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ON (m,n)–JORDAN CENTRALIZERS IN RINGS AND ALGEBRAS

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Abstract. Let $m \geq 0, n \geq 0$ be fixed integers with $m+n \neq 0$ and let R be a ring. It is our aim in this paper to investigate additive mapping $T : R \to R$ satisfying the relation $(m+n)T(x^2) = mT(x)x + nxT(x)$ for all $x \in R$.

This research is a continuation of our earlier work ([11]). Throughout, R will represent an associative ring with center $Z(R)$. Given an integer $n \geq 2$, a ring R is said to be $n-$torsion free, if for $x \in R$, $nx = 0$ implies $x = 0$. As usual the commutator $xy - yx$ will be denoted by $[x,y]$. We define $[y,x]_n$ inductively as follows: $[y,x]_1 = [y,x]$, $[y,x]_{n+1} = [[y,x]_n, x]$. We shall use the commutator identities $[xy,z] = [x,z]\,y + x\,[y,z]$ and $[x,yz] = [x,y]\,z + y\,[x,z]$, for all $x,y,z \in R$. A mapping F, which maps a ring R into itself, is called commuting on R in case $[F(x),x] = 0$ holds for all $x \in R$. Recall that a ring R is prime if for $a,b \in R$, $aRb = (0)$ implies that either $a = 0$ or $b = 0$, and is semiprime in case $aRa = (0)$ implies $a = 0$. An additive mapping $D : R \to R$, where R is an arbitrary ring, is called a derivation if $D(xy) = D(x)y + xD(y)$ holds for all pairs $x,y \in R$, and is called a Jordan derivation in case $D(x^2) = D(x)x + xD(x)$ is fulfilled for all $x \in R$. A derivation D is inner in case there exists $a \in R$, such that $D(x) = [a,x]$ holds for all $x \in R$. Every derivation is a Jordan derivation. The converse is in general not true. A classical result of Herstein ([8]) asserts that any Jordan derivation on a prime ring with $char(R) \neq 2$ is a derivation. A brief proof of Herstein's result can be found

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in [3]. Cusack ([7]) generalized Herstein's result to $2-$torsion free semiprime rings (see also [4] for an alternative proof). We denote by Q_r, C and RC Martindale right ring of quotients, extended centroid, and central closure of a semiprime ring R, respectively.For the explanation of Q_r, C, and RC we refer the reader to [2]. An additive mapping $T : R \to R$ is called a left centralizer in case $T(xy) = T(x)y$ holds for all pairs $x, y \in R$. In case R has the identity element $T : R \to R$ is a left centralizer iff T is of the form $T(x) = ax$ for all $x \in R$, where $a \in R$ is a fixed element. For a semiprime ring R all left centralizers are of the form $T(x) = qx$ for all $x \in R$, where q is a fixed element of Q_r (see Chapter 2 in [2]). An additive mapping $T : R \to R$ is called a left Jordan centralizer in case $T(x^2) = T(x)x$holds for all $x \in R$. The definition of right centralizer and right Jordan centralizer should be self-explanatory. We call $T : R \to R$ a two-sided centralizer in case T is both a left and a right centralizer. In case $T : R \to R$ is a two-sided centralizer, where R is a semiprime ring with extended centroid C, then there exists an element $\lambda \in C$ such that $T(x) = \lambda x$ for all $x \in R$ (see Theorem 2.3.2 in [2]). Zalar ([14]) has proved that any left (right) Jordan centralizer on a $2-$torsion free semiprime ring is a left (right) centralizer. Molnár ([9]) has proved that in case we have an additive mapping $T : A \to A$, where A is a semisimple H^*-algebra, satisfying the relation $T(x^3) = T(x)x^2$ $(T(x^3) = x^2T(x))$ for all $x \in A$, then T is a left (right) centralizer. Let us recall that a semisimple H^*-algebra is a complex semisimple Banach$^*-$algebra whose norm is a Hilbert space norm such that $(x, yz^*) = (xz, y) = (z, x^*y)$ is fulfilled for all $x, y, z \in A$ (see [1]). For results concerning centralizers in rings and algebras we refer to [10–13] where further references can be found. Let X be a real or complex Banach space and let $L(X)$ and $F(X)$ denote the algebra of all bounded linear operators on X and the ideal of all finite rank operators in $L(X)$, respectively. An algebra $A(X) \subset L(X)$ is said to be standard in case $F(X) \subset A(X)$. Let us point out that any standard operator algebra is prime, which is a consequence of Hahn-Banach theorem. In case X is a real or complex Hilbert space we denote by A^* the adjoint operator of $A \in L(X)$.

