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GRAPHS AND SYMMETRIC DESIGNS CORRESPONDING TO DIFFERENCE SETS IN GROUPS OF ORDER 96

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Abstract. Using the list of 2607 so far constructed (96,20,4) difference sets as a source, we checked the related symmetric designs upon isomorphism and analyzed their full automorphism groups. New (96,20,4,4) and (96,19,2,4) regular partial difference sets are constructed, together with the corresponding strongly regular graphs.

1. Introduction

There are 231 groups of order 96. Seven of them are abelian. These groups will be referred to as in the "SmallGroups" library of the software package GAP ([9]). For instance, the group of order 96 with the catalogue number (*cn*) 68 in that GAP library is denoted by $[96, 68]$.

Definition 1.1. *A* (v, k, λ) difference set *is a subset* $\Delta \subseteq G$ *of size* k *in a group* G *of order* v *with the property that the multiset* $\left\{xy^{-1} \mid x, y \in \Delta, x \neq y\right\}$ *contains each nonidentity element of* G *exactly* λ *times.*

In case a set $\Delta \subseteq G$ is a difference set in a group G, then its *translate* (or "*shift*") $\Delta x = \{dx \mid d \in \Delta\}$ by any element $x \in G$ is a difference set in G as well. Depending on the respective property of G, a difference set is called *abelian, cyclic* or *nonabelian.*

It is customary to view a group subset $S \subseteq G$ as a group ring $\mathbb{Z}G$ element $\underline{S} = \sum_{s \in S} s$ and to put $S^{(-1)} = \{s^{-1} \mid s \in S\}$. In that notation difference set $\Delta \subseteq G$ is defined as a subset of G that satisfies the fundamental equation

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$$\underline{\Delta} \cdot \underline{\Delta^{(-1)}} = k\underline{\{e\}} + \lambda \underline{G \setminus \{e\}} = (k-\lambda)\underline{\{e\}} + \lambda \underline{G} \tag{1.1}$$

in $\mathbb{Z}G$; e denotes the group identity element.

The *development* of a difference set $\Delta \subseteq G$ is the incidence structure $dev\Delta$ whose points are the elements of the group G and whose blocks are the translates $\{\Delta g \mid g \in G\}$. By this structure, difference sets are related to symmetric designs.

DEFINITION 1.2. *A* symmetric block design with parameters (v, k, λ) *is a finite incidence structure* $\mathcal{D} = (\mathcal{V},\mathcal{B})$ *consisting of* $|\mathcal{V}| = v$ *points and* $|\mathcal{B}| = v$ *blocks, where each block is incident with* k *points and any two distinct points are incident with exactly* λ *common blocks.*

An *automorphism* of a symmetric block design $\mathcal{D}$ is a permutation on $\mathcal{V}$ which sends blocks to blocks. The set of all automorphisms of $\mathcal{D}$ forms its *full automorphism group* commonly denoted by $\mathrm{Aut}\,\mathcal{D}$. If a subgroup $H \leq \mathrm{Aut}\,\mathcal{D}$ acts regularly on $\mathcal{V}$ (and $\mathcal{B}$), then $\mathcal{D}$ is called *regular* and H is called a *Singer group* of $\mathcal{D}$.

THEOREM 1.3 ([6]). *Let* G *be a finite group of order* v *and* Δ *a proper, non-empty subset of* G *with* k *elements. Then* Δ *is a* (v, k, λ) *difference set in* G *if and only if* $dev\Delta$ *is a symmetric* (v, k, λ) *design on which* G *acts regularly.*

DEFINITION 1.4. *Two difference sets* Δ^1 *in* G^1 *and* Δ^2 *in* G^2 *are* isomorphic *if the designs* $dev\Delta^1$ *and* $dev\Delta^2$ *are isomorphic.* Δ^1 *and* Δ^2 *are* equivalent *if there exists a group isomorphism* $\varphi : G^1 \to G^2$ *such that* $\varphi(\Delta^1) = \Delta^2 g$ *for a suitable* $g \in G^2$.

It is easy to see that equivalent difference sets Δ^1 and Δ^2 give rise to isomorphic symmetric designs $dev\Delta^1$ and $dev\Delta^2$. A difference set is said to be *genuinely nonabelian* if its development has no abelian group acting regularly on the point set.

