GLASNIK MATEMATIČKI
Vol. 45(65)(2010), 55 – 62

CHARACTERIZATIONS OF FINITE ABELIAN AND MINIMAL NONABELIAN GROUPS

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In memory of Semen L. Gramm (1916-2009)

Abstract. In this note we present the following characterizations of finite abelian and minimal nonabelian groups: (i) A group G is abelian if and only if $G' = \Phi(G)'$. (ii) A group G is either abelian or minimal nonabelian if and only if $\Phi(G)' = H'$ for all maximal subgroups H of G. We also prove a number of related results.

In this note G is a nonidentity finite group and p, q are distinct primes. Our notation is standard for finite group theory (see [3] and [6]).

In what follows we use freely some known properties of Φ-subgroups ([7], see also [3, §1]). For example, if $M \triangleleft G$, then $\Phi(M) \leq \Phi(G)$, and G is nilpotent if and only if $G' \leq \Phi(G)$ which is equivalent to normality of all maximal subgroups in G (Wielandt; see Lemma J(d)). It follows from Schur-Zassenhaus' theorem that orders $|G|$ and $|G/\Phi(G)|$ have the same prime divisors (see Lemma J(g)). If $P \in \mathrm{Syl}_p(G)$ is G-invariant, then $P \cap \Phi(G) = \Phi(P)$ ([1]; see Lemma J(a) and its proof following Lemma J (for more general result where P is a normal Hall subgroup of G, see [2, Theorem 3.1]). We also use the Miller-Moreno-Redei description of minimal nonabelian groups (see Lemma J(b,c); for proofs, see [3, Exercise 1.8a] and [6, Lemma 11.2]). For example, if G is a minimal nonabelian group, then $\Phi(G)$ is primary cyclic if G is nonnilpotent, and G is prime-power with $|G'| = p$ if G is nilpotent.

2010 *Mathematics Subject Classification.* 20D15.

Key words and phrases. Maximal subgroup, abelian, minimal nonabelian, minimal nonnilpotent and Frobenius groups, Frattini subgroup, derived subgroup.

If G is abelian, then $G' = \Phi(G)'$ and it appears that this property characterizes abelian groups (Theorem 1). If G is either abelian or minimal nonabelian, then $H' = \Phi(G)'$, and Theorem 3 shows that this property is characteristic for groups all of whose maximal subgroups are abelian.

Let Γ_1 (Γ_1^n)) be the set of maximal (nonnormal maximal) subgroups of a group G. A group G is not nilpotent if and only it the set Γ_1^n is not empty (Lemma J(d)).

In Lemma J we collected most known results cited in what follows so that our note is self contained modulo Lemma J.

LEMMA J. *Let G be a finite group.*

(a) *R. Baer ([1]). If $P \in \mathrm{Syl}(G)$ is G-invariant, then $\Phi(P) = P \cap \Phi(G)$.*

(b) *(Redei; see [3, Exercise 1.8a].) If G is a nilpotent minimal nonabelian group, then G is a p-group, $|G'| = p, \mathrm{Z}(G) = \Phi(G)$ is of index p^2 in G and one of the following holds:*

   (i) *$p = 2$ and G is the ordinary quaternion group,*

   (ii) *$G = \langle a, b \mid a^{p^m} = b^{p^n} = 1,\ a^b = a^{1+p^{m-1}},\ m > 1\rangle$ is metacyclic of order p^{m+n},*

   (iii) *$G = \langle a, b \mid a^{p^m} = b^{p^n} = c^p = 1,\ c = [a, b],\ [a, c] = [b, c] = 1\rangle$ is nonmetacyclic of order p^{m+n+1}.*

   If $\exp(G) = p$, then $p > 2$ and $|G| = p^3$.