We proceed with the following definition.

Definition 1. *Let $m \geq 0, n \geq 0$ be fixed integers with $m + n \neq 0$ and let R be a ring. An additive mapping $T : R \to R$ will be called an $(m, n)-$Jordan centralizer in case*

$$(m + n)T(x^2) = mT(x)x + nxT(x) \tag{1}$$

holds for all $x \in R$.

Obviously, $(1, 0)$–Jordan centralizer is a left Jordan centralizer, $(0, 1)$–Jordan centralizer is a right Jordan centralizer, and in case $(1, 1)$–Jordan centralizer we have the relation

$$2T(x^2) = T(x)x + xT(x), x \in R. \tag{2}$$

Vukman ([11]) has proved that in case there exists an additive mapping $T : R \to R$, where R is a $2-$torsion free semiprime ring, satisfying the relation (2), then T is a two-sided centralizer. The above observations lead to the following conjecture.

CONJECTURE 2. *Let $m \geq 1, n \geq 1$ be some integers, let R be a semiprime ring with suitable torsion restrictions, and let $T : R \to R$ be an $(m,n)-$Jordan centralizer. In this case T is a two-sided centralizer.*

In this paper we prove some results related to the above conjecture. First we prove the following proposition.

PROPOSITION 3. *Let $m \geq 0, n \geq 0$ be some integers with $m + n \neq 0$, let R be a ring and let $T : R \to R$ an $(m,n)-$Jordan centralizer. In this case we have*

$$\begin{aligned} &2(m+n)^2T(xyx) \\ &\quad = mnT(x)xy + m(2m+n)T(x)yx - mnT(y)x^2 + 2mnxT(y)x \\ &\qquad - mnx^2T(y) + n(m+2n)xyT(x) + mnyxT(x), \end{aligned} \tag{3}$$

for all pairs $x, y \in R$.

PROOF. The linearization of the relation (1) gives

$$(m+n)T(xy+yx) = mT(x)y + mT(y)x + nxT(y) + nyT(x), x, y \in R. \tag{4}$$

Putting in the above relation $(m+n)(xy+yx)$ for y we obtain

$$\begin{aligned} &(m+n)^2T(x^2y + yx^2 + 2xyx) \\ &\quad = m(m+n)T(x)(xy+yx) + m(m+n)T(xy+yx)x \\ &\qquad + n(m+n)xT(xy+yx) + n(m+n)(xy+yx)T(x), x, y \in R. \end{aligned}$$

Applying first the relation (4) and then the relation (1) we obtain

$$\begin{aligned} &2(m+n)^2T(xyx) + (m+n)mT(x^2)y + (m+n)mT(y)x^2 \\ &\quad + (m+n)nx^2T(y) + (m+n)nyT(x^2) \\ &= m(m+n)T(x)(xy+yx) + m(mT(x)y + mT(y)x + nxT(y) + nyT(x))x \\ &\quad + nx(mT(x)y + mT(y)x + nxT(y) + nyT(x)) + n(m+n)(xy+yx)T(x), \\ &\qquad x, y \in R. \end{aligned}$$

$$\begin{aligned} &2(m+n)^2T(xyx) + m(mT(x)x + nxT(x))y + (m+n)mT(y)x^2 \\ &\quad + (m+n)nx^2T(y) + ny(mT(x)x + nxT(x)) \\ &= m(m+n)T(x)(xy+yx) + m(mT(x)y + mT(y)x + nxT(y) + nyT(x))x \\ &\quad + nx(mT(x)y + mT(y)x + nxT(y) + nyT(x)) + n(m+n)(xy+yx)T(x), \\ &\qquad x, y \in R. \end{aligned}$$

