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FINITE 2-GROUPS WITH EXACTLY ONE MAXIMAL SUBGROUP WHICH IS NEITHER ABELIAN NOR MINIMAL NONABELIAN

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Abstract. We shall determine the title groups G up to isomorphism. This solves the problem Nr.861 for $p = 2$ stated by Y. Berkovich in [2]. The resulting groups will be presented in terms of generators and relations. We begin with the case $\mathrm{d}(G) = 2$ and then we determine such groups for $\mathrm{d}(G) > 2$. In these theorems we shall also describe all important characteristic subgroups so that it will be clear that groups appearing in distinct theorems are non-isomorphic. Conversely, it is easy to check that all groups given in these theorems possess exactly one maximal subgroup which is neither abelian nor minimal nonabelian.

1. Introduction and some elementary results

We consider here only finite p-groups and our notation is standard. A p-group G is called an A_2-group if all maximal subgroups of G are either abelian or minimal nonabelian and at least one maximal subgroup of G is minimal nonabelian. Such groups are completely determined in [2, §71].

Suppose that G is a p-group all of whose maximal subgroups are metacyclic except one (which is non-metacyclic). If $p > 2$ and $|G| > p^4$, then Y. Berkovich has shown (with a short and elegant proof) that G must be a so called L_3-group, i.e., $\Omega_1(G)$ is of order p^3 and exponent p and $G/\Omega_1(G)$ is cyclic of order $\geq p^2$ (see [3, Proposition A.40.12]). However if $p = 2$, then the problem of determination of such groups is much more difficult and this was done in [2, §87]. All these results will be used very heavily in the present work.

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In this paper we continue with this idea of classifying p-groups all of whose maximal subgroups, but one, have a certain strong property. Here we determine up to isomorphism the 2-groups G all of whose maximal subgroups, but one, are abelian or minimal nonabelian. We begin with the case $\mathrm{d}(G) = 2$ (Theorems 2.1, 2.2 and 2.3). Actually, a detailed investigation of such groups has already begun with Lemma 76.5 in [2]. Then we determine such groups with $\mathrm{d}(G) > 2$ (Theorems 3.1, 3.2 and 3.3). All resulting groups will be presented in terms of generators and relations but we shall also describe all important characteristic subgroups of these groups for two reasons. One reason is that only the knowledge of the subgroup structure of these groups will make our theorems useful for applications. Another reason is that with this knowledge we see that 2-groups appearing in distinct theorems are non-isomorphic.

Conversely, it is easy to check that all groups given in these theorems indeed possess exactly one maximal subgroup which is neither abelian nor minimal nonabelian.

The corresponding problem for $p > 2$ is open but we think that this problem is within the reach of the present methods in finite p-group theory.

We shall list here some elementary results which are used very often in the proof of our theorems. In particular, Propositions 1.2 and 1.4 will be used many times without quoting.

Proposition 1.1 (L. Rédei, see [2, Lemma 65.1 and 65.2]). *A p-group G is minimal nonabelian if and only if* $\mathrm{d}(G) = 2$ *(minimal number of generators of G is 2) and $|G'| = p$. In that case $\Phi(G) = Z(G)$.*

A 2-group G is metacyclic and minimal nonabelian if and only if G is minimal nonabelian and $|\Omega_1(G)| \leq 4$ in which case either $G \cong Q_8$ (a quaternion group of order 8) or $G = \langle a, b \mid a^{2^m} = b^{2^n} = 1,\ a^b = a^{1+2^{m-1}} \rangle$, where $m \geq 2,\ n \geq 1$.

If a 2-group G is non-metacyclic and minimal nonabelian, then $G = \langle a, b \mid a^{2^m} = b^{2^n} = 1,\ [a, b] = c,\ c^2 = [a, c] = [b, c] = 1 \rangle$, where $m \geq n \geq 1$ and $m \geq 2$. In this case $\Omega_1(G) \cong \mathrm{E}_8$ and $G' = \langle c \rangle \cong \mathrm{C}_2$ is a maximal cyclic subgroup in G.

Proposition 1.2. *Let $H = \langle a, b \rangle$ be a two-generator p-group with H' of order p. Then $\Phi(H) = \langle a^p, b^p, [a, b] \rangle$.*

Proof. By Proposition 1.1, H is minimal nonabelian and $\Phi(H) = \mathrm{Z}(H)$. We have $S = \langle a^p, b^p, [a, b] \rangle \leq \Phi(H)$ and H/S is elementary abelian. Hence $S = \Phi(H)$. □

Proposition 1.3 ([1, Lemma 1.1]). *Let G be a nonabelian p-group with an abelian maximal subgroup. Then $|G| = p|G'||\mathrm{Z}(G)|$.*

Proposition 1.4 (Exercise 1.6(a) in [1]). *Let G be a nonabelian p-group. Then the number of abelian maximal subgroups is* $0, 1$ or $p+1$. *If G has more than one abelian maximal subgroup, then $|G'| = p$.*

Proof. Suppose that G possesses two distinct abelian maximal subgroups H and K. Set $D = H \cap K$ so that $D \leq \mathrm{Z}(G)$ and $G/D \cong \mathrm{E}_{p^2}$. Since G is nonabelian, we have $D = \mathrm{Z}(G)$ and then Proposition 1.3 implies $|G'| = p$. There are exactly $p+1$ maximal subgroups of G which contain D and they are all abelian. Suppose that G possesses an abelian maximal subgroup M which does not contain D. Then $G = DM$ is abelian, a contradiction. □

Proposition 1.5 (A. Mann, see Exercise 1.69(a) in [1]). *Let G be a p-group and let $H \neq K$ be two distinct maximal subgroups of G. Then $|G' : (H'K')| \leq p$.*

Proof. We have $H' \trianglelefteq G$, $K' \trianglelefteq G$ and $H'K' \leq H \cap K$. Thus $H/(H'K')$ and $K/(H'K')$ are two distinct abelian maximal subgroups of $G/(H'K')$. By Proposition 1.4, we have either $G' = H'K'$ (and then $G/(H'K')$ is abelian) or $|G' : (H'K')| = p$. □

Proposition 1.6 (O. Taussky, see [1, Corollary 36.7]). *Let G be a nonabelian 2-group. If $|G : G'| = 4$, then G is of maximal class and so G possesses a cyclic maximal subgroup.*

Proposition 1.7 ([1, Proposition 10.17]). *Let G be a p-group with a nonabelian subgroup B of order p^3 such that $\mathrm{C}_G(B) \leq B$. Then G is of maximal class.*

2. The title groups with $\mathrm{d}(G) = 2$

Theorem 2.1. *Let G be a two-generator 2-group with exactly one maximal subgroup H which is neither abelian nor minimal nonabelian. If G has an abelian maximal subgroup A, then $\Gamma_1 = \{A, M, H\}$ is the set of maximal subgroups of G, where M is minimal nonabelian, A and M are both metacyclic, $\mathrm{d}(H) = 3$ and we have more precisely:*

$$G = \langle a, x \mid [a, x] = v,\ v^4 = 1,\ v^2 = z,\ v^x = v^{-1},\ v^a = v^{-1},$$
$$x^2 \in \langle z \rangle,\ a^{2^m} \in \langle z \rangle,\ m \geq 2 \rangle,$$

where $G' = \langle v \rangle \cong \mathrm{C}_4$, $\mathrm{K}_3(G) = [G, G'] = \langle z \rangle \cong \mathrm{C}_2$, $E = \langle v, x \rangle \cong \mathrm{D}_8$ or Q_8, $E \trianglelefteq G$, $G = E\langle a \rangle$, $G/E \cong \mathrm{C}_{2^m}$, $\Phi(G) = G'\langle a^2 \rangle$ is abelian, $H = E\langle a^2 \rangle$, $Z(G) = \langle a^2, z \rangle$, $A = \langle ax, v \rangle$ is an abelian maximal subgroup of G, $M = \langle v, a \rangle$ is metacyclic minimal nonabelian of order 2^{m+2} and $|G| = 2^{m+3}$, $m \geq 2$.

Proof. If G has more than one abelian maximal subgroup, then all three maximal subgroups of G are abelian, a contradiction. Hence A is a unique abelian maximal subgroup of G. It follows that $\Gamma_1 = \{A, M, H\}$, where M is minimal nonabelian. The subgroup $A \cap M = \Phi(G)$ is abelian, $\Phi(M) < \Phi(G)$ and $|\Phi(G) : \Phi(M)| = 2$. Also, $\Phi(M) = Z(M) \le Z(G)$ so that for an element $m \in M - A$, $C_{\Phi(G)}(m) = Z(M)$. If $C_A(m) > Z(M)$, then $G = M * C$ with $C = C_G(M)$ and $M \cap C = Z(M)$, contrary to $d(G) = 2$. It follows that $C_A(m) = Z(M) = Z(G)$ and so $|G : Z(G)| = 8$. From $|G| = 2|Z(G)||G'|$ (Proposition 1.3) follows that $|G'| = 4$. For each $x \in G - A$, $C_A(x) = Z(M)$ and so $x^2 \in Z(M) = Z(G)$ which implies that x inverts $A/Z(M)$. If $A/Z(M) \cong E_4$, then $\Phi(G) = \mho_1(G) \le Z(M)$, a contradiction. Hence $A/Z(M) \cong C_4$ and since $m \in M - A$ inverts $A/Z(M)$, we have $G/Z(M) \cong D_8$. Taking $a \in A - \Phi(G)$, then $\langle a \rangle$ covers $A/Z(M)$ and so $v = [a, m] \in \Phi(G) - Z(M)$. We get

$$1 = [a, m^2] = [a, m][a, m]^m = vv^m \text{ and so } v^m = v^{-1}$$

which implies that $o(v) = 4$. Indeed, if $o(v) = 2$, then $[v, m] = 1$ which contradicts the fact that $C_{\Phi(G)}(m) = Z(M)$. We get $G' = \langle v \rangle \cong C_4$ and $[v, m] = v^2$ implies that $M' = \langle v^2 \rangle$. Since v^2 is a square in M, it follows that M is metacyclic (Proposition 1.1). In particular, $|\Omega_1(\Phi(G))| \le 4$ which together with $A/Z(M) \cong C_4$ gives $\Omega_1(A) \le \Phi(G)$ and so A is also metacyclic. Here $H = \Phi(G)\langle am \rangle$ is our third maximal subgroup of G. From $C_{\Phi(G)}(am) = Z(M)$ follows that H is nonabelian with $Z(H) = Z(M)$ and so Proposition 1.3 gives $|H'|$=2. If $d(H) = 2$, then Proposition 1.1 gives that H would be minimal nonabelian, a contradiction. Hence $d(H) \ge 3$ and so H is the only maximal subgroup of G which is nonmetacyclic. In fact $d(H) = 3$ since $\Phi(G)$ is metacyclic. We are in a position to use Theorem 87.12 in [2] for $n = 2$ since $G' \cong C_4$. This gives the generators and relations described in our theorem, where we have used the notation from Theorem 87.12 in [2]. □

THEOREM 2.2. *Let G be a 2-group with $d(G) = 2$ which has exactly one maximal subgroup H which is neither abelian nor minimal nonabelian. If the other two maximal subgroups H_1 and H_2 are minimal nonabelian with $H_1' = H_2'$, then one of the following holds:*

(a) *G is one of the groups given in Theorem 87.10 in [2].*
(b) *G is the group of order 2^5 given in Theorem 87.14 in [2].*
(c)

$$G = \langle h, x \mid h^{2^n} = 1,\, n \ge 2,\, [h, x] = s,\, s^2 = 1,\, [s, h] = z,$$
$$z^2 = [z, h] = [z, x] = [x, s] = 1,\, x^2 \in \langle z \rangle\rangle,$$

where $|G| = 2^{n+3}$, $G' = \langle z, s \rangle \cong E_4$, $K_3(G) = [G, G'] = \langle z \rangle \cong C_2$, $\Phi(G) = G'\langle h^2 \rangle$ is abelian and maximal subgroups of G are $H_1 = \langle G', h \rangle$, $H_2 = \langle G', xh \rangle$ (both are nonmetacyclic minimal nonabelian) and $H = \langle x, s, h^2 \rangle$ with $d(H) = 3$ and $H_1' = H_2' = H' = \langle z \rangle$.

