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COMPOSITION SERIES OF THE INDUCED REPRESENTATIONS OF $SO(5)$ USING INTERTWINING OPERATORS

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Abstract. Let F be a p-adic field of characteristic zero. We determine the composition series of the induced representations of $SO(5, F)$.

1. Introduction

In this paper we investigate composition series of the parabolicaly induced representations of the split connected group $SO(5, F)$, where F is a p-adic field of characteristic zero, and determine the set $\widetilde{SO(5, F)}$ of equivalence classes of irreducible representations of $SO(5, F)$ (modulo cuspidal representations). It is of interest to know whether the induced representation reduces or not, and to derive its composition series if it reduces. Similar examples of admissible duals of some other low - rank groups can be found in [3, 7, 9]. In the paper [6] we determine the unitary dual of $SO(5, F)$.

We expect that our results, besides being interesting by themselves, will play a role in determining unitary dual of some low-rank metaplectic groups.

In the next section we establish notation and review some standard facts from the representation theory of $SO(5, F)$. In the third section our main results are stated and proved. We determine composition series of the representations supported in the minimal parabolic subgroup, using rather new and powerful intertwining operator methods ([7, 8, 10]), combined with the method of Jacquet modules ([4, 13, 14]). In the last section we obtain the reducibility points of the representations with cuspidal support in the maximal parabolic subgroups. These reducibility points follow directly from the

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results of F. Shahidi, who has described reducibility in terms of $L-$functions ([10, 11]).

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2. Preliminaries

Let G be the F-points of a reductive group defined over F, where F is a p-adic field of characteristic zero. We denote by $R(G)$ the Grothendieck group of the category of admissible representations of finite length of G. In computations we write shortly σ for the semi-simplification of an admissible representation of finite length σ of G.

The odd special orthogonal group $SO(2n+1,F)$ is the group

$$SO(2n+1,F) = \{g \in SL(2n+1,F) : \ {}^{\tau}gg = I_{2n+1}\}$$

where ${}^{\tau}g$ denotes the transposed matrix of g with respect to the second diagonal. Let $R(S) = \bigoplus_{n \geq 0} R(SO(2n+1,F))$.

The character $|det(g)|_F$ of $GL(n,F)$, where $| \ |_F$ is the modulus of F, is denoted by ν. Set $R = \bigoplus_{n \geq 0} R(GL(n,F))$. If π is a representation of $GL(n,F)$ and $0 \leq k \leq n$, the normalized Jacquet module of π with respect to the standard parabolic subgroup whose Levi factor is $GL(k,F) \times GL(n-k,F)$ is denoted by $r_{(k)}(\pi)$. For $\pi \in R(GL(n,F))$, define $m^*(\pi) = \sum_{k=1}^{n} r_{(k)}(\pi)$ (the sum of all semi-simplifications). Obviously, one may consider $m^*(\pi) \in R \otimes R$. If π_1 is an admissible representation of $GL(k,F)$ and π_2 an admissible representation of $GL(n-k,F)$, we write $\pi_1 \times \pi_2$ for the representation of $GL(n,F)$ that is parabolically induced from $\pi_1 \otimes \pi_2$.

We fix a minimal parabolic subgroup P_{min} of $SO(2n+1,F)$ consisting of all upper triangular matrices in the group. A standard parabolic subgroup P of $SO(2n+1,F)$ is a parabolic subgroup of $SO(2n+1,F)$ containing P_{min}. Every standard parabolic subgroup has Levi factor isomorphic to $GL(n_1,F) \times \cdots \times GL(n_k,F) \times SO(2(n-|\alpha|)+1)$, where $\alpha = (n_1, \ldots, n_k)$ is a sequence of the positive integers with $\sum_{i=1}^{k} n_i = |\alpha|$, $|\alpha| \leq n$. We denote such parabolic subgroups by P_α and their Levi factors by M_α (recall that $P_\alpha = M_\alpha N_\alpha$ is a Levi decomposition of P_α, where N_α denotes the unipotent radical).

