{4.54}$$

$$\bigcup_i p_i^{-1}(H_i) = V. \tag{4.55}$$

Note that, for every i,

$$\overline{p_i(A)} \cap H_i = \emptyset. \tag{4.56}$$

Indeed, since H_i is an open set, $\overline{p_i(A)} \cap H_i \neq \emptyset$ implies that also $p_i(A) \cap H_i \neq \emptyset$ and thus, $\emptyset \neq A \cap p_i^{-1}(H_i) \subseteq A \cap V$, which is a contradiction.

We will now define, by induction on i, a sequence of closed sets $C_i \subseteq X_i$ and a sequence of mappings $h_i \colon C_i \to P$, which have the following properties.

$$\overline{H_i} \subseteq C_i \subseteq \overline{G_i} \cup \overline{H_i}, \tag{4.57}$$

$$G_i \cap p_i(A) \subseteq \operatorname{Int} C_i, \tag{4.58}$$

$$p_{ii+1}^{-1}(C_i) \subseteq C_{i+1}, \tag{4.59}$$

$$h_{i+1} | p_{ii+1}^{-1}(C_i) = h_i p_{ii+1} | p_{ii+1}^{-1}(C_i), \tag{4.60}$$

$$h_i | (C_i \setminus \overline{H_i}) = g_i | (C_i \setminus \overline{H_i}). \tag{4.61}$$

We begin the induction by putting $C_1 = \overline{G_1} \cup \overline{H_1}$. We define h_1 on $\overline{G_1}$ by $h_1 | \overline{G_1} = g_1$. We then extend it to C_1 using the fact that $\dim X_1 \leq P$ and thus also $\dim C_1 \leq P$. Now assume that we have already defined the sets $C_1, \dots, C_i$ and the mappings $h_1, \dots, h_i$. In order to define C_{i+1}, note that (4.54) and (4.56) (for $i+1$) yield $p_{ii+1}^{-1}(\overline{H_i}) \cap \overline{p_{i+1}(A)} = \emptyset$. Therefore, one can find an open set $W \subseteq X_{i+1}$ such that

$$\overline{p_{i+1}(A)} \subseteq W, \tag{4.62}$$

$$\overline{W} \subseteq X_{i+1} \setminus p_{ii+1}^{-1}(\overline{H_i}). \tag{4.63}$$

Put

$$C_{i+1} = p_{ii+1}^{-1}(C_i) \cup (\overline{G_{i+1} \cap W}) \cup \overline{H_{i+1}}. \tag{4.64}$$

Cleary, (4.59) is fulfilled. (4.57) for $i+1$ is a consequence of (4.57) for i, of (4.54) and of the analogous relation for G_i and G_{i+1}. Furthermore, (4.58) holds because

$$G_{i+1} \cap p_{i+1}(A) \subseteq G_{i+1} \cap W \subseteq \operatorname{Int} C_{i+1}. \tag{4.65}$$

We define h_{i+1} on $p_{ii+1}^{-1}(C_i)$ by (4.60). We also put

$$h_{i+1} | (\overline{G_{i+1} \cap W}) = g_{i+1} | (\overline{G_{i+1} \cap W}). \tag{4.66}$$

In order to verify that (4.66) is compatible with (4.60), we will show that both formulas on the intersection

$$S = p_{ii+1}^{-1}(C_i) \cap (\overline{G_{i+1} \cap W}) \tag{4.67}$$

yield the same mapping $g_i p_{ii+1}|S$. Indeed, $p_{ii+1}^{-1}(\overline{H_i}) \cap (\overline{G_{i+1} \cap W}) \subseteq p_{ii+1}^{-1}(\overline{H_i}) \cap \overline{W} = \emptyset$ and thus, $S \subseteq p_{ii+1}^{-1}(\overline{G_i}) \cap (\overline{G_{i+1} \cap W}) \subseteq p_{ii+1}^{-1}(\overline{G_i})$. By Remark 2, $g_{i+1}|p_{ii+1}^{-1}(\overline{G_i}) = g_i p_{ii+1}|p_{ii+1}^{-1}(\overline{G_i})$. Therefore, (4.66) yields $h_{i+1}|S = g_{i+1}|S = g_i p_{ii+1}|S$. On the other hand, by (4.63), $S \subseteq p_{ii+1}^{-1}(C_i) \backslash p_{ii+1}^{-1}(\overline{H_i}) = p_{ii+1}^{-1}(C_i \backslash \overline{H_i})$ and