2. Brief history of searching for (96,20,4) difference sets

Solving the problem of difference set existence in groups of order 96 has lasted for more than four decades. The abelian case was considered first and positive result was obtained in the case of three groups.

In groups $[96,231] \cong \mathbb{Z}_2^4 \times \mathbb{Z}_6$ and $[96,220] \cong \mathbb{Z}_2^3 \times \mathbb{Z}_{12}$ difference sets were obtained by McFarland's construction ([19]). Using vector spaces of dimension $d+1$ over finite fields of order q, that construction works for abelian groups with an elementary abelian subgroup of order q^{d+1} and yields difference sets of the so called McFarland series with parameters

$$v = q^{d+1}\left(1 + \frac{q^{d+1}-1}{q-1}\right),\ k = q^d \frac{q^{d+1}-1}{q-1},\ \text{and } \lambda = q^d \frac{q^d - 1}{q-1}; \tag{2.1}$$

here $q = p^f$ is a prime power and d is a positive integer. Putting $d = 1$ and $q = 4$ in (2.1) gives the parameter set (96,20,4).

In group [96,161] $\cong \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_{12}$ a difference set was constructed by Arasu and Sehgal in 1995 ([5]).

The abelian case was finally solved in 1996 by ruling out the existence of difference sets in groups [96,46] $\cong \mathbb{Z}_4 \times \mathbb{Z}_{24}$ and [96,176] $\cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{24}$ by Arasu, Davis, Jedwab, Ma and McFarland, ([4]). Groups [96,2] $\cong \mathbb{Z}_{96}$ and [96,59] $\cong \mathbb{Z}_2 \times \mathbb{Z}_{48}$ were ruled out long before by the result of Turyn ([24]). A summary of the abelian case is obviously the following: an abelian group G has a (96,20,4) difference set if and only if the exponent of G is not larger than 12.

In the nonabelian case this exponent bound is violated. Precisely, nonabelian groups [96,*cn*] for *cn* in {10, 14, 20, 51, 52, 54, 64, 177, 188, 190, 191} have difference sets and exponent 24. Major contributions to deciding the existence status of difference sets in 224 nonabelian groups appeared as follows.

In 1985 Dillon ([8]) generalized McFarland's construction to work for a larger set of groups, i.e., groups containing an elementary abelian normal subgroup of order q^{d+1} in their center. In such a way difference sets in groups [96,218] and [96,230] were constructed. Besides, [8] provided a result which was used for ruling out groups [96,6], [96,81], [96,110] and [96,207].

In 1999, at the International Conference on Geometry in Haifa, Klin presented two nonisomorphic (96,20,4) difference sets in group [96,226] ([16]).

Afterwards the problem was solved partially, step by step, through results of O. A. AbuGhneim and K. W. Smith ([1–3]) on one side, and our results [11] and [12] on another side. By the beginning of 2006 exactly 20 nonabelian cases remained undecided. On the opened cases of the groups [96,*cn*] for *cn* in {3, 65, 66, 67, 68, 69, 73, 74, 187, 189, 191, 192, 193, 198, 199, 200, 201, 202, 203, 204} both teams continued working.

The record of AbuGhneim/Smith progress, a continuously updated documentation, can be found on the web site [20]. Our approach to the problem was through design construction and the use of Theorem 1.3. For symmetric (96,20,4) designs construction we used the well known *method of tactical decompositions* ([15]) based on the assumption that a certain group acts on the design as its automorphism group. In that approach the choice of an appropriate group is of great importance. In one such attempt we managed to solve the problem of the existence of a difference set in the group [96,68]. That construction we describe in Section 3.

Finally in May 2006, K. W. Smith put an end to the problem with the conclusion that the group [96,*cn*] contains a difference set if and only if *cn* belongs to the set $\mathcal{A}$ = {10, 13, 14, 20, 41, 51, 52, 54, 64, 68, 70, 71, 72, 75, 77, 78, 79, 83, 84, 85, 86, 87, 88, 90, 91, 92, 94, 95, 96, 97, 98, 99, 101, 103, 105, 129, 130, 131, 133, 135, 136, 141, 142, 143, 144, 145, 146, 147, 151, 152, 159, 160, 161, 162, 164, 165, 166, 167, 168, 169, 170, 171, 172, 173, 174, 175, 177, 185, 186, 188, 190, 191, 194, 195, 196, 197, 202, 205, 206, 209, 210, 212, 218, 219, 220, 221, 223, 225, 226, 227, 228, 229, 230, 231}.