Suppose that $\pi_1, \ldots, \pi_k$ are the representations of groups $GL(n_1,F), \ldots, GL(n_k,F)$ and σ a representation of $SO(2(n - \sum_{i=1}^{k} n_i) + 1, F)$. Then we consider $\pi_1 \otimes \cdots \otimes \pi_k \otimes \sigma$ as a representation of M_α, where $\alpha = (n_1, \ldots, n_k)$. Following [13], normalized induction is written as

$$\pi_1 \times \cdots \times \pi_k \rtimes \sigma = Ind_{P_\alpha}^{GL(n,F)}(\pi_1 \otimes \cdots \otimes \pi_k \otimes \sigma).$$

If σ is a representation of $SO(2n+1,F)$, the normalized Jacquet module of σ with respect to P_α is denoted by $s_\alpha(\sigma)$. In this way we get a group homomorphism $R(SO(2n+1,F)) \to R(M_\alpha)$. In a similar way as before, for a smooth representation σ of $SO(2n+1,F)$ of finite length, set $\mu^*(\sigma) = \sum_{k=0}^{n} s_{(k)}(\sigma)$. We can consider $\mu^*(\sigma) \in R \otimes R(S)$. Then Frobenius reciprocity in this setting tells:

$$Hom_{SO(2n+1,F)}(\pi, \pi_1 \times \cdots \times \pi_k \rtimes \sigma) \simeq Hom_{M_\alpha}(s_\alpha(\pi), \pi_1 \otimes \cdots \otimes \pi_k \otimes \sigma).$$

If σ is a representation of $SO(5,F)$, the normalized Jacquet module $s_\alpha(\sigma)$ is denoted by $s_{min}(\sigma)$ if $\alpha = (1,1)$ (minimal parabolic subgroup, P_{min}), by $s_{Sieg}(\sigma)$ if $\alpha = (2)$ (Siegel parabolic subgroup, P_{Sieg}) or by $s_{(1)}(\sigma)$ if $\alpha = (1)$ (Heisenberg parabolic subgroup, $P_{(1)}$).

Let π_i be representations of $GL(n_i,F)$, $1 \le i \le 2$, and σ a representation of $SO(2n+1,F)$. We shortly recall some well-known properties that are helpful while working with Jacquet modules of the induced representations and determining their composition series ($\tilde{}$ denotes contragredient):

- Representations $\pi_1 \times \pi_2$ and $\pi_2 \times \pi_1$ have the same composition series. Also, if $\pi_1 \times \pi_2$ is irreducible, then $\pi_1 \times \pi_2 \simeq \pi_2 \times \pi_1$.
- $\widetilde{\pi_1 \times \pi_2} \simeq \widetilde{\pi_1} \times \widetilde{\pi_2}$.
- Representations $\pi \rtimes \sigma$ and $\widetilde{\pi} \rtimes \sigma$ have the same composition series and $\widetilde{\pi \rtimes \sigma} \simeq \widetilde{\pi} \rtimes \widetilde{\sigma}$.

For an admissible representation π of a reductive group G, Aubert dual of π is denoted by $\widehat{\pi}$. We list some basic properties ([1, Théorème 1.7]):

(a) If π is irreducible cuspidal representation, then $\widehat{\pi} = \pi$,
(b) $\widehat{\widehat{\pi}} = \pi$,
(c) $\widehat{\pi_1 \times \pi_2} = \widehat{\pi}_1 \times \widehat{\pi}_2$ and $s_{min}(\widehat{\pi}) = Ad(w)\widehat{s_{min}(\pi)}$, where w is the longest element of Weyl group of G.