$$h_i p_{ii+1}|p_{ii+1}^{-1}(C_i \backslash \overline{H_i}) = g_i p_{ii+1}|p_{ii+1}^{-1}(C_i \backslash \overline{H_i}), \tag{4.68}$$

because $p_{ii+1}(p_{ii+1}^{-1}(C_i \backslash \overline{H_i})) \subseteq C_i \backslash \overline{H_i}$ and by (4.61), $h_i|(C_i \backslash \overline{H_i}) = g_i|(C_i \backslash \overline{H_i})$. Hence, definition (4.60) yields $h_{i+1}|S = h_i p_{ii+1}|S = g_i p_{ii+1}|S$. Finally, $\dim C_{i+1} \leq P$, because $\dim X_{i+1} \leq P$. Therefore, h_{i+1} extends to all of C_{i+1}.

To verify (4.61) for $i+1$, note that

$$C_{i+1} \backslash \overline{H_{i+1}} \subseteq (p_{ii+1}^{-1}(C_i) \backslash \overline{H_{i+1}}) \cup (\overline{G_{i+1} \cap W}). \tag{4.69}$$

By (4.66), h_{i+1} coincides with g_{i+1} on the second summand. Now note that, by (4.60), $h_{i+1}|p_{ii+1}^{-1}(C_i) = h_i p_{ii+1}|p_{ii+1}^{-1}(C_i)$. Since $p_{ii+1}^{-1}(\overline{H}_i) \subseteq \overline{H_{i+1}}$, we see that

$$p_{ii+1}^{-1}(C_i) \backslash \overline{H_{i+1}} \subseteq p_{ii+1}^{-1}(C_i \backslash \overline{H_i}) \subseteq p_{ii+1}^{-1}(\overline{G_i}). \tag{4.70}$$

However, by (4.61) for i, we conclude that on the first summand $h_i p_{ii+1}$ coincides with $g_i p_{ii+1}$. On the other hand, $g_{i+1}|p_{ii+1}^{-1}(\overline{G_i}) = g_i p_{ii+1}|p_{ii+1}^{-1}(\overline{G_i})$ and thus, (4.70) shows that on the first summand g_{i+1} also coincides with $g_i p_{ii+1}$.

Now note that (4.57) and (4.58) imply

$$\bigcup_i p_i^{-1}(\operatorname{Int} C_i) = X. \tag{4.71}$$

Indeed, (4.55) and $H_i \subseteq \operatorname{Int} C_i$ imply that the left side of (4.71) contains $V = X \backslash A$. Moreover, since $A \subseteq U$, (4.58) and (1.2) imply that the left side of (4.71) also contains A.

We now define a mapping $h\colon X \to P$ by putting $h(x) = h_i p_i(x)$, for $x \in p_i^{-1}(C_i)$. Notice that $x \in p_i^{-1}(C_i)$ implies $p_{i+1}(x) \in p_{ii+1}^{-1}(C_i)$ and thus, by (4.60), $h_{i+1} p_{i+1}(x) = h_i p_{ii+1} p_{i+1}(x) = h_i p_i(x)$, which shows that h is well defined. It is a continuous mapping, because it is given by continuous mappings on the open sets $p_i^{-1}(\operatorname{Int} C_i)$ which form a covering of X.