The procedure and the results are fully documented in [20]. Unlike our approach, for majority of the groups the AbuGhneim/Smith construction is exhaustive and the list of difference sets obtained is complete.

Smith put together all 2607 constructed inequivalent difference sets in the list '**DS96**' available at [20]. The list is prepared for everyone else's use (GAP users) and reference. It contains 55 abelian and 2552 nonabelian difference sets. Many of the latter are genuinely nonabelian. The list is exhaustive with the following possible exceptions:

1^0 The groups [96,64], [96,70], [96,71], [96,72] and [96,227] do not have normal subgroups of sizes 2 or 3 so that the search technique of AbuGhneim/Smith did not work. In this case the list contains difference sets obtained by our non-exhaustive approach.

2^0 The groups [96,218], [96,220], [96,230] and [96,231] have large automorphism groups which ruins the feasibility of AbuGhneim/Smith computer search.

3. A difference set in the group $[96, 68]$

This section gives an example of the construction of a $(96, 20, 4)$ symmetric design and a corresponding difference set in the group $[96, 68]$. For the design construction we use the procedure described in our papers [11] and [12].

Let's consider the automorphism group $G_0 = [48, 3] \cong \mathbb{Z}_4^2 \cdot \mathbb{Z}_3$,

$$G_0 = \langle a, b, c \mid a^4 = b^4 = c^3 = 1, [a, b] = 1, a^c = a^3 b^3, b^c = a \rangle \tag{3.1}$$

(for p, q arbitrary group elements $p^q = qpq^{-1}$), and its action in six orbits of the length 16 on a $(96, 20, 4)$ symmetric design. In such a case it is accustomed to denote the points of design by $\mathrm{I}_1, \mathrm{I}_2, \ldots, \mathrm{I}_{16}$, $\mathrm{I} = 1, 2, \ldots, 6$. Further, in the course of design construction it is convenient to use a group G_0 generators' permutation representation of degree 16. Here we use the representation given in (3.2).

$$G_0 \cdots \begin{cases} a = (\ 1\ 2\ 3\ 4\ )\ (\ 5\ 8\ 9\ 10\ )\ (\ 6\ 13\ 14\ 15\ )\ (\ 7\ 12\ 16\ 11\ ) \\ b = (\ 1\ 5\ 6\ 7\ )\ (\ 2\ 8\ 13\ 12\ )\ (\ 3\ 9\ 14\ 16\ )\ (\ 4\ 10\ 15\ 11\ ) \\ c = (\ 2\ 5\ 11\ )\ (\ 3\ 6\ 14\ )\ (\ 4\ 7\ 8\ )\ (\ 9\ 10\ 15\ )\ (\ 12\ 13\ 16\ ) \end{cases} \tag{3.2}$$

The numbers $1, 2, \ldots, 16$ are then observed as points of point orbits and they appear as indices of the points of our design.

The possible dispersion (cardinality) of the points lying on the blocks of each block orbit into point orbits can be represented by orbit matrices. The entries of these matrices satisfy the well-known equations ([15]). In our case, the corresponding calculations give a single orbit matrix (3.3).

(3.3)

16	16	16	16	16	16	
0	4	4	4	4	4	16
4	0	4	4	4	4	16
4	4	0	4	4	4	16
4	4	4	0	4	4	16
4	4	4	4	0	4	16
4	4	4	4	4	0	16

Design construction is equivalent to the orbit matrix "indexing". Indexing means determining precisely which points from every point orbit lie on a representative block of each block orbit. As design representative blocks (six of them, each representing one block orbit) we take blocks stabilized by the subgroup $\langle c\rangle \leq G_0$. Therefore, these blocks are to be composed from $\langle c\rangle$-point orbits as a whole. The representation (3.2) implies that a selection of 4 points in each point orbit is accomplished using the fixed point and one of five $\langle c\rangle$-orbits of length three. From (3.3) and (3.2) we easily see that there are 5^5 possibilities for a selection of 20 points of a representative block. In the procedure of indexing, on each level, every possible selection of orbit representative block is submitted to all the necessary λ-balance checking (as required by the definition of a symmetric (v, k, λ) design), so indexing is necessarily performed by computer.