(c) *(Miller-Moreno; see [3, Lemma 10.8]) If G is a nonnilpotent minimal nonabelian group, then $|G| = p^a q^b$, $G = P \cdot Q$, where $P \in \mathrm{Syl}_p(G)$ is cyclic, $Q = G' \in \mathrm{Syl}_q(G)$ is a minimal normal subgroup of $G, \mathrm{Z}(G) = \Phi(G)$ has index p in P.*

(d) *(Wielandt) A group G is nilpotent if and only if $G' \leq \Phi(G)$ (or, what is equivalent, $G/\Phi(G)$ is nilpotent).*

(e) *If G is nilpotent and noncyclic, then G/G' is also noncyclic.*

(f) *([7]) If a Sylow p-subgroup is normal in $G/\Phi(G)$, then a Sylow p-subgroup is normal in G. If H is normal in G, then $\Phi(H) \leq \Phi(G)$.*

(g) *$\pi(G/\Phi(G)) = \pi(G)$, where $\pi(G)$ is the set of prime divisors of the order of G.*

(h) *([4]) If $\mathrm{N}_F(F \cap H) \neq F \cap H \neq \mathrm{N}_H(F \cap H)$ for any two distinct $F, H \in \Gamma_1$, then either G is nilpotent or $G/\Phi(G)$ is (nonnilpotent) minimal nonabelian.*

(i) *If G is a nonabelian p-group such that $G' \leq \mathrm{Z}(G)$ has exponent p, then $\Phi(G) \leq \mathrm{Z}(G)$.*

(j) *(Fitting; see [3, corollary 6.5]) Let Q be a normal abelian Sylow q-subgroup of a group G. Then $Q \cap \mathrm{Z}(G)$ is a direct factor of G.*

Let us prove Lemma J(a). Since $\Phi(P) \leq D = \Phi(G) \cap P$ (Lemma J(f)), it suffices to prove the reverse implication. To this end, one may assume that $\Phi(P) = \{1\}$. Let H be a p'-Hall subgroup of G. Then $P = D \times L$, where L

is H-admissible (Maschke). In that case, $G = (HL) \cdot D$, a semidirect product with kernel D, so $D = \{1\}$ since $D \leq \Phi(G)$. Thus, $\Phi(P) = \Phi(G) \cap P$.

Let us prove Lemma J(i). Take $x, y \in G$. Since G is of class 2, we have $1 = [x, y]^p = [x, y^p]$ so that $\mho_1(G) \leq \mathrm{Z}(G)$, and we obtain $\Phi(G) = G'\mho_1(G) \leq \mathrm{Z}(G)$.

THEOREM 1. *The following conditions for a group G are equivalent:*

(a) $G' = \Phi(G)'$.
(b) *G is abelian.*

PROOF. Obviously, (b) $\Rightarrow$ (a) so it remains to prove the reverse implication. We have $G' = \Phi(G)' \leq \Phi(G)$ so G is nilpotent (Lemma J(d)); then $G = P_1 \times \cdots \times P_k$, where $P_1, \dots, P_k$ are Sylow subgroups of G. We have

$$\Phi(G) = \Phi(P_1) \times \cdots \times \Phi(P_k), \tag{1}$$

$$G' = P_1' \times \cdots \times P_k', \tag{2}$$

$$\Phi(G)' = \Phi(P_1)' \times \cdots \times \Phi(P_k)'. \tag{3}$$

Equality (1) follows from Lemma J(a) and (2), (3) are known consequences of (1). Since $\Phi(G)' = G'$, it follows from (2) and (3) that $\Phi(P_i)' = P_i'$ so P_i satisfies the hypothesis for $i = 1, \dots, k$. To complete the proof, it suffices to show that P_i is abelian for $i = 1, \dots, k$, so one may assume that $k = 1$, i.e., G is a p-group. Assume that G is nonabelian of the least possible order. We get $\Phi(G)' = G' > \{1\}$ so that $\Phi(G)$ is nonabelian. Let $R < \Phi(G)'$ be G-invariant of index p. Since $R < \Phi(G)' = G' < \Phi(G)$, we get

$$\Phi(G/R)' = (\Phi(G)/R)' = \Phi(G)'/R = G'/R = (G/R)',$$

whence the nonabelian group G/R satisfies the hypothesis so we must have $R = \{1\}$, by induction. In that case, $|\Phi(G)'| = |G'| = p$ so $G' \leq \mathrm{Z}(G)$. By Lemma J(i), $\Phi(G) \leq \mathrm{Z}(G)$ so that $\Phi(G)$ is abelian, contrary to what has been said above. □