We take a momment to recall Langlands classification for odd special orthogonal groups. For each irreducible essentially square integrable representation δ of $GL(n,F)$ there is an $e(\delta) \in \mathbb{R}$ such that $\delta = \nu^{e(\delta)}\delta^u$, where δ^u is unitarizable. We use the letter D to denote the set of equivalence classes of all irreducible essentially square integrable representations of $GL(n,F)$, $n \ge 1$. Let $D_+ = \{\delta \in D : e(\delta) > 0\}$. Further, let $\delta_1, \ldots, \delta_k \in D_+$ such that $e(\delta_1) \ge e(\delta_2) \ge \cdots \ge e(\delta_k)$ and σ an irreducible tempered representation of $SO(2n+1,F)$, $n \in \mathbb{N}$. Then the representation $\delta_1 \times \delta_2 \times \cdots \times \delta_k \rtimes \sigma$ has an unique irreducible quotient, which we denote by $L(\delta_1, \delta_2, \ldots, \delta_k, \sigma)$. This irreducible quotient is called Langlands quotient of the representation $\delta_1 \times \delta_2 \times \cdots \times \delta_k \rtimes \sigma$. Every irreducible representation π of $SO(2n+1,F)$, $n \in \mathbb{N}$, is isomorphic to some $L(\delta_1, \delta_2, \ldots, \delta_k, \sigma)$.

The following version of Casselmans square-integrability criterion is frequently used:

Let π be an admissible irreducible representation of $SO(2n+1,F)$ and let P_α be any standard parabolic subgroup minimal with respect to the property that

$s_\alpha(\pi) \neq 0$. Write $\alpha = (n_1, \ldots, n_k)$ and let σ be any irreducible subquotient of $s_\alpha(\pi)$. Then we can write $\sigma = \rho_1 \otimes \rho_2 \otimes \cdots \otimes \rho_k \otimes \rho$.

If all of the following inequalities:

$$\begin{aligned} n_1 e(\rho_1) &> 0, \\ n_1 e(\rho_1) + n_2 e(\rho_2) &> 0, \\ &\vdots \\ n_1 e(\rho_1) + n_2 e(\rho_2) + \cdots + n_k e(\rho_k) &> 0 \end{aligned}$$

hold for every α and σ as above, then π is a square integrable representation.

Also, if π is a square integrable representation, then all of given inequalities hold for any α and σ as above. The criterion for tempered representations is given by replacing every inequality above with $\geq$.

With $Spin(2n+1, F)$ we denote a simply - connected double covering of $SO(2n+1, F)$ as algebraic groups (for details see [12]) and let $f : Spin(2n+1, \overline{F}) \twoheadrightarrow SO(2n+1, \overline{F})$ be the central isogeny. In the exact sequence

$$1 \to \{\pm 1\} \hookrightarrow Spin(2n+1, F) \xrightarrow{f} SO(2n+1, F) \xrightarrow{\delta} F^\times/(F^\times)^2$$

homomorphism δ is called spinor norm. Spinor norm δ enables us to view every character of $F^\times/(F^\times)^2$ (i.e., every quadratic character of $F^\times$) as a character of $SO(2n+1, F)$. So, for the quadratic character ζ of $F^\times$, $\nu^{\alpha_1}\zeta \times \nu^{\alpha_2}\zeta \rtimes 1 \cong \zeta(\nu^{\alpha_1} \times \nu^{\alpha_2} \rtimes 1)$. Observe that, for $n = 1$, f gives an isomorphism between $SO(3, F)$ and $PGL(2, F)$.

In the same way as in [9, Chapter 2] we get the next two useful technical results:

- Fix an admissible representation π of $GL(2, F)$, suppose that π is of finite length. Let $m^*(\pi) = 1 \otimes \pi + \sum_i \pi_i^1 \otimes \pi_i^2 + \pi \otimes 1$, where $\sum_i \pi_i^1 \otimes \pi_i^2$ is a decomposition into a sum of irreducible representations. Now we have:

$$\begin{aligned} \mu^*(\pi \rtimes \sigma) &= 1 \otimes \pi \rtimes \sigma + \sum_i \pi_i^1 \otimes \pi_i^2 \rtimes \sigma + \sum_i \widetilde{\pi}_i^2 \otimes \pi_i^1 \rtimes \sigma + \\ &\quad + \pi \otimes \sigma + \widetilde{\pi} \otimes \sigma + \sum_i \pi_i^1 \times \widetilde{\pi}_i^2 \otimes \sigma \end{aligned}$$