The indexing procedure ends up successfully with a great number of symmetric designs constructed. After isomorphic structures reduction it turns out that there are exactly 4 nonisomorphic symmetric designs admitting the specified action of G_0. Among them we point to this *regular* one, say D_0 :

$$2_1 2_2 2_5 2_{11} 3_1 3_3 3_6 3_{14} 4_1 4_4 4_7 4_8 5_1 5_9 5_{10} 5_{15} 6_1 6_{12} 6_{13} 6_{16}$$
$$1_1 1_{12} 1_{13} 1_{16} 3_1 3_9 3_{10} 3_{15} 4_1 4_3 4_6 4_{14} 5_1 5_2 5_5 5_{11} 6_1 6_4 6_7 6_8$$
$$1_1 1_3 1_6 1_{14} 2_1 2_4 2_7 2_8 4_1 4_2 4_5 4_{11} 5_1 5_{12} 5_{13} 5_{16} 6_1 6_9 6_{10} 6_{15}$$
$$1_1 1_9 1_{10} 1_{15} 2_1 2_3 2_6 2_{14} 3_1 3_{12} 3_{13} 3_{16} 5_1 5_4 5_7 5_8 6_1 6_2 6_5 6_{11}$$
$$1_1 1_4 1_7 1_8 2_1 2_{12} 2_{13} 2_{16} 3_1 3_2 3_5 3_{11} 4_1 4_9 4_{10} 4_{15} 6_1 6_3 6_6 6_{14}$$
$$1_1 1_2 1_5 1_{11} 2_1 2_9 2_{10} 2_{15} 3_1 3_4 3_7 3_8 4_1 4_{12} 4_{13} 4_{16} 5_1 5_3 5_6 5_{14}$$

The subgroup $\langle a, b\rangle \leq G_0$ generates all the blocks of the design.

Taking an incidence matrix of the design D_0 as an input, the computer program by V. Tonchev ([23]) gives out the order of the full automorphism group $\mathrm{Aut}\, D_0$, as well as a permutation representation of degree 96 of $\mathrm{Aut}\, D_0$ generators. The latter (given in Table 1) enables us to further analyse the properties of $\mathrm{Aut}\, D_0$. $\mathrm{Aut}\, D_0 \cong [576, 5550]$, in terms of generators and relations

$$\begin{aligned}\mathrm{Aut}\, D_0 = \langle a,b,c,d,e,f \,| & a^4 = b^4 = [a,b] = c^2 = [a,c] = [b,c] = 1, d^3 = 1, \\ & a^d = a^3b^3, b^d = a, [c,d] = 1, e^3 = 1, a^e = a^3b^3, \\ & b^e = a, [c,e] = 1, [d,e] = f^2 = 1, a^f = a^3b^2, \\ & b^f = a^2b, [c,f] = 1, d^f = e^2d^2, [e,f] = 1\rangle,\end{aligned}$$

acts transitively on the point set of D_0. Therefore, once a permutation representation of $\mathrm{Aut}\, D_0$ generators of degree 96 has been chosen (Table 1) and set {1,2,. . . ,96} taken as the point set of our design, D_0 can be represented by a single block, for instance

(3.4)

$$D_0 := [1, 12, 13, 16, 33, 41, 42, 47, 49, 51, 54, 62, 65, 66, 69, 75, 81, 84, 87, 88].$$

r1:=(2,13)(4,15)(5,16)(7,9)(8,10)(11,12)(17,81)(18,93)(19,83)(20,95)(21,96)(22,86)(23,89)
(24,90)(25,87)(26,88)(27,92)(28,91)(29,82)(30,94)(31,84)(32,85)(34,45)(36,47)(37,48)(39,41)
(40,42)(43,44)(49,65)(50,77)(51,67)(52,79)(53,80)(54,70)(55,73)(56,74)(57,71)(58,72)(59,76)
(60,75)(61,66)(62,78)(63,68)(64,69)