PROOF. Here $A = H_1 \cap H_2 = \Phi(G)$ is a maximal normal abelian subgroup of G. Set $H_1' = H_2' = \langle z \rangle \le A$ so that $H_1/\langle z \rangle$ and $H_2/\langle z \rangle$ are two distinct abelian maximal subgroups in $G/\langle z\rangle$. It follows that $H/\langle z\rangle$ is also abelian and so $H' = \langle z\rangle$ since H is nonabelian. If $\mathrm{d}(H) = 2$, then (by Proposition 1.1) H would be minimal nonabelian, a contradiction. Thus $\mathrm{d}(H) \geq 3$.

If $G/\langle z\rangle$ is abelian, then $G' = \langle z\rangle$ and so $G = H_1\mathrm{C}_G(H_1)$ which gives $\mathrm{d}(G) = 3$, a contradiction. Hence $G/\langle z\rangle$ is nonabelian and so $(G/\langle z\rangle)' \cong \mathrm{C}_2$ since $G/\langle z\rangle$ has three distinct abelian maximal subgroups. Thus $|G'| = 4$ and $G' \leq A = \Phi(G)$. Taking $h_1 \in H_1 - A$ and $h_2 \in H_2 - A$, we have $\langle h_1, h_2\rangle = G$ and so $s = [h_1, h_2] \in G' - \langle z\rangle$. If $G' \cong C_4$, then z is a square in H_1 and H_2 and so both H_1 and H_2 are metacyclic (Proposition 1.1). Since $\mathrm{d}(H) \geq 3$, G has exactly one nonmetacyclic maximal subgroup. But then Theorem 87.12 in [2] for $n = 2$ implies that G has an abelian maximal subgroup, a contradiction. Thus $G' = \langle s, z\rangle \cong \mathrm{E}_4$, where s is an involution. If $s \in \mathrm{Z}(G)$, then $G/\langle s\rangle$ would be abelian (because $\langle h_1, h_2\rangle = G$ and $s = [h_1, h_2]$), a contradiction. Hence $s \notin \mathrm{Z}(G)$ and so $s \notin \mathrm{Z}(H_1)$ or $s \notin \mathrm{Z}(H_2)$. Without loss of generality we may assume that $s \notin \mathrm{Z}(H_1)$.

Suppose that z is a square in H_1, i.e., there is $v \in H_1$ such that $v^2 = z$. Suppose at the moment that $v \in H_1 - A$ in which case $\langle v, G'\rangle = \langle v, s\rangle \cong \mathrm{D}_8$ since $s^v = sz$. It follows that $\langle v, G'\rangle = H_1$. Since $\mathrm{C}_G(H_1) \leq H_1$ (otherwise, $\mathrm{d}(G) = 3$), G would be of maximal class (Proposition 1.7), a contradiction (noting that 2-groups of maximal class have a cyclic maximal subgroup). Thus $v \in A = \Phi(G)$. In that case both H_1 and H_2 are metacyclic (Proposition 1.1) which together with $\mathrm{d}(H) \geq 3$ allows us to use §87, part 2^0 in [2]. If G has a normal elementary abelian subgroup of order 8, then we get groups in part (a) of our theorem. If G has no normal elementary abelian subgroup of order 8, then we get the group of order 2^5 given in part (b) of our theorem.

Now we assume that z is not a square in H_1 which implies that H_1 is nonmetacyclic (Proposition 1.1). If h_1 is an involution, then $s^{h_1} = sz$ shows that $\langle h_1, s\rangle \cong \mathrm{D}_8$ and so $\langle h_1, s\rangle = H_1$ is metacyclic, a contradiction. Hence $\mathrm{o}(h_1) = 2^n$, $n \geq 2$. Set $u = h_1^{2^{n-1}}$ so that $u \in \mathrm{Z}(H_1)$ and $u \notin G'$ since z is not a square in H_1 and $s^{h_1} = sz$. We have $E = \Omega_1(H_1) = \langle z, s, u\rangle \cong \mathrm{E}_8$, $E \trianglelefteq G$ and $E \leq A$ which implies that H_2 is nonmetacyclic and therefore z is also not a square in H_2. Since $H_1 = E\langle h_1\rangle = \langle h_1, s\rangle$, we have $|H_1| = 2^{n+2}$, $|G| = 2^{n+3}$ and $A = \langle h_1^2, s, z\rangle$ is abelian of order 2^{n+1} and type $(2^{n-1}, 2, 2)$. Also, H_1 is a splitting extension of G' by $\langle h_1\rangle \cong \mathrm{C}_{2^n}$, $n \geq 2$. Since $\mathrm{d}(G/G') = 2$, we get that G/G' is abelian of type $(2^n, 2)$. We get $G = FH_1$, where $F \cap H_1 = G'$ and $|F : G'| = 2$ so that $F = \langle G', x\rangle$ with $\mathrm{o}(x) \leq 4$. In fact, $x^2 \in \langle z\rangle$. Indeed, if F is not elementary abelian, then $\mho_1(F) \cong \mathrm{C}_2$ and $\mho_1(F) \leq G'$. But $F \trianglelefteq G$ and so $\mho_1(F) \leq \mathrm{Z}(G)$ which implies that $\mho_1(F) = \langle z\rangle$ since $G' \not\leq \mathrm{Z}(G)$. Since $\langle xh_1\rangle G'/G'$ is another cyclic subgroup of index 2 in G/G' (distinct from H_1/G'), $M = \langle G', xh_1\rangle$ is a maximal subgroup of G distinct from H_1 and $M' = \langle z\rangle$ (since $H_2' = H' = \langle z\rangle$). If $G' \leq \mathrm{Z}(M)$, then M would be abelian, a contradiction. We get $s^{xh_1} = sz$ and so $M = \langle xh_1, s\rangle$ is minimal nonabelian which gives $M = H_2$. We may set $h_2 = xh_1$, where $[h_1, h_2] = s$

and $G' = \langle s, z\rangle$. From $s^{xh_1} = sz$ follows

$$s^x = (s^{xh_1})^{h_1^{-1}} = (sz)^{h_1^{-1}} = s^{h_1^{-1}} z = (sz)z = s$$

and so F is abelian. From $s = [h_1, h_2]$ follows

$$s = [h_1, h_2] = [h_1, xh_1] = [h_1, h_1][h_1, x]^{h_1} = [h_1, x]^{h_1}$$

and so $[h_1, x] = sz$. We have $\Phi(G) = G'\langle h_1^2\rangle$ is abelian and $H = F\langle h_1^2\rangle$, where

$$[h_1^2, x] = [h_1, x]^{h_1}[h_1, x] = (sz)^{h_1}(sz) = (szz)(sz) = z.$$

Since $H/\langle z\rangle = H/H'$ is abelian of type $(2, 2, 2^{n-1})$, we have $\mathrm{d}(H) = 3$. Replacing s with $s' = sz$, we get $[h_1, x] = s'$ and writing again s instead of s', we may write $[h_1, x] = s \in G' - \langle z\rangle$. Also, writing h instead of h_1 we obtain the relations of part (c) of our theorem. ◻

THEOREM 2.3. *Let G be a 2-group with $\mathrm{d}(G) = 2$ which has exactly one maximal subgroup H which is neither abelian nor minimal nonabelian. If the other two maximal subgroups H_1 and H_2 are minimal nonabelian with $H_1' \neq H_2'$, then one of the following holds:*

(a) *G is a group of order 2^6 with $n = 2$ given in Theorem 87.15(a) in [2].*

(b) *G is a group of order 2^{m+4} $(m \geq 2)$ with $n = 2$ given in Theorem 87.16 in [2].*

(c)

$$G = \langle h, x \mid [h, x] = v,\ v^4 = 1,\ v^h = vz_1,\ v^x = v^{-1},\ v^2 = z_1z_2,\ z_1^2 = z_2^2 = 1,$$
$$[z_1, h] = [z_1, x] = [z_2, h] = [z_2, x] = 1,\ x^2 \in \langle z_1, z_2\rangle,\ h^{2^m} = (z_1z_2)^{\epsilon}\rangle,$$

where $\epsilon = 0, 1$, $m \geq 2$, $|G| = 2^{m+4}$, $G' = \langle v, z_1\rangle \cong \mathrm{C}_4 \times \mathrm{C}_2$, $\mathrm{K}_3(G) = [G, G'] = \langle z_1, z_2\rangle \cong \mathrm{E}_4$ and $\langle z_1, z_2\rangle \leq \mathrm{Z}(G)$. Moreover, maximal subgroups of G are $H_1 = G'\langle h\rangle$, $H_2 = G'\langle hx\rangle$ and $H = G'\langle x, h^2\rangle$, where H_1 and H_2 are both nonmetacyclic minimal nonabelian with $H_1' = \langle z_1\rangle$, $H_2' = \langle z_2\rangle$ and $H' = \langle z_1, z_2\rangle \cong \mathrm{E}_4$ $(\mathrm{d}(H) = 3)$.

PROOF. Set $\langle z_1\rangle = H_1'$ and $\langle z_2\rangle = H_2'$ so that $W = \langle z_1, z_2\rangle \cong \mathrm{E}_4$, $W \leq \mathrm{Z}(G)$ and $W \leq A = H_1 \cap H_2 = \Phi(G)$, where A is abelian. Here $\{H_1/\langle z_1\rangle, H_2/\langle z_1\rangle, H/\langle z_1\rangle\}$ is the set of maximal subgroups of $G/\langle z_1\rangle$ and $H_1/\langle z_1\rangle$ is abelian and two-generated, $H_2/\langle z_1\rangle$ is minimal nonabelian and so $H/\langle z_1\rangle$ must be nonabelian. If $H/\langle z_1\rangle$ is minimal nonabelian, then by a result of N. Blackburn (Theorem 44.5 in [1]), $G/\langle z_1\rangle$ would be metacyclic. But then, by another result of N. Blackburn (Lemma 44.1 and Corollary 44.6 in [1]), G is also metacyclic, contrary to $W \leq G'$ and $W \cong \mathrm{E}_4$. Hence $H/\langle z_1\rangle$ is neither abelian nor minimal nonabelian. By Theorem 2.1, $\mathrm{d}(H/\langle z_1\rangle) = 3$, $(H/\langle z_1\rangle)' \cong \mathrm{C}_2$ and $G'/\langle z_1\rangle \cong \mathrm{C}_4$. Similarly, considering $G/\langle z_2\rangle$, we get $(H/\langle z_2\rangle)' \cong \mathrm{C}_2$ and $G'/\langle z_2\rangle \cong \mathrm{C}_4$. It follows that G' is abelian of type $(4, 2)$ with $\mho(G') = \langle z_1z_2\rangle$. On the other hand, $\{H_1/W, H_2/W, H/W\}$ is the set of maximal subgroups of the nonabelian group G/W, where both H_1/W and

H_2/W are abelian. Hence H/W is also abelian and so $H' \le W$. By the above, H' is distinct from $\langle z_1\rangle$ and $\langle z_2\rangle$ and so either $H' = W$ or $H' = \langle z_1z_2\rangle$.