Collecting terms we arrive at

$$\begin{aligned}&2(m+n)^2T(xyx)\\&\quad = mnT(x)xy + m(2m+n)T(x)yx - mnT(y)x^2 + 2mnxT(y)x\\&\qquad - mnx^2T(y) + n(m+2n)xyT(x) + mnyxT(x),\ \ x, y \in R.\end{aligned}$$

which completes the proof. ∎

In particular for $y = x$ the relation (3) reduces to the relation below which will be considered latter on.

$$\begin{aligned}&2(m+n)^2T(x^3)\\&\quad = m(2m+n)T(x)x^2 + 2mnxT(x)x + n(2n+m)x^2T(x), x \in R.\end{aligned} \tag{5}$$

The result below proves Conjecture 2 in case R is a prime ring.

Theorem 4. *Let $m \geq 1, n \geq 1$ be fixed integers and let R be a prime ring with $char(R) \neq 6mn(m+n)$. Suppose $T : R \to R$ is a $(m, n)-$Jordan centralizer. If $Z(R)$ is nonzero, then T is a two-sided centralizer.*

In the proof of Theorem 4 we shall use the result below proved by Brešar and Hvala ([6]).

Theorem 5. *Let $n > 1$ be an integer and let R be a prime ring such that $char(R) = 0$ or $char(R) \geq n$. Let $f_1, ..., f_n : R \to R$ be additive mappings satisfying the relation*

$$f_1(x)x^{n-1} + xf_2(x)x^{n-2} + ... + x^{n-1}f_n(x) = 0$$

for all $x \in R$. If $Z(R)$ is nonzero, then there exist elements $a_1, a_2, ..., a_{n-1} \in RC + C$ and additive mappings $\zeta_1, ..., \zeta_n : R \to C$, such that

$$\begin{aligned}f_1(x) &= xa_1 + \zeta_1(x),\\ f_k(x) &= -a_{k-1}x + xa_k + \zeta_k(x), k = 2, ..., n-1,\\ f_n(x) &= -a_{n-1}x + \zeta_n(x),\end{aligned}$$

for all $x \in R$. Moreover, $\zeta_1 + ... + \zeta_n = 0$.

Proof of Theorem 4. Putting $(m+n)x^2$ for x in (1) and applying (1) we obtain

$$\begin{aligned}(m+n)^3T(x^4) &= m(m+n)^2T(x^2)x^2 + n(m+n)^2x^2T(x^2)\\ &= m(m+n)(mT(x)x + nxT(x))x^2\\ &\quad + n(m+n)x^2(mT(x)x + nxT(x))\\ &= m^2(m+n)T(x)x^3 + mn(m+n)xT(x)x^2\\ &\quad + mn(m+n)x^2T(x)x + n^2(m+n)x^3T(x).\end{aligned}$$

We have therefore

$$(6)\qquad \begin{aligned}(m+n)^3T(x^4) &= m^2(m+n)T(x)x^3 + mn(m+n)xT(x)x^2\\ &\quad + mn(m+n)x^2T(x)x + n^2(m+n)x^3T(x),\ x\in R.\end{aligned}$$