It remains to prove that $h|A = g|A$. First note that, by (4.53) and (4.52), $p_i^{-1}(\overline{H_i}) \subseteq p_i^{-1}(V_i) \subseteq V = X \backslash A$ and thus, $\overline{H_i} \cap p_i(A) = \emptyset$. Moreover, by (4.58), $G_i \cap p_i(A) \subseteq C_i$. Therefore, $G_i \cap p_i(A) \subseteq C_i \backslash \overline{H_i}$. Consequently, by

(4.61), $h_i|(G_i \cap p_i(A)) = g_i|(G_i \cap p_i(A))$. Let $a \in A$ be an arbitrary point. There is an i such that $a \in p_i^{-1}(G_i)$ and thus, $p_i(a) \in G_i \cap p_i(A)$. By (4.58), we conclude that $p_i(a) \in C_i$ and thus, $h(a) = h_i p_i(a)$. Since $h_i|(G_i \cap p_i(A)) = g_i|(G_i \cap p_i(A))$, we see that $h(a) = g_i p_i(a)$. On the other hand, $a \in p_i^{-1}(G_i)$ implies that also $g(a) = g_i p_i(a)$. Consequently, $h(a) = g(a)$. □

We precede the proof of Theorem 1 by a simple technical lemma.

LEMMA 5. *Let $\boldsymbol{X} = (X_i, p_{ii+1})$ be an inverse sequence of stratifiable spaces with limit X. Then there exists a sequence $\boldsymbol{X}^* = (X_i^*, p_{ii+1}^*)$ with limit X^* and surjective projections $p_i^*\colon X^* \to X_i^*$ such that X_i^* is the direct sum of X_i and of a discrete space D_i, the mapping $p_{ii+1}^*|X_{i+1} = p_{ii+1}$ and the natural inclusion $X \to X^*$ embeds X as a closed subset of X^*.*

PROOF. We construct X_i^* and p_{ii+1}^* by induction on i beginning with $X_1^* = X_1$. By definition, $X_{i+1}^* = X_{i+1} \sqcup D_{i+1}$, where D_{i+1} is a discrete space which admits a surjection $D_{i+1} \to (X_i \backslash p_{ii+1}(X_{i+1})) \cup D_i$. If a point $x^* \in X^*$ does not belong to X, then there is an i for which $p^*(x^*) \in D_i$ and therefore, it has a neighborhood which misses X. Hence, X is closed in X^*. □

PROOF OF THEOREM 1. We will first prove the assertion under the additional assumption that the projections $p_i\colon X \to X_i$ are surjective. Consider a closed set $A \subseteq X$ and a mapping $f\colon A \to P$. We must show that f extends to all of X. Note that X is stratifiable and therefore, the homotopy extension theorem applies. Consequently, it suffices to produce a mapping $g\colon A \to P$, which is homotopic to f and extends to all of X. Choose an open covering $\mathcal{V}$ of P such that any two $\mathcal{V}$-near mappings into P are homotopic. It suffices to find a mapping $g\colon A \to P$, which is $\mathcal{V}$-near to f and extends to all of X. Since polyhedra are ANE's for stratifiable spaces, the mapping f extends to an open neighborhood U of A, $f\colon U \to P$. This enables us to apply Theorem 2 and obtain a mapping $g\colon U \to P$ such that g and f are $\mathcal{V}$-near and g admits a filtered factorization through $\boldsymbol{X}$. However, Theorem 3 implies that $g|A$ extends to all of X.

The case of arbitrary projections p_i is reduced to the case of surjective projections using Lemma 5. Indeed, members X_i^* of the sequence $\boldsymbol{X}^*$ are stratifiable and $\dim X_i^* \leq P$. Since the projections p_i^* are surjective, the already established case of the theorem yields the conclusion that $\dim X^* \leq P$. However, X is a closed subset of X^*, and therefore, the latter relation shows that also $\dim X \leq P$. □

COROLLARY 1. *If $\boldsymbol{X} = (X_i, p_{ii+1})$ is an inverse sequence of polyhedra X_i of dimension $\dim X_i \leq n$, then the limit X has dimension $\dim X \leq n$.*

REMARK 5. In the proof of Theorem 1 for metrizable spaces [18] the first axiom of countability played an important role. In general, stratifiable spaces do not satisfy that axiom. An easy example is given by a simplicial complex which consists of infinitely many 1-simplexes exiting out of one common vertex.