Suppose that $H' = \langle z_1z_2\rangle$. Then $\{H_1/\langle z_1z_2\rangle, H_2/\langle z_1z_2\rangle, H/\langle z_1z_2\rangle\}$ is the set of maximal subgroups of $G/\langle z_1z_2\rangle$, where $H_1/\langle z_1z_2\rangle$ and $H_2/\langle z_1z_2\rangle$ are minimal nonabelian and $H/\langle z_1z_2\rangle$ is abelian. By results of §71 in [2], $G/\langle z_1z_2\rangle$ is metacyclic, contrary to $G'/\langle z_1z_2\rangle \cong \mathrm{E}_4$. We have proved that $H' = W = \langle z_1, z_2\rangle$. Take $h_1 \in H_1 - A$, $h_2 \in H_2 - A$ so that we have $\langle h_1, h_2\rangle = G$. If $v = [h_1, h_2] \in W$, then v is an involution in $\mathrm{Z}(G)$ and $G/\langle v\rangle$ is abelian, $G' \le \langle v\rangle$, a contradiction. Hence $v \notin W$ and so $v \in G' - W$ is of order 4 with $v^2 = z_1z_2$ and $\langle v\rangle$ is not normal in G (and so also $\langle vz_1\rangle$ is not normal in G). Indeed, if $\langle v\rangle$ is normal in G, then $G/\langle v\rangle$ would be abelian since $[h_1, h_2] = v$ and $\langle h_1, h_2\rangle = G$. In particular, $\langle v\rangle$ cannot be normal in both H_1 and H_2 and so we may assume without loss of generality that $\langle v\rangle$ is not normal in H_1. Hence $[h_1, v] = z_1$ which gives $v^{h_1} = vz_1$ and $H_1 = \langle h_1, v\rangle$.

Suppose that H_1 is metacyclic. Then there is $h_1' \in H_1$ such that $(h_1')^2 = z_1$. If $h_1' \in H_1 - A$, then $v^{h_1'} = vz_1$ and so $H_1 = \langle h_1', v\rangle$ is of order 2^4. But then $|G| = 2^5$ and $|G'| = 2^3$ imply (using Proposition 1.6) that G is of maximal class, a contradiction. It follows that $h_1' \in A$ and then $(h_1'v)^2 = (h_1')^2v^2 = z_1(z_1z_2) = z_2$ which implies that H_2 is also metacyclic. We are in a position to use §87, 2^0 in [2]. By Theorems 87.9 and 87.10 in [2], G has no normal elementary abelian subgroup of order 8 (since $|G'| = 8$). We have $\Phi(G') \ne \{1\}$ and $\mathrm{Z}(G) \ge W$ is noncyclic. If $G/\Phi(G')$ has no normal elementary abelian subgroup of order 8, then G is isomorphic to a group of order 2^6 given in Theorem 87.15(a) in [2] for $n = 2$. If $G/\Phi(G')$ has a normal elementary abelian subgroup of order 8, then G is a group of order 2^{m+4}, $m \ge 2$, given in Theorem 87.16 in [2] for $n = 2$.

Suppose that H_1 is nonmetacyclic. If there is an element $l \in H_1 - A$ such that $l^2 \in G'$, then $v^l = vz_1$ gives that $H_1 = \langle l, v\rangle = G'\langle l\rangle$ is nonmetacyclic minimal nonabelian of order 2^4. But in that case $|G| = 2^5$ and $|G'| = 2^3$ imply (using Proposition 1.6) that G is of maximal class, a contradiction. It follows that $\Omega_1(H_1) = \Omega_1(A) \cong \mathrm{E}_8$ and so H_2 is also nonmetacyclic minimal nonabelian. Since $h_2^2 \in H_1$, we have $[h_1, h_2^2] = z_1^\eta$, $\eta = 0, 1$. We compute

$$z_1^\eta = [h_1, h_2^2] = [h_1, h_2][h_1, h_2]^{h_2} = vv^{h_2},$$

and so $v^{h_2} = v^{-1}z_1^\eta = v(z_1z_2)z_1^\eta = vz_1^{\eta+1}z_2$. If $\eta = 0$, then $v^{h_2} = v(z_1z_2)$, contrary to $H_2' = \langle z_2\rangle$. Thus $\eta = 1$ and so $v^{h_2} = vz_2$ which implies that $H_2 = \langle h_2, v\rangle = G'\langle h_2\rangle$. Also, $v^{h_1} = vz_1$ implies that $H_1 = \langle h_1, v\rangle = G'\langle h_1\rangle$ and since $h_1^2 \notin G'$, we have $H_1/G' \cong \mathrm{C}_{2^m}$, $m \ge 2$, and then also $H_2/G' \cong \mathrm{C}_{2^m}$.

Since $\mathrm{d}(G/G') = 2$, we see that G/G' is abelian of type $(2^m, 2)$, $m \ge 2$. We may set $G = FH_1$ with $F \cap H_1 = G'$ and $|F : G'| = 2$. Since $\langle h_1\rangle$ covers $H_1/G' \cong \mathrm{C}_{2^m}$, $v^{h_1} = vz_1$ and neither z_1 nor z_2 are squares of any element in $A = G'\langle h_1^2\rangle = \Phi(G)$, we get $h_1^{2^m} = (z_1z_2)^\epsilon$, $\epsilon = 0, 1$. We may set $h_2 = h_1x$

with $x \in F - G'$ so that from $v^{h_2} = vz_2$ follows

$$vz_2 = v^{h_2} = (v^{h_1})^x = (vz_1)^x = v^x z_1$$

and so $v^x = v(z_1z_2) = v^{-1}$ which gives $x^2 \in \langle z_1, z_2\rangle \leq \mathrm{Z}(G)$. From $v = [h_1, h_2]$ follows $v = [h_1, h_1x] = [h_1, x][h_1, h_1]^x = [h_1, x]$. Finally, we have $H_2 = G'\langle h_1x\rangle$ and $H = F\langle h_1^2\rangle$, where $F' = \langle [v, x]\rangle = \langle z_1z_2\rangle$ and

$$[h_1^2, x] = [h_1, x]^{h_1}[h_1, x] = v^{h_1}v = (vz_1)v = z_1v^2 = z_1(z_1z_2) = z_2$$

and so indeed $H' = \langle z_1, z_2\rangle \cong \mathrm{E}_4$ which shows that H is neither abelian nor minimal nonabelian. Writing h instead of h_1, we have obtained the relations given in part (c) of our theorem. □

3. The title groups with $\mathrm{d}(G) > 2$

We turn now to the case $\mathrm{d}(G) \geq 3$. Since G possesses at least one minimal nonabelian maximal subgroup, it follows that in this case $\mathrm{d}(G) = 3$. It is well known that the number of abelian maximal subgroups in a nonabelian 2-group G is $0, 1$ or 3 (Proposition 1.4). According to this fact we shall subdivide our study of the title groups with $\mathrm{d}(G) = 3$.

Theorem 3.1. *Let G be a 2-group with $\mathrm{d}(G) = 3$ which has exactly one maximal subgroup which is neither abelian nor minimal nonabelian. If G possesses more than one abelian maximal subgroup, then one of the following holds:*

(a) *$G = Q * Z$, where $Q \cong \mathrm{Q}_8$, $Z \cong \mathrm{C}_{2^n}$, $n \geq 3$ and $Q \cap Z = \mathrm{Z}(Q)$.*
(b) *$G = Q \times Z$, where $Q \cong \mathrm{Q}_8$ and $Z \cong \mathrm{C}_{2^n}$, $n \geq 2$.*
(c) *$G = D \times Z$, where $D \cong \mathrm{D}_8$ and $Z \cong \mathrm{C}_{2^n}$, $n \geq 2$.*

Proof. By our assumption, G has exactly three abelian maximal subgroups. This implies $|G'| = 2$ and G possesses exactly three maximal subgroups which are minimal nonabelian. Let H be a minimal nonabelian maximal subgroup of G. Since $H' = G' \cong \mathrm{C}_2$, we get $G = H\mathrm{Z}(G)$, where $\mathrm{Z}(G) \cap H = \mathrm{Z}(H) = \Phi(H) = \Phi(G)$. All three maximal subgroups of G containing $\mathrm{Z}(G)$ are abelian.

If G is a title group with $|G'| = 2$, then the similar arguments (as above) imply that G possesses more than one abelian maximal subgroup.

In what follows H will denote a fixed maximal subgroup of G which is minimal nonabelian. Suppose that there is an involution $c \in \mathrm{Z}(G) - \mathrm{Z}(H)$. Then $G = H \times \langle c\rangle$ and so each maximal subgroup of G which does not contain $\langle c\rangle$ is isomorphic to $G/\langle c\rangle \cong H$ and so is minimal nonabelian, a contradiction. Hence there are no involutions in $\mathrm{Z}(G) - H$ which implies that $\Omega_1(\mathrm{Z}(H)) = \Omega_1(\mathrm{Z}(G))$ so that $\mathrm{d}(\mathrm{Z}(H)) = \mathrm{d}(\mathrm{Z}(G))$. It follows that for each $x \in \mathrm{Z}(G) - H$, $x^2 \in \mathrm{Z}(H) - \Phi(\mathrm{Z}(H))$. Suppose that $|G| = 2^4$. Then each nonabelian maximal subgroup of G is isomorphic to D_8 or Q_8 and so is minimal nonabelian, a contradiction. Hence $|G| \geq 2^5$ and in particular H is not isomorphic to Q_8 or D_8 .

(i) First assume that H is metacyclic. Since H is not isomorphic to Q_8, it follows that H is a "splitting" metacyclic group and so we may set:

$$H = \langle a, b \mid a^{2^m} = b^{2^n} = 1,\ a^b = az,\ z = a^{2^{m-1}} \rangle,$$

where $m \geq 2$, $n \geq 1$, $m + n \geq 4$, $H' = \langle z \rangle$, $|H| = 2^{m+n}$ and $|G| = 2^{m+n+1}$. We have $Z(H) = \langle a^2 \rangle \times \langle b^2 \rangle = \Phi(H) = \Phi(G)$ and for each $x \in Z(G) - H$, $x^2 \in \langle a^2, b^2 \rangle - \langle a^4, b^4 \rangle$ since $\langle a^4, b^4 \rangle = \Phi(Z(H))$.

Suppose that $n = 1$ so that b is an involution, $m \geq 3$, $H \cong M_{2^{m+1}}$ and $Z(H) = \langle a^2 \rangle$. Hence $Z(G) \cong C_{2^m}$ is cyclic and therefore we may choose $c \in Z(G) - H$ such that $c^2 = a^{-2}$ which gives $(ca)^2 = c^2a^2 = 1$. Since $[ca, b] = z$, we get $D = \langle ca, b \rangle \cong D_8$, $\langle ca, b \rangle \cap Z(G) = \langle z \rangle$ which together with $|G : Z(G)| = 4$ gives $G = DZ(G)$. But $D * \Omega_2(Z(G))$ contains a subgroup $Q \cong Q_8$ and so $G = Q*Z(G)$ with $Z(G) \cong C_{2^m}$, $m \geq 3$, $Q \cap Z(G) = Z(Q) = \langle z \rangle$ and we have obtained the groups stated in part (a) of our theorem.

It remains to treat the case $n \geq 2$. Suppose that there is an involution $x \in G - H$. We know that $x \notin Z(G)$ and so $[a, x] \neq 1$ or $[b, x] \neq 1$. Obviously, $\langle a, b, x \rangle = G$.