- Fix an admissible representation π of $GL(1, F)$ and an admissible representation σ of $SO(3, F)$. We have:

$$\begin{aligned} \mu^*(\sigma) &= 1 \otimes \sigma + \sum_i \sigma_i^1 \otimes \sigma_i^2, \\ \mu^*(\pi \rtimes \sigma) &= 1 \otimes \pi \rtimes \sigma + \pi \otimes \sigma + \widetilde{\pi} \otimes \sigma + \sum_i \sigma_i^1 \otimes \pi \rtimes \sigma_i^2 + \\ &\quad + \sum_i \pi \times \sigma_i^1 \otimes \sigma_i^2 + \sum_i \sigma_i^1 \times \widetilde{\pi} \otimes \sigma_i^2. \end{aligned}$$

Here and subsequently, St_G and 1_G denote the Steinberg and the trivial representation of some reductive group G. Set of the unitary characters of $F^\times$ will be denoted by $\widehat{F^\times}$, while the set of not necessarily unitary characters will be denoted by $\widetilde{F^\times}$.

In the next proposition we list some well-known reducibility results. For instance, they can be found in [14, Chapter 11].

PROPOSITION 2.1. *Let χ, χ_1, χ_2 and $\zeta \in \widetilde{F^\times}$, where $\zeta^2 = 1_{F^\times}$ (i.e., where ζ is a quadratic character).*

The representation $\chi_1 \times \chi_2$ of $GL(2,F)$ reduces if and only if $\chi_1 = \nu^{\pm 1}\chi_2$. We have: $\nu^{\frac{1}{2}}\chi \times \nu^{-\frac{1}{2}}\chi = \chi St_{GL(2)} + \chi 1_{GL(2)}$.

The representation $\chi \rtimes 1$ of $SO(3,F)$ reduces if and only if $\chi^2 = \nu^{\pm 1}$. We have: $\nu^{\frac{1}{2}}\zeta \rtimes 1 = \zeta St_{SO(3)} + \zeta 1_{SO(3)}$.

Remark: from now on, quadratic characters will be denoted by ζ or ζ_i, $i \geq 1$.

3. Representations with support in minimal parabolic subgroup

First we have to determine the reducibility points of the principal series representations. It is an result of Keys [5] that unitary principal series for $SO(2n+1,F)$ are irreducible, so we investigate non-unitary principal series.

Decomposition of the long intertwining operator gives us almost all of the representations whose composition series we have to determine. All the other cases are analyzed separately. We recall basic properties:

The intertwining operator $(GL(2))$ $\chi_1 \times \chi_2 \to \chi_2 \times \chi_1$ has a pole (of order one) if and only if $\chi_1 = \chi_2$.

The intertwining operator $(SO(3))$ $\chi \rtimes 1 \to \chi^{-1} \rtimes 1$ has a pole (of order one) if and only if $\chi = \chi^{-1}$, i.e., $\chi^2 = 1_{F^\times}$.

First, in case (A), we consider non-unitary principal series that reduce on its $GL(2)-$part. After that, in case (B) we consider non-unitary principal series that reduce on its $SO(3)-$part.

(A) Let χ be the unitary character of $F^\times$ and $s \in \mathbb{R}, s > 0$.

Let $\nu^s \chi St_{GL(2)} \rtimes 1 \xrightarrow{A(s)} \nu^{-s}\chi^{-1} St_{GL(2)} \rtimes 1$ be a standard long intertwining operator, obtained by a meromorphic continuation of the integral intertwining operator.