r2:=(2,5,11)(3,6,14)(4,7,8)(9,10,15)(12,13,16)(18,21,27)(19,22,30)(20,23,24)(25,26,31)
(28,29,32)(34,37,43)(35,38,46)(36,39,40)(41,42,47)(44,45,48)(50,53,59)(51,54,62)(52,55,56)
(57,58,63)(60,61,64)(66,69,75)(67,70,78)(68,71,72)(73,74,79)(76,77,80)(82,85,91)(83,86,94)
(84,87,88)(89,90,95)(92,93,96)

r3:=(1,2,3,4)(5,8,9,10)(6,13,14,15)(7,12,16,11)(17,18,19,20)(21,24,25,26)(22,29,30,31)
(23,28,32,27)(33,34,35,36)(37,40,41,42)(38,45,46,47)(39,44,48,43)(49,50,51,52)(53,56,57,58)
(54,61,62,63)(55,60,64,59)(65,66,67,68)(69,72,73,74)(70,77,78,79)(71,76,80,75)(81,82,83,84)
(85,88,89,90)(86,93,94,95)(87,92,96,91)

r4:=(1,17)(2,29)(3,19)(4,31)(5,32)(6,22)(7,25)(8,26)(9,23)(10,24)(11,28)(12,27)(13,18)
(14,30)(15,20)(16,21)(33,49)(34,61)(35,51)(36,63)(37,64)(38,54)(39,57)(40,58)(41,55)(42,56)
(43,60)(44,59)(45,50)(46,62)(47,52)(48,53)(65,81)(66,93)(67,83)(68,95)(69,96)(70,86)(71,89)
(72,90)(73,87)(74,88)(75,92)(76,91)(77,82)(78,94)(79,84)(80,85)

TABLE 1. Generators of $\mathrm{Aut}\, D_0$, GAP-*cn* :[576,5550]

The inspection of $\mathrm{Aut}\, D_0$ reveals its regular subgroups to be groups $[96, 68]$, $[96, 83]$, $[96, 130]$, and $[96, 161]$. If we express group $[96, 68]$ in terms of generators and relations as

$$\begin{aligned}[96,68] \cong H_{[96,68]} = \langle x,y,z,w \,|\, & x^4 = y^4 = [x,y] = 1, z^2 = [x,z] = [y,z] = 1, \\ & w^3 = 1, x^w = x^3y^3, y^w = x, [z,w] = 1 \,\rangle\,,\end{aligned}$$

and put $H_{[96,68]} = \left\{ w^l x^p y^j z^k \mid l = 0,1,2; p,j = 0,\dots,3; k = 0,1 \right\}$, then by identifying the points of D_0 with the elements of $H_{[96,68]}$ we obtain difference set $\Delta_{[96,68]}$ corresponding to the representative block (3.4) of D_0:

$$\begin{aligned}\Delta_{[96,68]} = & 1 + x + y + x^3y^3 + z + xyz + x^3z + y^3z \\ & + w(x + y + y^2 + x^3y + xz + x^2y^2z + x^2y^3z + x^3y^3z) \\ & + w^2(xy + xy^3 + x^3y + x^3y^3).\end{aligned}$$

More details on this identification can be found in [14] or [25].

Note that in the case of group $[96,68]$ McFarland's (Dillon's) construction cannot be applied since $Z([96,68]) \cong \mathbb{Z}_2$.

The obtained difference set $\Delta_{[96,68]}$ is not genuinely nonabelian because abelian group $[96,161]$ acts regularly on D_0.

4. Preliminaries on partial difference sets and strongly regular graphs

The notion of a partial difference set (PDS for short) generalizes that of a difference set.

Definition 4.1. *Let H be a group of order v. A k-subset $S \subset H$ is called a* (v,k,λ,μ) partial difference set *if the multiset* $\left\{xy^{-1} \mid x,y \in S, x \neq y\right\}$ *contains each nonidentity element of S exactly λ times and it contains each nonidentity element of $H \setminus S$ exactly μ times.*

Using the notation of group ring $\mathbb{Z}H$, a (v,k,λ,μ) partial difference set $S \subset H$ in group H can be described as a subset for which the equation

$$\underline{S} \cdot \underline{S^{(-1)}} = k\underline{\{e\}} + \lambda\underline{S \setminus \{e\}} + \mu\underline{(H \setminus S) \setminus \{e\}}$$

holds.