Suppose that $[a, x] \neq 1$. Since $\langle a \rangle \trianglelefteq G$, $\langle a, x \rangle$ is minimal nonabelian of order 2^{m+1}. Because $|G| = 2^{m+n+1}$ and $n \geq 2$, $\Phi(G)\langle a, x \rangle = \langle a, x, b^2 \rangle$ is a maximal subgroup of G which is neither abelian nor minimal nonabelian. Assume at the moment that $[b, x] \neq 1$. In that case $\langle b \rangle \times \langle z \rangle$ is normal in G and so $\langle b, x \rangle$ is a nonmetacyclic minimal nonabelian subgroup of G of order 2^{n+2}. It follows that $\langle b, x \rangle$ must be maximal in G with $|G| = 2^{n+3}$ (and so $m = 2$). But the case of a nonmetacyclic minimal nonabelian maximal subgroup in G will be studied in part (ii) of this proof. Hence we may assume $[x, b] = 1$ so that $[x, ab] = [x, a][x, b] = z$ which implies that $\langle x, ab \rangle$ is minimal nonabelian and so $\langle x, ab \rangle$ must be maximal in G. Now, $\langle ab \rangle$ covers $H/\langle a \rangle \cong C_{2^n}$, $n \geq 2$, and so $o(ab) \geq 2^n$. We get

$$(ab)^{2^n} = a^{2^n} b^{2^n} [b, a]^{2^{n-1}(2^n-1)} = a^{2^n}.$$

If $n \geq m$, then $(ab)^{2^n} = 1$ and so $o(ab) = 2^n$ and $\langle ab \rangle \cap \langle a \rangle = \{1\}$. Since $\langle ab, z \rangle \trianglelefteq G$, we see that $\langle ab, x \rangle$ is a nonmetacyclic minimal nonabelian subgroup of order 2^{n+2}. In that case $\langle ab, x \rangle$ must be maximal in G (with $m = 2$) and again this will be studied in part (ii) of this proof. It follows that we may assume $n < m$ and we set in that case $s = m - n \geq 1$. From $(ab)^{2^n} = a^{2^n}$ and $o(a^{2^n}) = 2^s$ follows that $o(ab) = 2^{n+s} = 2^m$ and $\langle ab \rangle \geq \langle z \rangle$ so that $\langle ab \rangle \trianglelefteq G$. Hence $\langle ab, x \rangle$ is metacyclic minimal nonabelian of order 2^{m+1} and so $\langle ab, x \rangle$ must be maximal in G. From $|G| = 2^{m+n+1}$ follows $n = 1$, contrary to our assumption.

We may assume $[a, x] = 1$ and so we must have $[b, x] \neq 1$. Since $\langle b \rangle \times \langle z \rangle \trianglelefteq G$, $\langle b, x \rangle$ is nonmetacyclic minimal nonabelian of order 2^{n+2}. If $\langle b, x \rangle$ is maximal in G, then this case will be treated in part (ii) of this proof. Thus we may assume that $\langle b, x\rangle$ is not maximal in G and so $M = \Phi(G)\langle b, x\rangle$ is maximal in G and M is neither abelian nor minimal nonabelian. It follows that the subgroup $\langle ab, x\rangle$ (with $[ab, x] = z$) being minimal nonabelian must be also a maximal subgroup in G. Since $\langle ab\rangle$ covers $H/\langle a\rangle$, we have $o(ab) \geq 2^n$, $n \geq 2$ and $(ab)^{2^n} = a^{2^n}$. If $n \geq m$, then $o(ab) = 2^n$ and $\langle ab\rangle \cap \langle a\rangle = \{1\}$ and so $\langle ab, x\rangle$ is nonmetacyclic minimal nonabelian of order 2^{n+2}. In that case $\langle ab, x\rangle$ is maximal in G (with $m = 2$) and again this will be treated in part (ii) of this proof. We may assume that $n < m$ and then $o(ab) = 2^m$, $\langle ab\rangle \geq \langle z\rangle$ and so $\langle ab, x\rangle$ is metacyclic minimal nonabelian of order 2^{m+1}. But then $|G| = 2^{m+n+1}$ implies $n = 1$, contrary to our assumption.

We have proved that we may assume that there are no involutions in $G - H$. If there is $c \in Z(G) - H$ such that $c^2 = h^2$ for some $h \in H$, then the abelian subgroup $\langle h, c\rangle$ is noncyclic since $\langle h\rangle$ and $\langle c\rangle$ are two distinct cyclic subgroups of $\langle h, c\rangle$ of the same order. But $\langle h, c\rangle \cap H = \langle h\rangle$ and so there is an involution in $\langle h, c\rangle - H$, a contradiction. It follows that not every element in $\mho_1(H) = Z(H)$ is a square of an element in H. By Proposition 26.23 in [1], H is not a powerful 2-group which implies that $H' = \langle z\rangle \not\leq \mho_2(H)$. This forces $m = 2$ and c^2 is not a square in H for any $c \in Z(G) - H$. We compute for any integers i, j:

$$(a^i b^j)^2 = a^{2i} b^{2j} [b^j, a^i] = a^{2i} b^{2j} z^{ij}.$$

We get that $a^{2i} b^{2j} \in \mho_1(H) = Z(H)$ is a square in H if and only if i or j is even. Therefore, for any $c \in Z(G) - H$, $c^2 = a^{2i} b^{2j}$, where both i and j are odd and then (since $m = 2$ and so $a^2 = z$) $c^2 = z b^{2j}$, where j is odd. Consider the nonabelian subgroup $S = \langle a, b^{-j} c\rangle$, where $(b^{-j} c)^2 = b^{-2j} c^2 = b^{-2j} z b^{2j} = z$ and so $S \cong Q_8$. Hence $G = \langle S, c\rangle = S \times \langle c\rangle \cong Q_8 \times C_{2^n}$ with $n \geq 2$, where $S \times \langle b^2\rangle \cong Q_8 \times C_{2^{n-1}}$ is a unique maximal subgroup of G which is neither abelian nor minimal nonabelian. We have obtained the groups stated in part (b) of our theorem.

(ii) It remains to consider the case where H is nonmetacyclic minimal nonabelian. We may set:

$$H = \langle a, b \mid a^{2^m} = b^{2^n} = 1,\ [a, b] = z,\ z^2 = [a, z] = [b, z] = 1\rangle,$$

where we may assume $m \geq 2$, $n \geq 1$, since $|H| \geq 2^4$. Here $H' = \langle z\rangle$, $|H| = 2^{m+n+1}$ and so $|G| = 2^{m+n+2}$. Also, z is not a square in H, $Z(H) = \langle a^2\rangle \times \langle b^2\rangle \times \langle z\rangle = \Phi(H) = \Phi(G)$ and for each $x \in Z(G) - H$, $x^2 \in Z(H) - \Phi(Z(H))$.

(ii1) First assume $n = 1$ so that $Z(H) = \langle a^2\rangle \times \langle z\rangle$ and for an element $c \in Z(G) - H$, $c^2 = a^{2i} z^j$. Suppose that i is even and then j must be odd and so we may set in that case $c^2 = a^{4i'} z$ and we compute for an element $c' = a^{-2i'} c \in Z(G) - H$:

$$(c')^2 = (a^{-2i'} c)^2 = a^{-4i'} c^2 = a^{-4i'} a^{4i'} z = z.$$

This gives $G = H * \langle c' \rangle$ with $(c')^2 = z$ where $\langle z \rangle = H'$ and it is easy to see that in that case G is an A_2-group (see Proposition 71.1 in [2]), a contradiction.

We have proved that i must be odd. The subgroup $D = \langle a^{-i}c, b \rangle$ is minimal nonabelian since $[a^{-i}c, b] = [a, b]^{-i} = z$. We have also

$$(a^{-i}c)^2 = a^{-2i}c^2 = a^{-2i}a^{2i}z^j = z^j,$$

which shows that $D \cong \mathrm{D}_8$. But $\langle c \rangle \cap \langle z \rangle = \{1\}$, where $\langle z \rangle = \mathrm{Z}(D)$ and so $\langle D, c \rangle = \langle a^{-i}c, b, c \rangle = G = D \times \langle c \rangle$ with $\mathrm{o}(c) = 2^m$, $m \geq 2$. The subgroup $D \times \langle c^2 \rangle$ is a unique maximal subgroup of G which is neither abelian nor minimal nonabelian and we have obtained the groups stated in part (c) of our theorem.

(ii2) It remains to consider the case $n \geq 2$. In this case for an element $c \in \mathrm{Z}(G) - H$ we have $c^2 = a^{2i}b^{2j}z^k$, where at least one of the integers i, j, k is odd.

Suppose that both i and j are even so that in this case k is odd and we may set $c^2 = a^{4i'}b^{4j'}z$. For the element $c' = a^{-2i'}b^{-2j'}c$, we get

$$(c')^2 = a^{-4i'}b^{-4j'}c^2 = a^{-4i'}b^{-4j'}a^{4i'}b^{4j'}z = z,$$

and so $G = H * \langle c' \rangle$ with $(c')^2 = z$ and $\langle z \rangle = H'$ which gives that G is an A_2-group of Proposition 71.1 in [2], a contradiction.

Now assume that one of the integers i, j is even and the other one is odd. Note that i, j occur symmetrically and so we may assume that i is odd and j is even. In that case the subgroup $T = \langle a^{-i}b^{-j}c, b \rangle$ is minimal nonabelian since $[a^{-i}b^{-j}c, b] = [a, b]^{-i} = z$. Using the fact that $b^{-j} \in \mathrm{Z}(G)$ we get:

$$(a^{-i}b^{-j}c)^2 = a^{-2i}b^{-2j}c^2 = a^{-2i}b^{-2j}a^{2i}b^{2j}z^k = z^k.$$

Since $\Phi(T) = \langle b^2 \rangle \times \langle z \rangle$, we have $|T| = 2^{n+2}$. On the other hand $|G| = 2^{m+n+2}$ with $m \geq 2$ and so $\Phi(G)T$ is a maximal subgroup of G which is neither abelian nor minimal nonabelian. Consider now the minimal nonabelian subgroup $U = \langle ab, ac \rangle$, where $[ab, ac] = z$. We have

$$(ab)^2 = a^2b^2z, (ac)^2 = a^2c^2 = a^2 \cdot a^{2i}b^{2j}z^k = a^{2(i+1)}b^{2j}z^k,$$

where both $i + 1$ and j are even. We have

$$\Phi(U) = \langle a^2b^2z, a^{2(i+1)}b^{2j}z^k, z \rangle \leq \langle a^2b^2, z \rangle \Phi(\mathrm{Z}(H))$$

since $a^{2(i+1)}b^{2j} \in \Phi(\mathrm{Z}(H))$. But $\mathrm{Z}(H) = \langle a^2 \rangle \times \langle b^2 \rangle \times \langle z \rangle$ and so $\mathrm{d}(\mathrm{Z}(H)) = 3$ which gives $\Phi(U) < \mathrm{Z}(H) = \Phi(G)$. This shows that $\Phi(G)U$ is another maximal subgroup of G which is neither abelian nor minimal nonabelian, a contradiction.

It remains to consider the possibility that both i and j are odd. Then we consider the minimal nonabelian subgroup $V = \langle a^{-i}b^{-j}c, b \rangle$, where $[a^{-i}b^{-j}c, b] = [a, b]^{-i} = z$. We get $(a^{-i}b^{-j}c)^2 = (a^{-i}b^{-j})^2c^2 = a^{-2i}b^{-2j}z^{ij}c^2 = a^{-2i}b^{-2j}z \cdot a^{2i}b^{2j}z^k = z^{k+1}$, which shows that $|V| = 2^{n+2}$. But $|G| = 2^{m+n+2}$ with $m \geq 2$ and so $\Phi(G)V$ is a maximal subgroup of G

which is neither abelian nor minimal nonabelian. Now we consider a minimal nonabelian subgroup $W = \langle a, bc\rangle$, where $[a, bc] = z$. We compute:

$$(bc)^2 = b^2c^2 = b^2 \cdot a^{2i}b^{2j}z^k = a^{2i}b^{2(1+j)}z^k,$$

where i is odd and $1 + j$ is even. We have:

$$\Phi(W) = \langle a^2, (bc)^2, z\rangle = \langle a^2, a^{2i}b^{2(1+j)}z^k, z\rangle = \langle a^2, z\rangle\Phi(\mathrm{Z}(H)) < \mathrm{Z}(H) = \Phi(G)$$

since $b^{2(1+j)} \in \Phi(\mathrm{Z}(H))$. Hence $\Phi(G)W$ is another maximal subgroup of G which is neither abelian nor minimal nonabelian, a contradiction. □

In the rest of this paper we shall assume that G is a title group with $\mathrm{d}(G) = 3$ which possesses at most one abelian maximal subgroup. We know that in that case $|G : \mathrm{Z}(G)| \geq 8$ and $|G'| > 2$. Let H be a maximal subgroup of G which is minimal nonabelian. Then $\Phi(H) = \mathrm{Z}(H) \leq \Phi(G)$ and $|H : \Phi(H)| = 4$. Since $|G : \Phi(H)| = 8$, we must have also $\Phi(H) = \Phi(G)$. Let $K \neq H$ be another maximal subgroup of G which is minimal nonabelian. Then $\mathrm{Z}(K) = \Phi(K) = \Phi(G)$ which implies $\Phi(G) \leq \mathrm{Z}(G)$ and so $\Phi(G) = \mathrm{Z}(G)$. Let M be the unique maximal subgroup of G which is neither abelian nor minimal nonabelian. Since $|M : \Phi(G)| = 4$ and $\Phi(G) = \mathrm{Z}(G)$, we have $M = S * \Phi(G)$, where S is minimal nonabelian, $S \cap \Phi(G) = \Phi(S) < \Phi(G)$ and so $M' = S' \cong \mathrm{C}_2$ with $\mathrm{d}(M) \geq 3$.