On the other hand, putting in the relation (3) $y=(m+n)x^2$ and applying (1), we obtain

$$\begin{aligned}2(m+n)^3T(x^4) &= mn(m+n)T(x)x^3 + m(2m+n)(m+n)T(x)x^3\\ &\quad - mn(m+n)T(x^2)x^2 + 2mn(m+n)xT(x^2)x\\ &\quad - mn(m+n)x^2T(x^2) + n(m+2n)(m+n)x^3T(x)\\ &\quad + mn(m+n)x^3T(x)\\ &= 2m(m+n)^2T(x)x^3 - mn(mT(x)x+nxT(x))x^2\\ &\quad + 2mnx(mT(x)x+nxT(x))x - mnx^2(mT(x)x+nxT(x))\\ &\quad + 2n(m+n)^2x^3T(x)\\ &= (2m(m+n)^2-m^2n)T(x)x^3 + mn(2m-n)xT(x)x^2\\ &\quad + mn(2n-m)x^2T(x)x + (2n(m+n)^2-mn^2)x^3T(x),\\ &\qquad x\in R.\end{aligned}$$

We have therefore

(7)
$$\begin{aligned}2(m+n)^3T(x^4) &= (2m(m+n)^2-m^2n)T(x)x^3 + mn(2m-n)xT(x)x^2\\ &\quad + mn(2n-m)x^2T(x)x + (2n(m+n)^2-mn^2)x^3T(x).\end{aligned}$$

By comparing (6) with (7) we obtain

$$mn(2n+m)T(x)x^3 - 3mn^2xT(x)x^2 - 3m^2nx^2T(x)x + mn(2m+n)x^3T(x) = 0,$$

for all $x\in R$, which reduces according to the requirements of the theorem to

$$(2n+m)T(x)x^3 - 3nxT(x)x^2 - 3mx^2T(x)x + (2m+n)x^3T(x) = 0,\quad x\in R.$$

Now applying Theorem 5 one can conclude that

$$(8)\qquad (2n+m)T(x) = xa_1 + \zeta_1(x),\ x\in R,$$

$$(9)\qquad -3nT(x) = -a_1x + xa_2 + \zeta_2(x),\ x\in R,$$

$$(10)\qquad -3mT(x) = -a_2x + xa_3 + \zeta_3(x),\ x\in R,$$

$$(11)\qquad (2m+n)T(x) = -a_3x + \zeta_4(x),\ x\in R,$$

where $a_1, a_2, a_3 \in RC + C$, and $\zeta_1 ... \zeta_4 : R \to C$ are additive mappings with $\zeta_1 + ... + \zeta_4 = 0$. Combining the relations from (8) to (11) one obtains

$$(12)\qquad D_1(x) + D_2(x) + D_3(x) = 0,\quad x\in R,$$

where $D_i(x)$ stands for $[a_i, x]$. Note that D_i are derivations. Combining relations (8) and (11), and putting x^2 for x we obtain

$$3(m+n)T(x^2) = x^2a_1 - a_3x^2 + \zeta_1(x^2) + \zeta_4(x^2), x \in R. \tag{13}$$

Left multiplication of the relation (9) by x and right multiplication of the relation (10) by x gives

$$-3nxT(x) = -xa_1x + x^2a_2 + \zeta_2(x)x, \ x \in R, \tag{14}$$

$$-3mT(x)x = -a_2x^2 + xa_3x + \zeta_3(x)x, \ x \in R. \tag{15}$$

Combining (13), (14) and (15) we obtain

$$\begin{aligned} &3((m+n)T(x^2) - mT(x)x - nxT(x)) \\ &\quad = -xD_1(x) - D_2(x^2) - D_3(x)x + \zeta_1(x^2) + \zeta_2(x)x \\ &\qquad + \zeta_3(x)x + \zeta_4(x^2), x \in R, \end{aligned}$$

which reduces because of (1) to

$$\begin{aligned} &-xD_1(x) - D_2(x)x - xD_2(x) - D_3(x)x + \zeta_1(x^2) + \zeta_2(x)x \\ &\quad + \zeta_3(x)x + \zeta_4(x^2) = 0, x \in R. \end{aligned}$$