It is obvious that any (v,k,λ) difference set is a (v,k,λ,λ) partial difference set.

There are different possibilities to define equivalency between partial difference sets. Here we will call partial differential sets S_1 and S_2 in groups H_1 and H_2, respectively, *equivalent* if there exists a group isomorphism $\varphi : H_1 \to H_2$ which maps S_1 onto S_2.

A partial difference set S is *reversible* if $S = S^{(-1)}$. A reversible partial difference set S is called *regular* if $e \notin S$. It is easy to see (cf. [18]) that the following assertions hold.

Proposition 4.2. *Suppose that S is a reversible (v,k,λ,μ) PDS in a group H such that $e \in S$. Then $(S - e)$ is a regular $(v,k-1,\lambda-2,\mu)$ PDS in H. Conversely, if S is a regular PDS in H, then $(S + e\ )$ is a reversible PDS with the corresponding parameters.*

Proposition 4.3. *Suppose that Δ is a (v,k,λ) difference set in H, $x \in H$. Then*

(i) *Δx is a regular (v, k, λ, λ) PDS if and only if $x^{-1} \notin \Delta$ and Δx is a reversible set;*
(ii) *$\Delta x - e$ is a regular $(v, k-1, \lambda-2, \lambda)$ PDS if and only if $x^{-1} \in \Delta$ and Δx is a reversible set.*

Regular partial difference sets and strongly regular graphs are closely related through the concept of Cayley graphs.

DEFINITION 4.4. *A* strongly regular graph *(SRG)* with parameters *(v, k, λ, μ) is a graph with v vertices which is regular of valency k, i.e., every vertex is incident with k edges, such that any pair of adjacent vertices have exactly λ common neighbors and any pair of non-adjacent vertices have exactly μ common neighbors.*

DEFINITION 4.5. *For a group H and a set $S \subset H$ with the property that $e \notin S$ and $S = S^{(-1)}$, the* Cayley graph $\Gamma = Cay(H, S)$ over H with connection set S *is the graph with vertex set H so that the vertices x and y are adjacent if and only if $xy^{-1} \in S$.*

Accordingly, the edge set of a Cayley graph $\Gamma = Cay(H, S)$ over H with connection set S is $E := \{\{x, sx\} \mid x \in H, s \in S\}$. Γ is an undirected graph without loops. Our construction of strongly regular graphs will be based on the following important assertion about Cayley graphs, [6, p. 230] or [17].

THEOREM 4.6. *A Cayley graph $Cay(H, S)$ is a (v, k, λ, μ) strongly regular graph if and only if S is a (v, k, λ, μ) regular partial difference set in H.*

Equivalent regular PDSs obviously correspond to isomorphic strongly regular Cayley graphs. Note that for two inequivalent partial difference sets S_1 and S_2 in a group H, the graphs $Cay(H, S_1)$ and $Cay(H, S_2)$ can be isomorphic. Similarly, for two inequivalent partial difference sets S_1 and S_2 in groups H_1 and H_2 respectively, $|H_1| = |H_2|$, the graphs $Cay(H_1, S_1)$ and $Cay(H_2, S_2)$ can be isomorphic. Several examples of both such cases occur in our results.

For graph exploring we use GRAPE ([22]), a package which is a part of GAP.

5. STRUCTURES CORRESPONDING TO (96,20,4) DIFFERENCE SETS

We ran an analysis of the 'DS96' list to obtain combinatorial structures corresponding to the difference sets in it. The documentation of the structures obtained or structures themselves are available at the site

(5.1) `http://www.pmfst.hr/~vucicic/DifSets96`

in several files, each including helpful comments. The file names will be given as they appear in the context of this section.

5.1. *Symmetric designs.* Using the DESIGN package for GAP ([21]) we checked that the number of nonisomorphic symmetric designs corresponding to 'DS96' list is 574. Consequently, the list contains 574 nonisomorphic difference sets. The file 'NISD96' at (5.1) creates a list 'NISD' in which NISD[i] contains the i-th isomorphism class, i =1,. . . ,574. The entries of the list NISD[i] have the form [cn, n1], which denotes that the underlying design is the development of the difference set occupying the position 'n1' on the list DS96[cn].