THEOREM 3.2. *Let G be a 2-group with $\mathrm{d}(G) = 3$ which has exactly one maximal subgroup M which is neither abelian nor minimal nonabelian. If G possesses exactly one abelian maximal subgroup A, then $\Phi(G) = \mathrm{Z}(G)$, $G' \cong \mathrm{E}_4$, $M' \cong \mathrm{C}_2$, $\mathrm{d}(M) \geq 3$ and one of the following holds:*

(a) *If G has no normal elementary abelian subgroup of order 8, then G is one of the groups given in Theorem 87.8(e) in [2].*

(b) *If G has a normal subgroup $E \cong \mathrm{E}_8$ but $\Omega_1(G) > E$, then*

$$G = \langle t, t', c \mid t^2 = t'^2 = c^4 = 1,\ [t, t'] = c^2 = z,\ [c, t] = u,$$
$$u^2 = [c, t'] = [u, t] = [u, t'] = [u, c] = [z, t] = [z, t'] = 1\rangle,$$

where $|G| = 2^5$, $G' = \Phi(G) = \mathrm{Z}(G) = \langle z, u\rangle \cong \mathrm{E}_4$, $\Omega_1(G) = M = \langle t, t'\rangle G' \cong \mathrm{C}_2 \times \mathrm{D}_8$, $A = \langle t', c\rangle G'$ is abelian of type $(4, 2, 2)$, and other five maximal subgroups of G are nonmetacyclic minimal nonabelian.

(c) *If G has a normal subgroup $E \cong \mathrm{E}_8$, $\Omega_1(G) = E$ and $E \not\leq A$, then*

$$G = \langle a, b, t \mid a^{2^{m+1}} = b^4 = t^2 = 1,\ a^{2^m} = z,\ b^2 = u,\ [a, t] = u,\ [b, t] = z,$$
$$[u, a] = [u, t] = [a, b] = [z, t] = 1\rangle,$$

where $|G| = 2^{m+4}$, $m \geq 2$, $G' = \langle z, u\rangle \cong \mathrm{E}_4$, $\Phi(G) = \mathrm{Z}(G) = \langle a^2, u\rangle \cong \mathrm{C}_{2^m} \times \mathrm{C}_2$, $E \not\leq \mathrm{Z}(G)$, $A = \langle a, b\rangle$ is abelian of type $(2^{m+1}, 4)$, $M = \langle b, t\rangle * \langle a^2\rangle$, where $\langle b, t\rangle$ is the nonmetacyclic minimal nonabelian group of order 2^4 and other five maximal subgroups of G are minimal nonabelian.

(d) *If G has a normal subgroup $E \cong \mathrm{E}_8$, $\Omega_1(G) = E$ and $E \le A$, then*

$$G = \langle a, b, d \mid a^4 = b^2 = d^4 = 1,\ a^2 = d^2 = z,\ [a, d] = z,\ [a, b] = c,$$
$$c^2 = [c, d] = [c, a] = [c, b] = [b, d] = 1\rangle,$$

where $|G| = 2^5$, $G' = \Phi(G) = \mathrm{Z}(G) = \langle z, c\rangle \cong \mathrm{E}_4$, $A = \langle b, d\rangle\Phi(G)$, $M = \langle a, d, c\rangle \cong \mathrm{Q}_8 \times \mathrm{C}_2$, $E \not\le \mathrm{Z}(G)$ and other five maximal subgroups of G are minimal nonabelian.

Proof. Let $\Gamma_1 = \{M, A, H_1, ..., H_5\}$ be the set of maximal subgroups of G, where $H_1, ..., H_5$ are minimal nonabelian. By a result of A. Mann (Proposition 1.5), $|G' : (A'H_1')| = |G' : H_1'| \le 2$ and so $|G'| = 4$ (since $|G'| > 2$). If $G' = \langle v\rangle \cong \mathrm{C}_4$, then $H_1' = ... = H_5' = \langle v^2\rangle$. But then $G/\langle v^2\rangle$ is a nonabelian group with at least five abelian maximal subgroups $H_i/\langle v^2\rangle$, $i = 1, ..., 5$, a contradiction. Hence $G' \cong \mathrm{E}_4$. Since A/M' and M/M' are two abelian maximal subgroups of the nonabelian group G/M', it follows that there is exactly one minimal nonabelian maximal subgroup of G, say H_5, such that $H_5' = M'$. With similar arguments we see that we may assume that $H_1' = H_2'$, $H_3' = H_4'$, $H_5' = M'$ are three pairwise distinct subgroups of order 2 in $G' \cong \mathrm{E}_4$.

Suppose that G has no normal elementary abelian subgroup of order 8. Then $A, H_1, ..., H_5$ are metacyclic and so M is the only maximal subgroup of G which is nonmetacyclic. Since $\mathrm{d}(G) = 3$ and $G' \cong \mathrm{E}_4$, we see that G is isomorphic to one of the groups stated in Theorem 87.8(e) in [2] which gives part (a) of our theorem.

From now on we assume that G has a normal elementary abelian subgroup E of order 8. Suppose at the moment that G possesses an elementary abelian subgroup F of order 16. Obviously, F is a maximal elementary abelian subgroup in G. Indeed, if X is an elementary abelian subgroup of order 32 in G, then $|X \cap H_1| = 16$, a contradiction. Since $G' \le \mathrm{Z}(G)$, we have $G' \le F$ and so $F \trianglelefteq G$. If G/F is noncyclic, then there are at least three distinct maximal subgroups of G containing F and so at least one of them is minimal nonabelian, a contradiction. Hence G/F is cyclic and let $a \in G - F$ be such that $\langle a\rangle$ covers G/F. Suppose that $|G : F| = 2$ so that F is an abelian maximal subgroup in G. Since $\mathrm{C}_F(a) = \mathrm{Z}(G) = \Phi(G)$ and $|G/\Phi(G)| = 8$, we get $\mathrm{C}_F(a) = G' \cong \mathrm{E}_4$. By Lemma 99.2 in [3], $G \cong \mathrm{E}_4 \wr \mathrm{C}_2$ and so we may assume that a is an involution. Let $f_1, f_2 \in F - G'$ so that $F = \langle f_1, f_2\rangle \times G'$ and $G = \langle f_1, f_2, a\rangle$. We have $\langle a, f_1\rangle \cong \langle a, f_2\rangle \cong \mathrm{D}_8$ and $\langle a, f_1\rangle G'$ and $\langle a, f_2\rangle G'$ are two distinct maximal subgroups of G which are isomorphic to $\mathrm{D}_8 \times \mathrm{C}_2$ (and so they are neither abelian nor minimal nonabelian), a contradiction. We have proved that $G/F \cong \mathrm{C}_{2^m}$, $m \ge 2$. Since $a^2 \in \Phi(G) = \mathrm{Z}(G)$, a induces an involutory automorphism on F which together with $|G : \mathrm{Z}(G)| = 8$ implies $\mathrm{C}_F(a) = G'$. Since $a^2 \notin F$ and $\Omega_1(\langle a\rangle) \le \mathrm{Z}(G)$, we must have $\Omega_1(\langle a\rangle) \le F$ (because E_{32} is not a subgroup of G). Hence $\langle a\rangle \cap F = \langle a\rangle \cap G' = \langle z\rangle \cong \mathrm{C}_2$ and so $o(a) = 2^{m+1}$, $m \geq 2$, $|G| = 2^{m+4}$ and $Z(G) = \Phi(G) = G'\langle a^2\rangle$. We may set $F = \langle x, y, u, z\rangle$, where $G' = \langle u, z\rangle$, $[a, x] = u$, $[a, y] = z$, $G = \langle x, y, a\rangle$ and so the structure of G is completely determined. Now, $\langle a, y\rangle \cong \langle ax, y\rangle \cong M_{2^{m+2}}$ so that $\langle u\rangle \times \langle a, y\rangle$ and $\langle u\rangle \times \langle ax, y\rangle$ are two distinct maximal subgroups of G which are neither abelian nor minimal nonabelian, a contradiction.

We have proved that G does not possess an elementary abelian subgroup of order 16. Since $G' \leq Z(G)$, we have $G' < E \cong E_8$. Next suppose that $E < \Omega_1(G)$ so that there is an involution $t \in G - E$ with $C_E(t) = G'$ and $S = \langle E, t\rangle \trianglelefteq G$. If G/S is noncyclic, then S is contained in a maximal subgroup of G which is minimal nonabelian, a contradiction (with the structure of minimal nonabelian 2-groups). Hence G/S is cyclic and let $c' \in G - S$ be such that $\langle c'\rangle$ covers G/S. Let $t' \in E - G'$ so that $1 \neq [t, t'] = z \in G'$ and $\langle t, t'\rangle = D \cong D_8$. Also, $E_1 = \langle G', t\rangle$ is another elementary abelian normal subgroup of order 8 in G. All elements in $S - (E \cup E_1)$ are of order 4 and $v = tt'$ is one of them. We have $v^2 = z$, $S = D \times \langle u\rangle \cong D_8 \times C_2$, where $u \in G' - \langle z\rangle$ and also $Z(S) = G'$ with $Z(G) = \Phi(G) = G'\langle c'^2\rangle$ so that $G = \langle t, t', c'\rangle$ and $M = S\langle c'^2\rangle$ must be a unique maximal subgroup of G which is neither abelian nor minimal nonabelian. If $x \in G - M$, then x is either an involution or $\Omega_1(\langle x\rangle) \leq G'$. If x is an involution, then $[t, x] \neq 1$ (because E_{16} is not a subgroup of G) and so $\langle t, x\rangle \cong D_8$, $|G : S| = 2$, $|G| = 2^5$, and $\langle t, x\rangle G'$ would be another maximal subgroup of G which is neither abelian nor minimal nonabelian, a contradiction. We have proved that x is not an involution and so $\Omega_1(\langle x\rangle) \leq G'$. Indeed, $\Omega_1(\langle x\rangle) \leq Z(G)$ and so $\Omega_1(\langle x\rangle) \leq S$ which implies that $\Omega_1(\langle x\rangle) \leq Z(S) = G'$.

Now, $A \cap M$ is equal to one of three abelian maximal subgroups of M (containing $\Phi(G) = Z(G)$): $\langle E_1, c'^2\rangle$, $\langle E, c'^2\rangle$, $\langle G'\langle v\rangle, c'^2\rangle$, where A is the unique abelian maximal subgroup of G. We choose $c \in A - M$ (instead of c'), where $\langle c\rangle$ also covers G/S, $o(c) \geq 4$, $\Omega_1(\langle c\rangle) \leq G'$, $\Phi(G) = Z(G) = G'\langle c^2\rangle$, and c centralizes exactly one of the elements in the set $\{t, t', v = tt'\}$. Indeed, otherwise, $G/\langle z\rangle$ would be abelian since c generates G together with any two elements in the above set. But then $G' = \langle z\rangle$, a contradiction. Interchanging t and t' (if necessary), we may assume that $[c, t'] = 1$ or $[c, tt'] = 1$. In that case $[c, t] \neq 1$ and if $[c, t] = z$, then again $G/\langle z\rangle$ would be abelian, a contradiction. It follows that we may set $[c, t] = u \in G' - \langle z\rangle$.