THEOREM 4. *Let P be a polyhedron and let $\boldsymbol{X} = (X_i, p_{ii+1})$ be an inverse sequence of stratifiable spaces with limit X. If $\dim \boldsymbol{X} \leq P$, then also $\dim X \leq\ P$.*

The proof of Theorem 4 is a variation of the proof of Theorem 1. First note that $\dim \boldsymbol{X} \leq P$ implies $\dim \boldsymbol{X}^* \leq P$. Therefore, it suffices to consider the case when the projections $p_i \colon X \to X_i$ are surjective. This enables us to reduce the problem of extending $f \colon A \to P$ to all of X to the problem of extending $g|A$ to all of X, where $g \colon U \to P$ is a mapping which is defined on an open neighborhood U of A and admits a filtered factorization through $\boldsymbol{X}$. In other words we need the following variation of Theorem 3.

THEOREM 5. *Let $\boldsymbol{X} = (X_i, p_{ii+1})$ be an inverse sequence of paracompact perfectly normal spaces with limit X and surjective projections $p_i \colon X \to X_i$. Let P be a polyhedron, let $A \subseteq X$ be a closed set and $U \subseteq X$ an open set, $A \subseteq U$, and let $g \colon U \to P$ be a mapping which admits a filtered factorization through $\boldsymbol{X}$. If $\dim \boldsymbol{X} \leq P$, then there exists a mapping $h \colon X \to P$, which extends $g|A$.*

The proof of Theorem 5 is a variation of the proof of Theorem 3. In particular, the sets V, V_i and H_i are defined as in the previous case. One then defines, by induction on i, an increasing sequence of indices $l(i) \in \mathbb{N}$, a sequence of closed sets $C_{l(i)} \subseteq X_{l(i)}$ and a sequence of mappings $h_{l(i)} \colon C_{l(i)} \to P$ which satisfy the analogues of (4.57)–(4.61), where i has been replaced by $l(i)$ and $i+1$ by $l(i+1)$. To begin the induction we consider the mapping $g_1 \colon \overline{G_1} \to P$. Since $\dim \boldsymbol{X} \leq P$, there exists an index $l(1) \in \mathbb{N}$ such that $g_1 p_{1l(1)} | p_{1l(1)}^{-1}(\overline{G_1})$ extends to $C_{l(1)} = p_{1l(1)}^{-1}(\overline{G_1}) \cup \overline{H_{l(1)}}$. The induction step is obtained by a similar variation of the induction step in the proof of Theorem 3. As in (4.71), the union of the open sets $p_i^{-1}(\operatorname{Int} C_{l(i)})$ equals X and one defines h by putting $h(x) = h_i p_i(x)$, where $x \in p_i^{-1}(C_{l(i)})$. Finally, one verifies as before that $h|A = g|A$. □

REMARK 6. *R. Cauty* [2] *proved that every CW-complex embeds as a retract in a polyhedron. Therefore, for stratifiable spaces X, the homotopy extension property remains valid if one replaces polyhedra P by CW-complexes. It readily follows that for a polyhedron P and a CW-complex Q of the same homotopy type, the properties* $\dim X \leq P$ *and* $\dim X \leq Q$ *are equivalent. Since CW-complexes have the homotopy type of polyhedra, one easily concludes that Theorem* 1 *remains valid if one assumes that P is a CW-complex. An analogous remark applies to Theorem* 5.

Remark 7. *In the Fall of* 1999 *Professor Leonard R. Rubin visited Zagreb and presented his work with Philip J. Schapiro to the Topology seminar of the Mathematics Department. This visit gave the original impetus for the writing of the present paper. In the proof of the main results there is some overlap with ideas encountered in the Rubin – Schapiro proof.*

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Department of Mathematics
University of Zagreb
P.O.Box 335
10 002 Zagreb
Croatia
E-mail: `smardes@math.hr`