The following list counts the number of elements in the isomorphism classes NISD[i], i =1,. . . ,574.

[31, 24, 6, 38, 1, 10, 12, 12, 4, 30, 30, 3, 5, 2, 26, 25, 4, 20, 20, 8, 8, 20, 20, 8, 7, 11, 2, 1, 1, 1, 1, 2, 1, 1, 1, 1, 3, 5, 10, 17, 5, 5, 7, 1, 1, 1, 2, 3, 1, 1, 2, 8, 1, 1, 2, 3, 1, 8, 1, 2, 1, 8, 2, 1, 1, 3, 8, 2, 1, 1, 4, 4, 1, 2, 2, 2, 2, 3, 2, 1, 2, 1, 1, 1, 1, 1, 2, 2, 2, 1, 2, 4, 4, 4, 4, 4, 4, 4, 2, 2, 2, 2, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 9, 9, 9, 9, 9, 8, 9, 9, 8, 9, 8, 9, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 30, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 1]

Our analysis of the 574 full automorphism groups of the designs is documented in the file 'AutSD96' at (5.1). The file creates a list 'AutSD' of elements [cn, n1, n2]. An entry triple [cn, n1, n2] denotes that the development of the difference set on position 'n1' in the list DS96[cn] is a symmetric design D with $|\mathrm{Aut}\, D| =$ n2. The group [96,cn] is obviously a Singer subgroup of $\mathrm{Aut}\, D$. 'cn' takes all values from the set $\mathcal{A}$.

Table 2 contains orders of the automorphism groups that appear in the 'AutSD' list and the number of nonisomorphic designs having the full automorphism group of the given order. As expected, nonisomorphic designs with small automorphism groups are numerous, while few of them have large automorphism groups. In cases of one sole design D with the group $\mathrm{Aut}\, D$ of the given order, the number of corresponding inequivalent difference sets is given in the brackets. For instance, design D_1 with $|\mathrm{Aut}\, D_1| = 552960$ enabled us

$\|\mathrm{Aut}\, D\|$	No. of designs
96	250
192	236
288	15
384	31
576	6
768	9
864	1 (4)
1152	3
1536	8
1728	1 (12)

$\|\mathrm{Aut}\, D\|$	No. of designs
3072	3
3456	1 (12)
4608	1 (10)
6144	2
7680	1 (1)
9216	1 (38)
12288	2
138240	1 (6)
184320	1 (24)
552960	1 (31)

Table 2.

to construct 31 inequivalent difference set in 18 nonisomorphic Singer groups, see [12].

5.2. *Regular (96,20,4,4) and (96,19,2,4) partial difference sets.* By the results highlighted in Section 4 it can easily be verified ([13]) that one procedure for the search of regular partial difference sets, starting from a known difference set $\Delta \subseteq G$, can be performed in the following two steps:

(i) construction of all shifts Δx of $\Delta, x \in G$,
(ii) selection of those shifts which are reversible sets in G.

Then, each reversible shift which does not contain e is a regular (v, k, λ, λ) PDS, while each reversible shift that contains e yields a regular $(v, k-1, \lambda-2, \lambda)$ PDS $\Delta x \setminus \{e\}$.

To this procedure of "surveyed shifting" we have submitted the difference sets in 'DS96' list. At the end of the procedure we obtained 285 regular PDSs Δx in 9 groups as detailed in the following table. The full

[96,*cn*]→	64	70	71	186	190	195	197	226	227
No. of PDSs →	2	6	2	32	18	108	32	72	13

Table 3.

list 'RTID' of Δx identifiers of the form [*cn*, n1, n2] is in the file "Reversible_translates_list.txt" at (5.1). An identifier [*cn*, n1, n2] stands for the translate Δx corresponding to the difference set Δ occupying the position 'n1' in the list DS96[*cn*]; 'n2' is the position of element x as obtained in the GAP command `Elements(SmallGroup([96,`*cn*`]))`output. '*cn*' takes values from the set {64, 70, 71, 186, 190, 195, 197, 226, 227}.