First assume $[c, tt'] = 1$ which gives $[c, t'] = u$. We have $M = \langle t, t'\rangle\Phi(G)$ and $A = \langle c, tt'\rangle\Phi(G)$. It follows that the other five maximal subgroups $\Phi(G)T$ of G must be minimal nonabelian, where T is one of the minimal nonabelian subgroups: $\langle t, c\rangle$, $\langle t', c\rangle$, $\langle t, t'c\rangle$, $\langle t', tc\rangle$, $\langle tt', tc\rangle$. We have to show that in each of these cases $\Phi(T) \geq \Phi(G) = \langle G', c^2\rangle = \langle u, z, c^2\rangle$. We have $[t, c] = u$ and so $\Phi(\langle t, c\rangle) = \langle c^2, u\rangle$ which implies $\Omega_1(\langle c\rangle) \in \{\langle z\rangle, \langle uz\rangle\}$. We have $[t, t'c] = zu$ and so $\Phi(\langle t, t'c\rangle) = \langle (t'c)^2 = c^2u, zu\rangle$. Here if $o(c) \geq 8$, then $\Omega_1(\langle t'c\rangle) = \Omega_1(\langle c\rangle)$ and then $\Omega_1(\langle c\rangle) \in \{\langle z\rangle, \langle u\rangle\}$ which together with the above result gives $\Omega_1(\langle c\rangle) = \langle z\rangle$, and if $o(c) = 4$, then $c^2 = uz$. We have $[tt', tc] = z$ and so $\Phi(\langle tt', tc\rangle) = \langle (tt')^2 = z, (tc)^2 = c^2u, z\rangle = \langle c^2u, z\rangle$. If $\mathrm{o}(c) = 4$, then by the above $c^2 = uz$ and then $\Phi(\langle tt', tc\rangle = \langle z\rangle$, a contradiction. If $\mathrm{o}(c) \geq 8$, then by the above $\Omega_1(\langle c\rangle) = \langle z\rangle$ and since $\Omega_1(\langle c^2u\rangle) = \Omega_1(\langle c\rangle) = \langle z\rangle$ we get again $\Phi(\langle tt', tc\rangle) = \langle c^2u\rangle \not\geq \Phi(G)$, a contradiction.

Now assume $[c, t'] = 1$ and from before we know that $[t, t'] = z$ and $[c, t] = u$. We have here $M = \langle t, t'\rangle\Phi(G)$ and $A = \langle c, t'\rangle\Phi(G)$ so that the other five maximal subgroups must be minimal nonabelian. We have $[t, c] = u$ and so $\Phi(\langle t, c\rangle) = \langle c^2, u\rangle$ which gives $\Omega_1(\langle c\rangle) \in \{\langle z\rangle, \langle uz\rangle\}$. We have $[t, t'c] = zu$ and so $\Phi(\langle t, t'c\rangle) = \langle (t'c)^2 = c^2, zu\rangle$ which implies that $\Omega_1(\langle c\rangle) \in \{\langle u\rangle, \langle z\rangle\}$ which together with our previous result gives $\Omega_1(\langle c\rangle) = \langle z\rangle$. We have $[t', tc] = z$ and so $\Phi(\langle t', tc\rangle) = \langle (tc)^2 = c^2u, z\rangle$. If $\mathrm{o}(c) \geq 8$, then $(c^2u)^2 = c^4$ and $\langle c^4\rangle \geq \langle z\rangle$ since $\Omega_1(\langle c\rangle) = \langle z\rangle$. In this case $\Phi(\langle t', tc\rangle) \not\geq \Phi(G)$, a contradiction. Hence $\mathrm{o}(c) = 4$ and so $c^2 = z$. We have obtained a uniquely determined group of order 2^5 given in part (b) of our theorem.

From now on we may assume that $\Omega_1(G) = E \cong \mathrm{E}_8$.

(i) Assume that $\Omega_1(G) = E \not\leq \mathrm{Z}(G) = \Phi(G)$ and $E \not\leq A$, where A is the unique abelian maximal subgroup of G.

Then $A \cap E = G'$, A covers G/E and A is metacyclic. Since there are three maximal subgroups of G containing E, there is at least one of them, denoted with H, which is minimal nonabelian. If H/E is noncyclic, then there are two distinct maximal subgroups $X_1 \neq X_2$ of H containing E. In that case $E \leq X_1 \cap X_2 = \Phi(H) = \Phi(G) = \mathrm{Z}(G)$, a contradiction. Hence H/E is cyclic. Since $\mathrm{d}(G/E) = 2$, G/E is abelian of type $(2^m, 2)$, $m \geq 1$. Therefore $A/G' \cong G/E$ is of type $(2^m, 2)$, where $A \cap H/G' \cong \mathrm{C}_{2^m}$. Let a be an element in $A \cap H$ such that $\langle a\rangle$ covers $A \cap H/G'$. Noting that $\Omega_1(G) = E$, we have $\mathrm{o}(a) = 2^{m+1}$ and $\Omega_1(\langle a\rangle) = \langle z\rangle \leq G'$, where $z = a^{2^m}$. If $t \in E - G'$, then $[t, a] = u \in G' - \langle z\rangle$ because H is nonmetacyclic and therefore u is not a square in H. Since $A/G' \cong \mathrm{C}_{2^m} \times \mathrm{C}_2$, there is an element $b \in A - H$ such that $1 \neq b^2 \in G'$ and $b^2 \neq z$. Indeed, if $b^2 = z$, then taking an element v of order 4 in $\langle a\rangle$, we get $(bv)^2 = b^2v^2 = z^2 = 1$, where $bv \in A - H$, a contradiction. Hence we get $b^2 \in \{u, uz\}$. We have $\Phi(H) = \Phi(G) = \langle a^2, u\rangle$ and $G = \langle a, b, t\rangle$. If $[b, t] \in \langle u\rangle$, then $G/\langle u\rangle$ is abelian, a contradiction. Hence $[b, t] \in \{z, uz\}$, $\mathrm{o}(b) = 4$ and A is abelian of type $(4, 2^{m+1})$. We set $[b, t] = zu^\epsilon$ and $b^2 = uz^\eta$, $\epsilon, \eta = 0, 1$.

First suppose that $\mathrm{o}(a) > 4$ so that $\langle a^4\rangle \geq \langle z\rangle$. In that case $\Phi(\langle b, t\rangle) = \langle b^2, [b, t]\rangle \leq G' < \Phi(G) = \langle a^2, u\rangle$ and so $M = \langle b, t\rangle\Phi(G)$. The fact that $\Phi(\langle ab, t\rangle) = \langle (ab)^2 = a^2uz^\eta, [ab, t] = zu^{\epsilon+1}\rangle = \Phi(G)$ gives $\epsilon = 0$. We may assume that $b^2 = u$, i.e., $\eta = 0$. Indeed, if $b^2 = uz = u'$, then we replace $H = \langle a, t\rangle$ with $H_1 = \langle a' = ab, t\rangle$, where $\mathrm{o}(a') = \mathrm{o}(a)$, $\langle a'\rangle \geq \langle z\rangle$ and $[a', t] = uz = u'$ and so writing again a and u instead of a' and u', respectively, we have obtained the relations for groups G of order 2^{m+4} given in part (c) of our theorem.

It remains to examine the case $\mathrm{o}(a) = 4$. In this case $m = 1$, $a^2 = z$, G is a special group of order 2^5, where $\Phi(G) = \langle u, z\rangle$. We have $A = \langle a, b\rangle \cong \mathrm{C}_4 \times \mathrm{C}_4$ and $\langle a, t\rangle = H$ is the nonmetacyclic minimal nonabelian group of order 2^4. We have $[b, t] = zu^{\epsilon}$ with $\Phi(\langle b, t\rangle) = \langle uz^{\eta}, zu^{\epsilon}\rangle$, $[ab, t] = zu^{\epsilon+1}$ with $\Phi(\langle ab, t\rangle) = \langle uz^{\eta+1}, zu^{\epsilon+1}\rangle$, $[b, at] = zu^{\epsilon}$ with $\Phi(\langle b, at\rangle) = \langle uz^{\eta}, uz, zu^{\epsilon}\rangle$, and $[ab, at] = zu^{\epsilon+1}$ with $\Phi(\langle ab, at\rangle) = \langle uz^{\eta+1}, uz, zu^{\epsilon+1}\rangle$. If $\epsilon = \eta = 0$, then $\Phi(\langle ab, t\rangle) = \langle uz\rangle$ and $\Phi(\langle ab, at\rangle) = \langle uz\rangle$. If $\epsilon = \eta = 1$, then $\Phi(\langle b, t\rangle) = \langle uz\rangle$ and $\Phi(\langle b, at\rangle) = \langle uz\rangle$. It follows that in the above two cases our group G has two distinct maximal subgroups which are neither abelian nor minimal nonabelian, a contradiction. It follows that we must have $\epsilon \neq \eta$ in which case we may set $\eta = \epsilon + 1$. But in this case we check that each nonabelian maximal subgroup of G is minimal nonabelian and so G would be an A_2-group, a contradiction.

(ii) Assume that $\Omega_1(G) = E \not\leq \mathrm{Z}(G) = \Phi(G)$ and $E \leq A$, where A is the unique abelian maximal subgroup of G.

Since there are three maximal subgroups of G containing E, there is a maximal subgroup H of G containing E which is minimal nonabelian. Then H is nonmetacyclic with $\mathrm{Z}(H) \cap E = G'$ and H/E is cyclic. Taking an element $b \in E - G'$, we may set:

$$H = \langle a, b \mid a^{2^{\alpha}} = b^2 = 1,\ \alpha \geq 2,\ [a, b] = c,\ c^2 = [a, c] = [b, c] = 1\rangle,$$

where $\langle c\rangle = H'$, $\mathrm{Z}(H) = \langle c\rangle \times \langle a^2\rangle$, $|G| = 2^{\alpha+3}$, and setting $a^{2^{\alpha-1}} = z$ we have $G' = \langle z, c\rangle \cong \mathrm{E}_4$ since c is not a square in H. Here $\langle a, c\rangle$ (containing G') is an abelian normal subgroup of type $(2^{\alpha}, 2)$ in G having exactly two cyclic subgroups $\langle a\rangle$ and $\langle ac\rangle$ of order 2^{α}. Since $a^b = ac$, we have $\mathrm{N}_H(\langle a\rangle) = \langle a, c\rangle$, which implies that $N = \mathrm{N}_G(\langle a\rangle)$ covers G/H and $N \cap H = \langle a, c\rangle$. It follows that N is a nonabelian maximal subgroup of G (because $A \geq E$), where $N/G' \cong G/E$ is noncyclic abelian and so N/G' is of type $(2, 2^{\alpha})$. Hence there is $d \in N - H$ with $1 \neq d^2 \in G'$ and so $\mathrm{o}(d) = 4$. But d normalizes $\langle a\rangle$ and therefore $[d, a] \in \langle a\rangle \cap G' = \langle z\rangle$ which gives $[d, a] = z$. There are exactly three maximal subgroups of G containing E: $H = \langle a, b\rangle$, $\langle d, b\rangle\Phi(H)$, $\langle ad, b\rangle\Phi(H)$, where exactly one of two last subgroups is abelian. It follows that either $[d, b] = 1$ or $[ad, b] = 1$ in which case $[d, b] = c$ (since $[a, b] = c$). We may set $[d, b] = c^{\epsilon}$, where $\epsilon = 0, 1$ and note that $G = \langle a, b, d\rangle$.