LEMMA 2. *Let G be a noncyclic p-group and $D = \langle \Phi(H) \mid H \in \Gamma_1 \rangle$. Then $D \leq \Phi(G)$ and $|\Phi(G) : D| \leq p$ with equality only for $p > 2$.*

PROOF. Since all members of the set Γ_1 are G-invariant, we have $D \triangleleft G$ and $D \leq \Phi(G)$ (Lemma J(f)). Since all maximal subgroups of the quotient group G/D are elementary abelian, G/D is either elementary abelian or nonabelian of order p^3 and exponent p (Lemma J(d)). In the first case, $D = \Phi(G)$. In the second case, $p > 2$ and $|\Phi(G) : D| = |\Phi(G/D)| = p$. In both cases $|\Phi(G) : D| \leq p$. □

If $H \in \Gamma_1$, then $\Phi(G) < H$ so $\Phi(G)' \leq H'$. Below we consider the extreme case when $\Phi(G)' = H'$ for all $H \in \Gamma_1$.

THEOREM 3. *The following assertions for a group G are equivalent:*

(a) $\Phi(G)' = H'$ *for all* $H \in \Gamma_1$.
(b) G *is either abelian or minimal nonabelian.*

PROOF. Obviously, (b) $\Rightarrow$ (a). Therefore, it remains to prove the reverse implication. In what follows one may assume that G is nonabelian.

(i) The subgroups $\Phi(G)$, $\Phi(G)' = H'$ are G-invariant for all $H \in \Gamma_1$. If $\Phi(G)$ is abelian, then all maximal subgroups of G are also abelian, by hypothesis, and G is minimal nonabelian so (b) holds. Next we assume that $\Phi(G)' > \{1\}$; then all members of the set Γ_1 are nonabelian. Assume, in addition that G is a p-group. Let $R < \Phi(G)'$ be G-invariant of index p and set $\bar{G} = G/R$; then $\Phi(\bar{G})$ is nonabelian and $|\Phi(\bar{G})'| = p$. Take $H \in \Gamma_1$. We have

$$(4) \qquad \Phi(\bar{G})' = \Phi(G/R)' = (\Phi(G)/R)' = \Phi(G)'/R = H'/R = \bar{H}'$$

is of order p so, by Lemma J(i), $\Phi(\bar{H}) \leq \mathrm{Z}(\bar{H})$ and hence $\Phi(\bar{H}) \leq \mathrm{Z}(\Phi(\bar{G}))$ since $\Phi(\bar{H}) \leq \Phi(\bar{G}) \leq \bar{H}$. Setting $\bar{D} = \langle \Phi(\bar{H}) \mid H \in \Gamma_1 \rangle$, we conclude that $\bar{D} \leq \mathrm{Z}(\Phi(\bar{G}))$ so $\Phi(\bar{G})$ is abelian since $|\Phi(\bar{G}) : \bar{D}| \leq p$, by Lemma 2, contrary to what has been said already. Thus, if G is a p-group, then (a) $\Rightarrow$ (b).

In what follows we assume that G is not a prime-power group. Write $T = \Phi(G)'$; then $T = H'$ for all $H \in \Gamma_1$, by hypothesis. Therefore, all maximal subgroups of G/T are abelian so G/T is either abelian or minimal nonabelian. As in (i), we may assume that $\Phi(G)$ is nonabelian.