Applying (12) in the above relation we obtain

$$D_1(x)x + xD_3(x) + \zeta_1(x^2) + \zeta_2(x)x + \zeta_3(x)x + \zeta_4(x^2) = 0, x \in R,$$

which gives

$$[D_1(x)x + xD_3(x), x] = 0, \ x \in R.$$

The above relation can be written in the form

$$D_1(x)x^2 + x(D_3(x) - D_1(x))x - x^2D_3(x) = 0, x \in R.$$

From the above relation it follows according to Corollary 3. 4.in [6] that $D_1(x) = D_3(x) = 0$ for all $x \in R$, whence it follows that $D_2(x) = 0$ because of (12). In other words, we have

$$[a_1, x] = [a_2, x] = [a_3, x] = 0, x \in R.$$

Now applying the above relation in (9) we obtain

$$3n\,[T(x), x] = [a_1x, x] - [xa_2, x] = [a_1, x]\,x - x\,[a_2, x] = 0, x \in R.$$

We have therefore $3n\,[T(x), x] = 0, x \in R$, which reduces to

$$[T(x), x] = 0, x \in R$$

according to the requirements of the theorem. In other words, T is commuting on R. Now, one can replace in (1) $xT(x)$ by $T(x)x$, which gives $(m+n)T(x^2) = (m+n)T(x)x, x \in R$, whence it follows because of the requirements of the theorem that

$$T(x^2) = T(x)x$$

holds for all $x \in R$. Of course, we have also

$$T(x^2) = xT(x), x \in R.$$

In other words, T is a left and a right Jordan centralizer. By proposition 1.4. in [14] T is a left and a right centralizer, which completes the proof of the theorem. ☐

An additive mapping $D : R \to R$, where R is an arbitrary ring, is called a Jordan triple derivation in case

$$D(xyx) = D(x)yx + xD(y)x + xyD(x)$$

holds for all pairs $x, y \in R$. One can easily prove that any Jordan derivation on arbitrary $2-$torsion free ring is a Jordan triple derivation (see [3] for the details). Brešar ([5]) has proved that any Jordan triple derivation, which maps a $2-$torsion free semiprime ring into itself, is a Jordan derivation. These observations and Proposition 3 lead to the definition and the conjecture below.

Definition 6. *Let $m \geq 0, n \geq 0$ be some integers with $m + n \neq 0$, and let R be an arbitrary ring. An additive mapping $D : R \to R$ is called an $(m,n)-$Jordan triple centralizer in case*

$$\begin{aligned} 2(m+n)^2T(xyx) &= mnT(x)xy + m(2m+n)T(x)yx - mnT(y)x^2 \\ &\quad + 2mnxT(y)x - mnx^2T(y) + n(m+2n)xyT(x) \\ &\quad + mnyxT(x), \end{aligned}$$

holds for all pairs $x, y \in R$.

Conjecture 7. *Let $m \geq 1, n \geq 1$ be some integers, let R be a semiprime ring with suitable torsion restrictions, and let $T : R \to R$ be an $(m,n)-$Jordan triple centralizer. In this case T is a two-sided centralizer.*

We proceed with the following result.

Theorem 8. *Let X be Hilbert space over the real or complex field $\mathcal{K}$, let $A(X) \subset L(X)$ be a standard operator algebra which is closed under the adjoint operation, and let $m \geq 1, n \geq 1$ be some integers. Suppose there exists an additive mapping $T : A(X) \to L(X)$ satisfying the relation*

$$(16)\quad 2(m+n)^2T(A^3) = m(2m+n)T(A)A^2 + 2mnAT(A)A + n(2n+m)A^2T(A)$$

for all $A \in A(X)$. In this case T is of the form $T(A) = \lambda A$, for all $A \in A(X)$ and some $\lambda \in \mathcal{K}$. In particular, T is linear and continuous.

Let us point out that in the theorem above we obtain as a result the continuity of T under purely algebraic assumptions, which means that Theorem 8 might be of some interest from the automatic continuity point of view.