First assume that $\alpha \geq 3$. If $d^2 = z$, then $\langle d, a\rangle \cong \mathrm{M}_{2^{\alpha+1}}$ and there are involutions in $\langle d, a\rangle - \langle a\rangle$, a contradiction. Hence $d^2 \in G' - \langle z\rangle$ in which case $\langle d, a^{2^{\alpha-2}}\rangle \cong \mathrm{C}_4 \times \mathrm{C}_4$ since $a^{2^{\alpha-2}} \in \mathrm{Z}(G)$. Hence, replacing d with $da^{2^{\alpha-2}}$ (if necessary), we may assume $d^2 = c$. If $\epsilon = 0$, then $A = \langle d, b\rangle\Phi(G)$ is an abelian maximal subgroup of G and we check that all other six maximal subgroups of G are minimal nonabelian and so G is an A_2-group, a contradiction. Hence we must have $\epsilon = 1$. We have $[b, d] = c$ and $\Phi(\langle b, d\rangle) = \langle c\rangle < \Phi(G)$. Also, $[ab, ad] = z$ and $\Phi(\langle ab, ad\rangle) = \langle a^2c, a^2cz, z\rangle = \langle a^2c\rangle < \Phi(G)$ since $(a^2c)^2 = a^4$

and $\langle a^4\rangle \geq \langle z\rangle$. Hence $\langle b,d\rangle\Phi(G)$ and $\langle ab,ad\rangle\Phi(G)$ are two distinct maximal subgroups of G which are neither abelian nor minimal nonabelian, a contradiction.

We have proved that we must have $\alpha = 2$ so that $a^2 = z$, G is special with $\Phi(G) = \langle z,c\rangle$ and $|G| = 2^5$. We have $[d,b] = c^{\epsilon}$ and if $\epsilon = 1$, then we replace d with $d' = ad$ so that $[d',b] = 1$ and $[a,d'] = z$. Writing again d instead of d', we may assume that $[d,b] = 1$ and $[a,d] = z$ (as before). If $d^2 = z$, then we obtain the group of order 2^5 given in part (d) of our theorem. It remains to analyze the cases $d^2 \in \{c,cz\}$. If $d^2 = c$, then $\langle b,ad\rangle \cong \mathrm{D}_8$ since $[b,ad] = c$ and $(ad)^2 = a^2d^2[d,a] = zcz = c$. This is a contradiction since $\Omega_1(G) \cong E_8$. Suppose that $d^2 = cz$. In that case we replace a with $a' = ab$, z with $z' = zc$ and d with $d' = db$. Then we get $a'^2 = (ab)^2 = zc = z'$, $d'^2 = (db)^2 = d^2 = cz = z'$, $[a',b] = [ab,b] = c$, $[a',d'] = [ab,db] = zc = z'$ and $[b,d'] = [b,db] = 1$ and so writing again a,z,d instead of a',z',d', respectively, we get again the group given in part (d) of our theorem.

(iii) We turn now to the difficult case, where $\Omega_1(G) = E \leq \mathrm{Z}(G) = \Phi(G)$.

Let $H_1 = H$ be a maximal subgroup of G which is minimal nonabelian and such that $H' \neq M'$. Since $\mathrm{Z}(H) = \mathrm{Z}(G) \geq E$, H is nonmetacyclic and we may set:

$$H = \langle a,b \mid a^{2^{\alpha}} = b^{2^{\beta}} = 1,\ [a,b] = c,\ c^2 = [a,c] = [b,c] = 1\rangle,$$

where $\alpha \geq 2$, $\beta \geq 2$, $\langle c\rangle = H'$, $\mathrm{Z}(H) = \Phi(G) = \langle c\rangle \times \langle a^2\rangle \times \langle b^2\rangle$ is abelian of type $(2^{\alpha-1},2^{\beta-1},2)$, and $|G| = 2^{\alpha+\beta+2}$. We have $G' < E = \langle a^{2^{\alpha-1}}, a^{2^{\beta-1}}, c\rangle \cong \mathrm{E}_8$.

We consider the group G/M', where $(G/M')' = G'/M' \cong \mathrm{C}_2$, $\mathrm{d}(G/M') = 3$, G/M' has exactly three abelian maximal subgroups A/M', M/M' and H_5/M' (since $H_5' = M'$) and other four maximal subgroups H_i/M', $i = 1,...,4$, are minimal nonabelian. Thus G/M' is an A_2-group of Proposition 71.1 in [2] which implies that there is an element $d \in G - H$ such that $[d,G] = M'$ and $1 \neq d^2 \in G'$. Since there are exactly three maximal subgroups of G containing $\langle d\rangle \cong \mathrm{C}_4$, at least one of them H^* is minimal nonabelian, where $H^* \geq E$ and so H^* is nonmetacyclic. It follows that $(H^*)'$ is a maximal cyclic subgroup in H^* and so $(H^*)' \neq \langle d^2\rangle$ which gives $G' = \langle (H^*)', d^2\rangle$. Taking an element $a^* \in (H \cap H^*) - \Phi(G)$, we get $H^* = \langle a^*, d\rangle$ and so $\Phi(H^*) = \langle (a^*)^2, d^2, (H^*)'\rangle = \langle (a^*)^2, G'\rangle = \Phi(G)$ and $E = \langle \Omega_1(\langle a^*\rangle), G'\rangle$. Set $\mathrm{o}(a^*) = 2^{\gamma}$, $\gamma \geq 2$, so that $\Phi(H^*) = \Phi(G)$ is of type $(2^{\gamma-1},2,2)$. On the other hand $\Phi(G)$ is of type $(2^{\alpha-1},2^{\beta-1},2)$. Interchanging the elements a and b (if necessary), we may assume that $\beta = 2$ and then $\gamma = \alpha$ so that $\Phi(G)$ is of type $(2^{\alpha-1},2,2)$ and $\mathrm{o}(b) = 4$. Since $[d,G] = M'$, we have $\langle [d,a^*]\rangle = M'$ and so $(H^*)' = M'$ and therefore $H^* = H_5$ and $d^2 \in G' - M'$. Because H^*/E is abelian of type $(2^{\alpha-1},2)$, it follows that $H \cap H^*/E$ is cyclic of order $2^{\alpha-1}$ and so $\langle a^*\rangle$ covers $H \cap H^*/E$. Since $H \cap H^* = \langle a^*\rangle \times G'$, H/G' is abelian of type $(2^{\alpha},2)$ which implies that $H = H_0(H \cap H^*)$ with $H_0 \cap (H \cap H^*) = G'$ and

$|H_0 : G'| = 2$. Hence there is an element $b^* \in H - H^*$ with $1 \neq (b^*)^2 \in G'$, $\langle a^*, b^*\rangle = H$, $(b^*)^2 \neq c$ (since $H' = \langle c\rangle$ and so c is not a square in H) and so $[a^*, b^*] = c$ and $o(b^*) = 4$. In addition, from $[d, G] = M'$ follows that either $[d, b^*] = m_0$ with $\langle m_0\rangle = M'$ or $[d, b^*] = 1$. Also $d^2 = cm_0^{\epsilon}$ and $(b^*)^2 = c^{\eta}m_0$, where $\epsilon, \eta = 0, 1$.

Assume that $[d, b^*] = 1$ in which case $d^2 \neq (b^*)^2$ (because if $d^2 = (b^*)^2$, then db^* is an involution in $G - H$, a contradiction) and so $\langle d, b^*\rangle \cong \mathrm{C}_4 \times \mathrm{C}_4$ and $\langle d^2, (b^*)^2\rangle = G'$. In that case, $[b^*a^*, d] = m_0$ since $[a^*, d] = m_0$ and we get $\Phi(\langle b^*a^*, d\rangle) = \langle (b^*a^*)^2 = c^{\eta}m_0 \cdot (a^*)^2 \cdot c = (a^*)^2c^{\eta+1}m_0, d^2 = cm_0^{\epsilon}, m_0\rangle = G'\langle (a^*)^2\rangle = \Phi(G)$, and so $\langle b^*a^*, d\rangle$ is a minimal nonabelian maximal subgroup of G with $\langle b^*a^*, d\rangle' = \langle m_0\rangle$ and $\langle b^*a^*, d\rangle \neq H^*$. This is a contradiction since $H^* = H_5$ is the only maximal subgroup of G which is minimal nonabelian and $(H^*)' = M' = \langle m_0\rangle$.

We have proved that $[d, b^*] = m_0$ which together with $\langle b^*, d\rangle \neq H^*$ implies that $\langle b^*, d\rangle\Phi(G) = M$ must be the unique maximal subgroup of G which is neither abelian nor minimal nonabelian. If $\epsilon = 0$ and $\eta = 1$, then $(b^*)^2 = cm_0$, $d^2 = c$ and therefore $(b^*d)^2 = cm_0 \cdot c \cdot m_0 = 1$ and b^*d would be an involution in $G - H$, a contradiction. Hence we have either $\epsilon = \eta$ or $\epsilon = 1$ and $\eta = 0$. We have $[a^*, b^*d] = cm_0$ and $\Phi(\langle a^*, b^*d\rangle) = \langle (a^*)^2, c^{\eta+1}m_0^{\epsilon}, cm_0\rangle = \Phi(G)$ implies that $\epsilon = 1$ and $\eta = 0$ is not possible and so $\epsilon = \eta$. Further, $[b^*, a^*d] = cm_0$ and so $\Phi(\langle b^*, a^*d\rangle) = \langle c^{\epsilon}m_0, (a^*)^2cm_0^{\epsilon+1}, cm_0\rangle$ forces that $\epsilon = \eta = 0$. But then $[a^*b^*, a^*d] = c$ and $\Phi(\langle a^*b^*, a^*d\rangle) = \langle (a^*)^2cm_0, c\rangle < \Phi(G)$ show that $\langle a^*b^*, a^*d\rangle\Phi(G)$ is another maximal subgroup of G which is neither abelian nor minimal nonabelian, a contradiction. Our theorem is proved. □

THEOREM 3.3. *Let G be a 2-group with $\mathrm{d}(G) = 3$ which has exactly one maximal subgroup M which is neither abelian nor minimal nonabelian. If G has no abelian maximal subgroups, then we get:*

$$G = \langle a, b, c \,|\, a^4 = b^4 = c^{2^n} = 1,\ a^2 = x,\ b^2 = y,\ c^{2^{n-1}} = z,\ [a, b] = z,\ [a, c] = y,$$
$$[b, c] = xy,\ [x, b] = [x, c] = [y, a] = [y, c] = [z, a] = [z, b] = 1\rangle,$$

where $|G| = 2^{n+4}$, $n \geq 3$, $G' = \langle x, y, z\rangle \cong \mathrm{E}_8$, $\mathrm{Z}(G) = \Phi(G) = G'\langle c^2\rangle$ is abelian of type $(2^{n-1}, 2, 2)$, $M = \Phi(G)\langle a, b\rangle = \langle c^2\rangle * \langle a, b\rangle$ with $\langle c^2\rangle \cap \langle a, b\rangle = \langle z\rangle = \langle a, b\rangle'$, $\langle c^2\rangle \cong \mathrm{C}_{2^{n-1}}$ and $\langle a, b\rangle$ is the nonmetacyclic minimal nonabelian group of exponent 4 and order 2^5 and all other six maximal subgroups of G are nonmetacyclic minimal nonabelian.

PROOF. We set $\Gamma_1 = \{H_1, H_2, ..., H_6, M\}$ to be the set of maximal subgroups of G, where $H_1, ..., H_6$ are minimal nonabelian. We know that $|M'| = 2$ and $\mathrm{d}(M) \geq 3$ (see the remark preceding Theorem 3.2). By a result of A. Mann (Proposition 1.5), $|G' : (H_1'H_2')| \leq 2$ and so $|G'| \leq 8$. However, if $|G'| = 2$, then we know that G has three abelian maximal subgroups (see the second paragraph of the proof of Theorem 3.1), a contradiction. Hence $|G'| = 4$ or $|G'| = 8$.