Analyzing the decomposition of the long intertwining operator $A(s)$ into the short intertwining operators in Diagram 1, which is commutative, we get for which $s > 0$ and unitary characters χ this intertwining operator is not an isomorphism (observe that i_s and i'_s are inclusions and depend holomorphically on s for all s): We directly get that either $A_1(s)$, $A_2(s)$, $A_3(s)$ have poles or given representations reduce only for $s = \frac{1}{2}, \chi^2 = 1_{F^\times}$ and $s = 1, \chi^2 = 1_{F^\times}$.

$$\begin{tikzcd}
\nu^s\chi St_{GL(2)} \rtimes 1 \arrow[r, "i_s"] \arrow[ddd, "A(s)"'] & \nu^{s+\frac{1}{2}}\chi \times \nu^{s-\frac{1}{2}}\chi \rtimes 1 \arrow[d, "A_1(s)"] \\
& \nu^{s+\frac{1}{2}}\chi \times \nu^{-s+\frac{1}{2}}\chi^{-1} \rtimes 1 \arrow[d, "A_2(s)"] \\
& \nu^{-s+\frac{1}{2}}\chi^{-1} \times \nu^{s+\frac{1}{2}}\chi \rtimes 1 \arrow[d, "A_3(s)"] \\
\nu^{-s}\chi^{-1} St_{GL(2)} \rtimes 1 \arrow[r, "i'_s"] & \nu^{-s+\frac{1}{2}}\chi^{-1} \times \nu^{-s-\frac{1}{2}}\chi^{-1} \rtimes 1
\end{tikzcd}$$

Diagram 1.

In all other cases operators $A_i(s)$, $i = 1, 2, 3$ are holomorphic and isomorphisms, so $A(s) = A_3(s)A_2\ (s)A_1(s)|_{\nu^s\chi St_{GL(2)} \rtimes 1}$ is an isomorphism and representation $\nu^s\chi St_{GL(2)} \rtimes 1$ is irreducible. Thus, we have proved the following result:

Proposition 3.1. *Let $\chi \in \widehat{F^\times}$, $s \in \mathbb{R}$, $s > 0$. The representations $\nu^s\chi St_{GL(2)} \rtimes 1$ and $\nu^s\chi 1_{GL(2)} \rtimes 1$ are irreducible unless $(s, \chi) = (\frac{1}{2}, \zeta)$ or $(s, \chi) = (1, \zeta)$, where $\zeta^2 = 1_{F^\times}$. In $R(S)$ we have $\nu^{s+\frac{1}{2}}\chi \times \nu^{s-\frac{1}{2}}\chi \rtimes 1 = \nu^s\chi St_{GL(2)} \rtimes 1 + \nu^s\chi 1_{GL(2)} \rtimes 1$. Also, if $(s, \chi) \neq (\frac{1}{2}, \zeta)$ and $(s, \chi) \neq (1, \zeta)$, then $\nu^s\chi St_{GL(2)} \rtimes 1 = L(\nu^s\chi St_{GL(2)}, 1)$ and*

$$\nu^s\chi 1_{GL(2)} \rtimes 1 = \begin{cases} L(\nu^{s+\frac{1}{2}}\chi, \nu^{\frac{1}{2}-s}\chi^{-1}, 1) & \text{if } s < \frac{1}{2}, \\ L(\nu\chi, \chi \rtimes 1) & \text{if } s = \frac{1}{2}, \\ L(\nu^{s+\frac{1}{2}}\chi, \nu^{s-\frac{1}{2}}\chi, 1) & \text{if } s > \frac{1}{2}. \end{cases}$$

So, for $s > 0$, there are two representations whose composition series we still have to determine: $\nu\zeta \times \zeta \rtimes 1$ and $\nu^{\frac{3}{2}}\zeta \times \nu^{\frac{1}{2}}\zeta \rtimes 1$.

(B) Let χ and ζ be the unitary characters, $s \in \mathbb{R}, s > 0$.

Let $\nu^s\chi \rtimes \zeta St_{SO(3)} \xrightarrow{B(s)} \nu^{-s}\chi^{-1} \rtimes \zeta St_{SO(3)}$ be a standard long intertwining operator, obtained by meromorphic continuation of integral intertwining operator, holomorphic for $s > 0$.