(ii) Suppose that G is nilpotent; then $G = P_1 \times \cdots \times P_k$, where $P_i \in \mathrm{Syl}_{p_i}(G)$, $i = 1, \ldots, k$, $k > 1$. It follows from (2) and (3) which are true for any nilpotent group that $\Phi(P_i)' = H_i'$ for all maximal subgroups H_i of P_i so P_i is either abelian or minimal nonabelian, by (i). Then $\Phi(P_i)$ is abelian for all i so that $\Phi(G)$ is abelian, in view of (1), contrary to the assumption. Thus, the theorem is true provided G is nilpotent. Next we assume that G is not nilpotent. In that case, by Lemma J(d), G/T is nonnilpotent.

(iii) It remains to consider the case where G/T is minimal nonabelian and nonnilpotent. It follows from the structure of G/T (Lemma J(c)) that $\Phi(G/T) = \Phi(G)/T$ is cyclic. Remembering that $\Phi(G)$ is nilpotent and $T = \Phi(G)'$, we conclude that $\Phi(G)$ is cyclic (Lemma J(e)) so abelian, a final contradiction. ∎

COROLLARY 4. *Suppose that a group* G *is neither abelian nor minimal nonabelian. Then there exists a nonabelian* $H \in \Gamma_1$ *such that* $\Phi(G)' < H'$.

PROOF. We have $\Phi(G)' \leq H'$ for all $H \in \Gamma_1$. Assume that $\Phi(G)' = H'$ for all nonabelian $H \in \Gamma_1$; then $\Phi(G)$ is nonabelian (take a nonabelian $H \in \Gamma_1$) so the set Γ_1 has no abelian members. In that case, $\Phi(G)' = H'$ for *all* $H \in \Gamma_1$, by hypothesis, so that G is minimal nonabelian (Theorem 3), contrary to the hypothesis. ∎

We claim that the following conditions for a group G are equivalent: (a) $H' \leq \Phi(G)$ for all $H \in \Gamma_1$, (b) either G is nilpotent or $G/\Phi(G)$ is nonnilpotent minimal nonabelian, (c) $N_F(F \cap H) \neq F \cap H \neq N_H(H \cap F)$ for all distinct $F, H \in \Gamma_1$. Obviously, (a) $\Leftrightarrow$ (b) and (b) $\Rightarrow$ (c). Next, (b) follows from (c), by Lemma J(h).

Below we use some known results on Frobenius groups (see [6, §10.2]). Recall that G is said to be a *Frobenius group* if there is a non-identity $H < G$ such that $H \cap H^x = \{1\}$ for all $x \in G - H$. In that case, $G = H \cdot N$ is a semidirect product with kernel N, Sylow subgroups of H are either cyclic or generalized quaternion.

LEMMA 5. *The following conditions for a nonnilpotent group G are equivalent:*

(a) *All members of the set Γ_1^n are abelian.*
(b) $G/\mathrm{Z}(G) = (U/\mathrm{Z}(G)) \cdot (Q_1/\mathrm{Z}(G))$ *is a Frobenius group with elementary abelian kernel* $Q_1/\mathrm{Z}(G) = (G/\mathrm{Z}(G))'$ *and a cyclic complement* $U/\mathrm{Z}(G)$, $U \in \Gamma_1$.

PROOF. If U, V are two distinct abelian maximal subgroups of a nonabelian group G, then $U \cap V = \mathrm{Z}(G)$.

Suppose that G satisfies condition (a). Given $H \in \Gamma_1^n$, set $H_G = \bigcap_{x \in G} H^x$; then $H_G = \mathrm{Z}(G)$, by the previous paragraph. Set $\bar{G} = G/\mathrm{Z}(G)$; then $\bar{G} = \bar{H} \cdot \bar{Q}_1$ is a Frobenius group with kernel $\bar{Q}_1$ and complement $\bar{H}$. Since $\bar{H}$ is abelian, it is cyclic. There is in $\bar{Q}_1$ an $\bar{H}$-invariant Sylow subgroup, by Sylow's theorem. Therefore, since $\bar{H}$ is maximal in $\bar{G}$, the subgroup $\bar{Q}_1$ is a p-group; moreover, $\bar{Q}_1$ is a minimal normal subgroup of $\bar{G}$ so it is elementary abelian. All nonnormal maximal subgroups of $\bar{G}$ are conjugate with $\bar{H}$ (Schur-Zassenhaus) so all members of the set Γ_1^n are conjugate in G (indeed, all members of the set Γ_1^n, being abelian, contain $\mathrm{Z}(G)$). Thus, (a) $\Rightarrow$ (b).