Proof of Theorem 8. Let us first consider the restriction of T on $F(X)$. Let A be from $F(X)$ and let $P \in F(X)$, $P^* = P$ be a projection with $AP = PA = A$. We have also $A^*P = PA^* = A^*$. From the relation (16) one obtains

$$2(m+n)^2T(P) = m(2m+n)T(P)P+2mnPT(P)P+n(2n+m)PT(P). \tag{17}$$

Right multiplication of the above relation by P gives

$$T(P)P = PT(P)P. \tag{18}$$

Similarly,

$$PT(P) = PT(P)P. \tag{19}$$

Combining (18) and (19) we obtain

$$T(P)P = PT(P). \tag{20}$$

Applying (18), (19) and (20) in (17) we obtain

$$T(P) = T(P)P = PT(P). \tag{21}$$

Putting $A+P$ for A in the relation (16) one obtains

$$\begin{aligned} &2(m+n)^2T(A^3+3A^2+3A+P)\\ &\quad = m(2m+n)T(A+P)(A^2+2A+P)+2mn(A+P)T(A+P)(A+P)\\ &\qquad + n(2n+m)(A^2+2A+P)T(A+P) \end{aligned}$$

which reduces to

$$\begin{aligned} &2(m+n)^2(3T(A^2)+3T(A))\\ &\qquad = m(2m+n)(T(P)A^2+2T(A)A+2T(P)A+T(A))\\ &\qquad\quad + 2mn(T(A)A+AT(P)A+T(P)A+AT(A)+T(A)+AT(P))\\ &\qquad\quad + n(2n+m)(2AT(A)+T(A)+A^2T(P)+2AT(P)). \end{aligned}$$

Putting in the above relation $-A$ for A and comparing the relation so obtained with the above relation we obtain

$$\begin{aligned} 6(m+n)^2T(A^2) &= m(2m+n)BA^2+4m(m+n)T(A)A+2mnABA\\ &\quad + 4n(m+n)AT(A)+n(2n+m)A^2B \end{aligned} \tag{22}$$

and

$$(m+n)T(A) = mBA+nAB \tag{23}$$

where B stands for $T(P)$. From the relation (23) one can conclude that T maps $F(X)$ into itself. Combining (22) with (23) we obtain

$$\begin{aligned} &6(m+n)(mBA^2+nA^2B)\\ &\qquad = m(2m+n)BA^2+4m(mBA+nAB)A+2mnABA\\ &\qquad\quad + 4nA(mBA+nAB)+n(2n+m)A^2B \end{aligned}$$

which reduces to $5mnBA^2 + 5mnA^2B - 10mnABA = 0$ and finally to

$$BA^2 + A^2B - 2ABA = 0,$$

which can be written in the form

$$[[B,A],A] = 0. \tag{24}$$

Let us denote by F_P the set $\{A;\ A \in F(X),\ AP = PA\}$. The set F_P is an algebra which is closed under the adjoint operation. According to (20) one can conclude that $B \in F_P$. Let us prove that F_P is semiprime. Suppose that

$$ACA = 0,$$

holds for some $A \in F_P$ and all $C \in F_P$. Putting in the above relation $C = A^*$ and multiplying the relation so obtained from the left side by A^*, we obtain $(A^*A)^*(A^*A) = 0$, whence it follows $A^*A = 0$, which gives $A = 0$. The linearization of the relation (24) gives

$$[[B,A],C] + [[B,C],A] = 0.$$

Putting AC for A in the above relation we obtain

$$\begin{aligned} 0 =& \ [[B,A],AC] + [[B,AC],A] \\ =& \ [[B,A],A]C + A[[B,A],C] + [[B,A]C + A[B,C],A] \\ =& \ A[[B,A],C] + [[B,A],A]C + [B,A][C,A] + A[[B,C],A] \\ =& \ [B,A][C,A]. \end{aligned}$$