Suppose that for some $H_i \neq H_j$, $H_i' = H_j'$. Then by a result of A. Mann (Proposition 1.5), we have $|G'| = 4$ and moreover we have $G' \cong \mathrm{E}_4$. Indeed, if $G' \cong \mathrm{C}_4$, then the nonabelian group $G/\Omega_1(G')$ would possess at least six abelian maximal subgroups $H_i/\Omega_1(G')$, $i = 1, ..., 6$, a contradiction. The group G/M' is obviously an A_2-group with $(G/M')' \cong \mathrm{C}_2$. By Proposition 71.1 in [2], G/M' has exactly three abelian maximal subgroups so that we may set $H_5' = H_6' = M' = \langle u \rangle$. In that case H_i', $i = 1, ..., 4$, cannot be all pairwise distinct and so we may set $H_2' = H_3' = H_4' = \langle v \rangle$ with $\langle u, v \rangle = G'$ and $G/\langle v \rangle$ has exactly three abelian maximal subgroups $H_i/\langle v \rangle$, $i = 2, 3, 4$. It follows that we must have $H_1' = \langle uv \rangle$ so that $G/\langle uv \rangle$ with $(G/\langle uv \rangle)' \cong \mathrm{C}_2$ has exactly one abelian maximal subgroup $H_1/\langle uv \rangle$. If $\mathrm{d}(M/\langle uv \rangle) = 2$, then $M/\langle uv \rangle$ is minimal nonabelian so that $G/\langle uv \rangle$ would be an A_2-group. But in that case (Proposition 71.1 in [2]) $G/\langle uv \rangle$ would have three abelian maximal subgroups, a contradiction. Hence we must have $\mathrm{d}(M/\langle uv \rangle) \geq 3$ in which case $M/\langle uv \rangle$ is a unique maximal subgroup of $G/\langle uv \rangle$ which is neither abelian nor minimal nonabelian. By Theorem 3.2, we must have $(G/\langle uv \rangle)' \cong \mathrm{E}_4$, a contradiction.

We have proved that all H_i' are pairwise distinct subgroups of order 2 in G'. This implies that $G' \cong \mathrm{E}_8$. If $M' = H_i'$ for some $i \in \{1, 2, ..., 6\}$, then considering G/M' we see that there must exist a maximal subgroup H_j, $j \neq i$, such that $M' = H_i' = H_j'$, a contradiction. Hence $\{H_1', ..., H_6', M'\}$ is the set of seven pairwise distinct subgroups of order 2 in G'. Since $G' \leq H_i$, all H_i $(i = 1, ..., 6)$ are nonmetacyclic minimal nonabelian. The group G/G' is abelian of rank 3. Suppose that there is an involution $t \in G - G'$. Then $F = G' \times \langle t \rangle \cong \mathrm{E}_{16}$ and G/F is noncyclic. But then there is a maximal subgroup H of G such that $H \geq F$ and H is minimal nonabelian, a contradiction. We have proved that $G' = \Omega_1(G)$. Set $T/G' = \Omega_1(G/G') \cong \mathrm{E}_8$. If G/T is noncyclic, then there is a maximal subgroup K of G such that $K \geq T$ and K is minimal nonabelian. But then $\mathrm{d}(K/G') = 3$, a contradiction. Hence G/T is cyclic and so G/G' is abelian of type $(2^m, 2, 2)$, $m \geq 1$.

(i) First assume $m = 1$, i.e., $T = G$, $G/G' \cong \mathrm{E}_8$ and G is a special group with $G' = \Omega_1(G) \cong \mathrm{E}_8$.

We shall determine the structure of $M > G'$. We have $M = G'S$, where $S = \langle a, b \rangle$ is minimal nonabelian and $G' \cap S = \Phi(S) < G'$. Set $\langle z \rangle = S' = M' \cong \mathrm{C}_2$. Suppose at the moment that $\Phi(S) = \langle z \rangle$ so that $S \cong \mathrm{Q}_8$. Then $G/\langle z \rangle$ is an A_2-group, where $M/\langle z \rangle \cong \mathrm{E}_{16}$ is a unique abelian maximal subgroup of $G/\langle z \rangle$ and $\mathrm{E}_4 \cong (G/\langle z \rangle)' \leq \mathrm{Z}(G/\langle z \rangle)$. But then Proposition 71.4(b) in [2] implies that $\Omega_1(G/\langle z \rangle) \cong \mathrm{E}_8$, a contradiction. We have proved that $\Phi(S) \cong \mathrm{E}_4$ and $\Omega_1(S) = \Phi(S)$. Hence S is the metacyclic minimal nonabelian group of order 16 and exponent 4. We may choose $a, b \in S - \Phi(S)$ so that $a^2 = z$, $b^2 = y$, $[a, b] = z$ and $\langle y, z \rangle = \Phi(S) = \Phi(M)$. Since $G' = \Phi(G)$, there is $c \in G - M$ such that $c^2 = x \in G' - \langle y, z \rangle$. We have $\langle x, y, z \rangle = G'$ and $\langle a, b, c \rangle = G$. All other six maximal subgroups (distinct from M) are nonmetacyclic minimal nonabelian. We have $\Phi(\langle a, c \rangle) = \langle z, x, [a, c] \rangle = G'$ so that $[a,c] = x^{\alpha}yz^{\beta}$. Also, $\Phi(\langle b,c\rangle) = \langle y,x,[b,c]\rangle = G'$ gives $[b,c] = x^{\gamma}y^{\delta}z$. Further

$$\Phi(\langle ab,c\rangle) = \langle y,x,[ab,c] = x^{\alpha+\gamma}y^{\delta+1}z^{\beta+1}\rangle = G',$$

which implies $\beta = 0$. From

$$\Phi(\langle b,ac\rangle) = \langle y,x^{\alpha+1}yz,[b,ac] = x^{\gamma}y^{\delta}\rangle = G',$$

we get $\gamma = 1$. From

$$\Phi(\langle ab,ac\rangle) = \langle y,x^{\alpha+1}yz,[ab,ac] = x^{\alpha+1}y^{\delta+1}\rangle = G',$$

follows $\alpha = 0$. Finally,

$$\Phi(\langle a,bc\rangle) = \langle z,y^{\delta+1}z,[a,bc] = zy\rangle = \langle y,z\rangle < G',$$

gives a contradiction since $G'\langle a,bc\rangle$ is another maximal subgroup of G (distinct from M) which is neither abelian nor minimal nonabelian.

(ii) Suppose that $T < G$, where $T/G' = \Omega_1(G/G') \cong \mathrm{E}_8$ and $\{1\} \neq G/T$ is cyclic so that G/G' is abelian of type $(2^m,2,2)$, $m \geq 2$.

The unique maximal subgroup of G containing T must be equal to M. There are normal subgroups U and V of G such that $G = UV$, $U \cap V = G'$, $U/G' \cong \mathrm{E}_4$ and V/G' is cyclic of order 2^m, $m \geq 2$. Let c be an element in $V - G'$ such that $\langle c\rangle$ covers V/G'. We have $\mathrm{o}(c) = 2^n$, $n \geq 3$, where $n = m+1$ (noting that $\Omega_1(G) = G'$). Set $\langle z\rangle = \Omega_1(\langle c\rangle)$ so that $z = c^{2^{n-1}}$ and $z \in G'$. Then $M = U\langle c^2\rangle$, $\Phi(G) = \mathrm{Z}(G) = G'\langle c^2\rangle$ is abelian of type $(2^{n-1},2,2)$ and $|G| = 2^{n+4}$. Let $a,b \in U - G'$ be such that $U = G'\langle a,b\rangle$, where $a^2,b^2 \in G'$ and $G = \langle a,b,c\rangle$. Since each maximal subgroup H_i $(i = 1,...,6)$ is nonmetacyclic and contains $\Phi(G)$ and z is a square in $\Phi(G)$, it follows that $H_i' \neq \langle z\rangle$ for all $i = 1,...,6$. This implies that $M' = \langle z\rangle$ and therefore $[a,b] = z$.

Now, $G/\langle z\rangle$ has the unique maximal abelian subgroup $M/\langle z\rangle$ and six minimal nonabelian maximal subgroups $H_i/\langle z\rangle$, $i = 1,...,6$, and so $G/\langle z\rangle$ is an A_2-group with the following properties. We have $\mathrm{d}(G/\langle z\rangle) = 3$ and so $G/\langle z\rangle$ is nonmetacyclic of order $2^{n+3} > 2^4$ since $n \geq 3$, $(G/\langle z\rangle)' \cong \mathrm{E}_4$, $G'/\langle z\rangle \leq \mathrm{Z}(G/\langle z\rangle)$ (since $G' \leq \mathrm{Z}(G)$) and $G/\langle z\rangle$ has a normal elementary abelian subgroup $\langle G',\Omega_2(\langle c\rangle)\rangle/\langle z\rangle$ of order 8. Hence $G/\langle z\rangle$ is an A_2-group of Proposition 71.4(b) in [2] which implies the fact that $\langle G',\Omega_2(\langle c\rangle)\rangle/\langle z\rangle = \Omega_1(G/\langle z\rangle)$. Set $a^2 = x$ and $b^2 = y$ and consider the abelian group $M/\langle z\rangle$. If the abelian subgroup $U/\langle z\rangle$ of order 16 and exponent ≤ 4 has rank > 2, then $\Omega_1(U/\langle z\rangle) > G'/\langle z\rangle$ which contradicts the above fact. Hence $U/\langle z\rangle \cong \mathrm{C}_4 \times \mathrm{C}_4$ which implies that $G' = \langle x,y,z\rangle$. Since all H_i are minimal nonabelian (containing $\Phi(G) = \langle c^2,x,y\rangle$), we get $[a,c] = x^{\alpha}yz^{\beta}$ and $[b,c] = xy^{\gamma}z^{\delta}$, where $\alpha,\beta,\gamma,\delta = 0,1$. From

$$\Phi(\langle ab,c\rangle) = \langle (ab)^2 = xyz, c^2, [ab,c] = x^{\alpha+1}y^{\gamma+1}z^{\beta+\delta}\rangle = \Phi(G)$$

and the fact that $\Omega_1(\langle c^2\rangle) = \langle z\rangle$ follows that $\alpha + 1 \neq \gamma + 1$ which gives $\gamma = \alpha + 1$ and $[b,c] = xy^{\alpha+1}z^{\delta}$.

Interchanging a and b and x and y, i.e., writing $a' = b, b' = a, x' = y, y' = x$, we get

$$a'^2 = b^2 = y = x',\ b'^2 = a^2 = x = y',\ [a', b'] = [b, a] = z,$$

$$[a', c] = [b, c] = xy^{\alpha+1}z^\delta = (x')^{\alpha+1}y'z^\delta,\ [b', c] = [a, c] = x^\alpha yz^\beta = x'y'^\alpha z^\beta.$$

Writing again a, b, x, y instead of a', b', x', y', respectively, we get

$$a^2 = x,\ b^2 = y,\ [a, b] = z,\ [a, c] = x^{\alpha+1}yz^\delta,\ [b, c] = xy^\alpha z^\beta,\ \beta, \delta = 0, 1,$$

which are the old relations in which α is replaced with $\alpha + 1$. This shows that we may assume $\alpha = 0$ and so we get $[a, c] = yz^\beta$ and $[b, c] = xyz^\delta$, β, $\delta = 0, 1$.

Finally, replacing c with $c' = ca^\delta b^\beta$, we get

$$c'^2 = c^2(a^\delta b^\beta)^2[a^\delta b^\beta, c] = c^2 l$$

with $l \in G'$ and so $c'^4 = c^4$, $\langle c'\rangle$ covers G/U and $\langle c'\rangle \cap G' = \langle z\rangle$. In addition we have

$$[a, c'] = [a, ca^\delta b^\beta] = [a, c][a, b]^\beta = yz^\beta z^\beta = y,$$
$$[b, c'] = [b, ca^\delta b^\beta] = [b, c][b, a]\delta = xyz^\delta z^\delta = xy.$$

Writing again c instead of c', we get the relations stated in our theorem. □

References

[1] Y. Berkovich, Groups of prime power order, Vol. 1, Walter de Gruyter, Berlin-New York, 2008.

[2] Y. Berkovich and Z. Janko, Groups of prime power order, Vol. 2, Walter de Gruyter, Berlin-New York, 2008.

[3] Y. Berkovich and Z. Janko, Groups of prime power order, Vol. 3, in preparation.

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