Conversely, every group such as in (b), satisfies condition (a). In fact, if $H \in \Gamma_1^n$, then $\mathrm{Z}(G) < H$. By (b), $H/\mathrm{Z}(G)$ is cyclic so H is abelian. □

The proof of Lemma 5 shows that if the set Γ_1^n has at least one abelian member, then the group G has the same structure as in Lemma 5(b).

COROLLARY 6. *A nonnilpotent group G satisfies $\Phi(G)' = H'$ for all $H \in \Gamma_1^n$ if and only if $\bar{G} = G/\Phi(G)'$ is as in Lemma 5(b).*

PROOF. Suppose that $\bar{G} = G/\Phi(G)'$ is as in Lemma 5(b) and $\bar{H}$ is a nonnormal maximal subgroup of $\bar{G}$; then $\bar{H}$ is cyclic, by hypothesis, so $H' \leq \Phi(G)'$. Since $\Phi(G) < H$, we get $\Phi(G)' \leq H'$ so $H' = \Phi(G)'$ and whence G satisfies the hypothesis.

Now suppose that $\Phi(G)' = H'$ for all $H \in \Gamma_1^n$. Then all nonnormal maximal subgroups of $G/\Phi(G)'$ are abelian so it is as in Lemma 5(b). □

Proposition 7. *Suppose that a nonnilpotent group G is not minimal nonabelian and such that $M' = \Phi(G)$ for all nonabelian $M \in \Gamma_1$. Then*

(a) *$G = P \cdot Q$ is a semidirect product with kernel $Q = G' \in \mathrm{Syl}_q(G)$, $P \in \mathrm{Syl}_p(G)$ is of order p, $\Phi(G) = \Phi(Q) > \{1\}$. Set $H = P\Phi(G)$. The set Γ_1 consists of two conjugacy classes of subgroups with representatives H and Q.*
(b) *$P^G = G$, where P^G is the normal closure of P in G.*
(c) *$G/\Phi(Q)$ is minimal nonabelian so that $Q/\Phi(Q)$ is a minimal normal subgroup of $G/\Phi(G)$.*
(d) *$H' < \Phi(Q)$ if and only if G is minimal nonnilpotent.*
(e) *If $\Phi(Q)$ is abelian, then either G is minimal nonnilpotent or a Frobenius group.*

Proof. By hypothesis, all maximal subgroups of $G/\Phi(G)$ are abelian so it is nonnilpotent and minimal nonabelian, by Lemma J(d,c), and (c) is proven. It follows that the set Γ_1 is the union of two conjugacy classes of subgroups. We have $\Phi(G/\Phi(G)) = \{1\}$ so that $|G/\Phi(G)| = pq^b$, where a subgroup of order p is not normal in $G/\Phi(G)$ (Lemma J(b,c)). By hypothesis, the set Γ_1 has a nonabelian member so $\Phi(G) > \{1\}$. By Lemma J(g), $\pi(G) = \pi(G/\Phi(G)) = \{p, q\}$. By Lemma J(f), $Q \in \mathrm{Syl}_q(G)$ is normal in G. We have $G = P \cdot Q$, where $P \in \mathrm{Syl}_p(G)$. By Lemma J(a), $\Phi(Q) = Q \cap \Phi(G)$. By the above, $\Phi(G) = P_1 \times \Phi(Q)$, where P_1 is a maximal subgroup of P. The subgroup $H = P\Phi(Q) \in \Gamma_1$ and all nonnormal maximal subgroups of G are conjugate in G (see the second sentence of this paragraph). Since P_1Q/Q is the unique normal maximal subgroup of G/Q, it follows that $G/Q \cong P$ is cyclic.