After GAP-testing on group automorphism, final result boils down to 144 inequivalent regular PDSs in 9 groups. In the same manner their identifiers are given in the file "Reversible_translates_list.txt", forming the list 'IEQRTID'. 115 PDSs are of cardinality 20 and 29 of cardinality 19. It turns out that

[96, *cn*]→	64	70	71	186	190	195	197	226	227
No. of inequivalent (96,20,4,4) PDSs	1	3	1	14	8	48	14	23	3
No. of inequivalent (96,19,2,4) PDSs	1	1	1	2	2	12	2	5	3

Table 4.

115 difference set shifts being (96,20,4,4) regular PDSs belong to 27 design-equivalency classes. In the NISD list these are the classes NISD[i] for i in $\{1$, 2, 3, 4, 5, 6, 9, 10, 11, 13, 15, 16, 17, 18, 19, 22, 23, 76, 81, 88, 250, 251, 252, 253, 333, 570, $571\}$.

5.3. *Strongly regular graphs with parameters (96,20,4,4) and (96,19,2,4).* Regarding isomorphism of the corresponding strongly regular Cayley graphs, our 144 PDSs split into 63 nonisomorphic SRG-classes (GRAPE-tested [22]). 52 graphs are with parameters (96,20,4,4) and 11 with parameters (96,19,2,4). Table 5 covers the case of valency 20, i.e., parameters (96,20,4,4). Each table row refers to nonisomorphic graphs Γ with $|\mathrm{Aut}\,\Gamma|$ indicated in the first colon, and the number of such graphs is given for each of the nine groups indicated in the heading row. The entries of the last colon summarize the number of nonisomorphic graphs with the full automorphism group of order cited in the first colon. The last row contains the colon sum, whether it be the number of nonisomorphic graphs for each observed group, or the total number of nonisomorphic (96,20,4,4) graphs constructed.

In the second row note the example of inequivalent regular PDSs (in different groups) giving isomorphic Cayley graphs. That situation does not occur only in rows 1, 3, and 8. In case of groups [96,186], [96,195], [96,197], and [96,226] the number of nonisomorphic graphs obtained is less than the number of inequivalent PDSs, cf. Table 4.

Among the obtained graphs with parameters (96,20,4,4) there are some graphs already known from the literature ([7] and [13]). The best known is a collinearity graph of $GQ(5,3)$.

The case of valency 19, i.e., parameters (96,19,2,4), is described in Table 6.

In the case of groups [96,195] and [96,227] the number of nonisomorphic graphs obtained is less than the number of inequivalent PDSs, cf. Table 4. So far only four SRGs with parameters (96,19,2,4) have been known ([7] and [13]). Their full automorphism groups are not of size 96, 786 or 1536, which means that at least 9 of our (96,19,2,4) graphs are new. Strongly regular graphs with parameters (96,19,2,4) are candidates for 5-chromatic SRGs ([10]). The GRAPE checking gives that none of here constructed graphs is 5-chromatic.

[96, cn]→ / $\lvert\mathrm{Aut}\,\Gamma\rvert$↓	64	70	71	186	190	195	197	226	227	Nonisomorphic
96						12		9		21
192				6		6	6	6		6
384						4				4
576				1		1	1	1		1
768		1		4	4	10	4	4		12
1536						2			1	2
3072		1			2	2			1	2
7680						1				1
11520				2	1	2	2	2		2
138240	1	1	1		1	1			1	1
↑ $\lvert\mathrm{Aut}\,\Gamma\rvert$	1	3	1	13	8	41	13	22	3	Total: 52

Table 5. Survey of SRGs with parameters (96,20,4,4)

[96, cn]→ / $\lvert\mathrm{Aut}\,\Gamma\rvert$↓	64	70	71	186	190	195	197	226	227	Nonisomorphic
96						2		3		5
288						1			1	1
786				2	1	3	2	2		3
1536						1				1
9216	1	1	1		1	1			1	1
↑ $\lvert\mathrm{Aut}\,\Gamma\rvert$	1	1	1	2	2	8	2	5	2	Total: 11

Table 6. Survey of SRGs with parameters (96,19,2,4)

The GRAPE-file '63srg96v.txt' containing records of our (96,20,4,4) and (96,19,2,4) graphs can be found at the site (5.1).

Note that even 49 nonisomorphic graphs, 41 of valency 20 and 8 of valency 19, can be represented as PDSs in the group [96,195].

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