We have therefore

$$[B,A][C,A] = 0.$$

The substitution CB for C in the above relation gives $[B,A]C[B,A] = 0$, for all pairs $A, C \in F_P$. Since F_P is semiprime we have

$$[B,A] = 0$$

for all $A \in F_P$. Now the relation (23) reduces to $T(A) = BA = AB$, which gives

$$T(A^2) = BA^2 = A^2B = (BA)A = A(AB) = T(A)A = AT(A).$$

Thus we have $T(A^2) = T(A)A = AT(A)$, for all $A \in F(X)$. In other words, T is a left and a right Jordan centralizer on $F(X)$. Since $F(X)$ is prime one can conclude by Proposition 1.4 in [14] that T is a two-sided centralizer. One can easily prove that T is of the form

$$T(A) = \lambda A$$

for any $A \in F(X)$ and some $\lambda \in \mathcal{K}$ (see [10] for the details). It remains to prove that the above relation holds on $A(X)$ as well. Let us introduce $T_1 : A(X) \to L(X)$ by $T_1(A) = \lambda A$ and consider $T_0 = T - T_1$. The mapping T_0 is, obviously, additive and satisfies the relation (16). Besides, T_0 vanishes on $F(X)$. Let $A \in A(X)$, let $P \in F(X)$, be an one-dimensional projection and $S = A + PAP - (AP + PA)$. Note that S can be written in the form $S = (I - P)A(I - P)$, where I denotes the identity operator on X. Since, obviously, $S - A \in F(X)$, we have $T_0(S) = T_0(A)$. Besides, $SP = PS = 0$. We have therefore the relation

$$2(m+n)^2T_0(S^3) = m(2m+n)T_0(S)S^2 + 2mnST_0(S)S + n(2n+m)S^2T_0(S).$$

Applying the above relation and the fact that $T_0(P) = 0$, $SP = PS = 0$, we obtain

$$\begin{aligned} &m(2m+n)T_0(S)S^2 + 2mnST_0(S)S + n(2n+m)S^2T_0(S) \\ &\quad = 2(m+n)^2T_0(S^3) = 2(m+n)^2T_0(S^3+P) = 2(m+n)^2T_0((S+P)^3) \\ &\quad = m(2m+n)T_0(S+P)(S+P)^2 + 2mn(S+P)T_0(S+P)(S+P) \\ &\qquad + n(2n+m)(S+P)^2T_0(S+P) \\ &\quad = m(2m+n)T_0(S)(S^2+P) + 2mn(S+P)T_0(S)(S+P) \\ &\qquad + n(2n+m)(S^2+P)T_0(S). \end{aligned}$$

We have therefore

$$\begin{aligned} &m(2m+n)T_0(S)S^2 + 2mnST_0(S)S + n(2n+m)S^2T_0(S) \\ &\quad = m(2m+n)T_0(S)(S^2+P) + 2mn(S+P)T_0(S)(S+P) \\ &\qquad + n(2n+m)(S^2+P)T_0(S), \end{aligned}$$

which reduces to

(25)
$$(2m+n)T_0(A)P + 2mPT_0(A)S + 2mST_0(A)P + 2mPT_0(A)P + 2mPT_0(A) = 0.$$

Multiplying the above relation from both sides by P we obtain

$$PT_0(A)P = 0. \tag{26}$$

Right multiplication of the relation (25) by P gives because of (26)

$$(2m+n)T_0(A)P + 2mST_0(A)P = 0. \tag{27}$$

Putting in the above relation $-A$ for A, and comparing the relation so obtained with the above relation, (let us recall that $S = (I - P)A(I - P)$) we obtain

$$T_0(A)P = 0.$$

Since P is an arbitrary one-dimensional projection, one can conclude that $T_0(A) = 0$, for any $A \in A(X)$. In other words, we have proved that T is of the form $T(A) = \lambda A$, for all $A \in A(X)$ and some $\lambda \in \mathcal{K}$. The proof of the theorem is complete. $\square$

It should be mentioned that in the proof of Theorem 8 we used some ideas and methods similar to those used by Molnár in [9].

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