As above, $P_1 \leq \Phi(G)$ since $|G/\Phi(G)| = pq^b$ (here P_1 is maximal in cyclic subgroup P). Next, $P_1Q = P_1 \times Q = M \in \Gamma_1$. Assume that $P_1 > \{1\}$ and M is nonabelian. We have $P_1 \not\leq M' = \Phi(G)$, a contradiction. Thus, if $P_1 > \{1\}$, then M is abelian. Similarly, $P_1 \not\leq (P\Phi(G))' = H'$ so that H is also abelian. It follows that G is minimal nonabelian, contrary to the hypothesis. Thus, $P_1 = \{1\}$ so that $|P| = p$ and $Q \in \Gamma_1$. This completes the proof of (a). By Lemma J(a), $\Phi(G) = \Phi(Q)$.

Since Q, the unique normal maximal subgroup of G, has index p in G, it follows that $P^G = G$, and the proof of (b) is complete.

Suppose that $H' < \Phi(G)$. Then H is abelian, by hypothesis. In that case, $\mathrm{C}_G(\Phi(G)) \geq H^G = G$ so $\Phi(G) = \mathrm{Z}(G)$ since $\mathrm{Z}(G/\Phi(G)) = \{1\}$, and we conclude that G is minimal nonnilpotent since $G/\Phi(G)$ is minimal nonabelian. The proof of (d) is complete.

Assume that $\Phi(Q)(= \Phi(G))$ is abelian and G is not minimal nonnilpotent. Then, by (d), $H = P\Phi(Q)$ is nonabelian so $H' = \Phi(Q)$, by hypothesis. In that case, by Lemma J(j), $\mathrm{N}_H(P) = P$ so that H is a Frobenius group since $|P| = p$. Since $G/\Phi(Q)$ is a Frobenius group, it follows that G is also a

Frobenius group. This completes the proof of (e) and thereby the proposition. ∎

Let a nonabelian group G be such that $H' = \Phi(H)$ for all nonabelian $H \leq G$. Then G has no minimal nonnilpotent subgroups, by Lemma J(d). It follows that G is nilpotent. Let $A \leq G$ be minimal nonabelian. Then A is a p-subgroup for some prime p and $A' = \Phi(A)$ has order p and index p^2 in A (Lemma J(b)) so that $|A| = |A'|p^2 = p^3$. Suppose that $P \in \mathrm{Syl}_2(G)$ is nonabelian. Then P is among the 2-groups described in [8] (see also [5, §90]). Now suppose that $G = Q \times H$, where $Q \in \mathrm{Syl}_q(G)$ is nonabelian and $H > \{1\}$. Assume that $Z \leq H$ is cyclic of order p^2. Then the subgroup $K = Q \times Z$ does not satisfy the hypothesis since $\Phi(Z) \not\leq K'$ and $\Phi(Z) \leq \Phi(K)$. Thus, either G is a prime power or $G = Q \times H$, where $Q \in \mathrm{Syl}_q(G)$ is nonabelian and $\exp(H) > 1$ is square free. It follows that if H is also nonabelian, then $\exp(Q) = q$ and we conclude that $q > 2$.

Problems

1. Study the nonabelian p-groups G of exponent $> p$ satisfying $\mho_1(\Phi(H)) = \mho_1(H)$ for all $H \leq G$ (obviously, we must have $p > 2$ since for any 2-group G we have $\Phi(G) = \mho_1(G)$).
2. Classify the p-groups G satisfying $H' = \Phi(G)'$ for all those $H \leq G$ that are neither abelian nor minimal nonabelian.
3. Suppose that G is a p-group with $|G/\Phi(G)| = p^d$, Given $i \leq d$, let Γ_i be the set of all normal subgroups N of G such that G/N is elementary abelian of order p^i. Study the p-groups G such that $H' = \Phi(G)'$ for all $H \in \Gamma_i$ (for $i = 1$, see Theorem 3).

Acknowledgements.
I am indebted to the referee for a number of constructive remarks.

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Received: 8.12.2008.
Revised: 5.5.2009. & 27.6.2009.