Analyzing the decomposition of the long intertwining operator $B(s)$ into the short intertwining operators in Diagram 2, which is commutative, we get for which $s > 0$ and unitary characters χ this intertwining operator is not an isomorphism (observe that j_s and j'_s are inclusions and depend holomorphically on s for all s): we directly get that either $B_1(s)$, $B_2(s)$, $B_3(s)$ have poles

$$\begin{tikzcd}
\nu^s\chi \rtimes \zeta St_{SO(3)} \arrow[r, "j_s"] \arrow[ddd, "B(s)"'] & \nu^s\chi \times \nu^{\frac{1}{2}}\zeta \rtimes 1 \arrow[d, "B_1(s)"] \\
 & \nu^{\frac{1}{2}}\zeta \times \nu^s\chi \rtimes 1 \arrow[d, "B_2(s)"] \\
 & \nu^{\frac{1}{2}}\zeta \times \nu^{-s}\chi^{-1} \rtimes 1 \arrow[d, "B_3(s)"] \\
\nu^{-s}\chi^{-1} \rtimes \zeta St_{SO(3)} \arrow[r, "j'_s"] & \nu^{-s}\chi^{-1} \times \nu^{\frac{1}{2}}\zeta \rtimes 1
\end{tikzcd}$$

Diagram 2.

or given representations reduce only for $s = \frac{1}{2}, \chi = \zeta$; $s = \frac{1}{2}, \chi^2 = 1_{F^\times}$ and $s = \frac{3}{2}, \chi = \zeta$.

In all other cases $B_i(s)$, $i = 1, 2, 3$ are holomorphic and isomorphisms, so $B(s) = B_3(s)B_2(s)\ B_1(s)|_{\nu^s\chi\rtimes\zeta St_{SO(3)}}$ is also an isomorphism and representation $\nu^s\chi \rtimes \zeta St_{SO(3)}$ is irreducible. Summarizing, we have the following:

Proposition 3.2. *Let $\chi \in \widehat{F^\times}$, $s \in \mathbb{R}$, $s > 0$, $\zeta \in \widehat{F^\times}$ such that $\zeta^2 = 1_{F^\times}$. The representations $\nu^s\chi \rtimes \zeta St_{SO(3)}$ and $\nu^s\chi \rtimes \zeta 1_{SO(3)}$ are irreducible unless $(s, \chi) = (\frac{3}{2}, \zeta)$ or $(s, \chi) = (\frac{1}{2}, \zeta_1)$, where $\zeta_1^2 = 1_{F^\times}$. In $R(S)$ we have $\nu^s\chi \times \nu^{\frac{1}{2}}\zeta \rtimes 1 = \nu^s\chi \rtimes \zeta St_{SO(3)} + \nu^s\chi \rtimes \zeta 1_{SO(3)}$. Also, if $(s, \chi) \neq (\frac{3}{2}, \zeta)$ and $(s, \chi) \neq (\frac{1}{2}, \zeta_1)$, then $\nu^s\chi \rtimes \zeta St_{SO(3)} = L(\nu^s\chi, \zeta St_{SO(3)})$ and*

$$\nu^s\chi \rtimes \zeta 1_{SO(3)} = \begin{cases} L(\nu^{\frac{1}{2}}\zeta, \nu^s\chi, 1) & \text{if } 0 < s < \frac{1}{2}, \\ L(\nu^s\chi, \nu^{\frac{1}{2}}\zeta, 1) & \text{if } s \geq \frac{1}{2}, \\ L(\nu^{\frac{1}{2}}\zeta, \chi \rtimes 1) & \text{if } s = 0. \end{cases}$$

So, for $s > 0$, there are three representations whose composition series we still have to determine: $\nu^{\frac{1}{2}}\zeta \times \nu^{\frac{1}{2}}\zeta \rtimes 1$, $\nu^{\frac{3}{2}}\zeta \times \nu^{\frac{1}{2}}\zeta \rtimes 1$ and $\nu^{\frac{1}{2}}\zeta_1 \times \nu^{\frac{1}{2}}\zeta_2 \rtimes 1$.

All together, it remains to determine composition series of the following four representations:
(i) $\nu^{\frac{1}{2}}\zeta \times \nu^{\frac{1}{2}}\zeta \rtimes 1$, (ii) $\nu^{\frac{3}{2}}\zeta \times \nu^{\frac{1}{2}}\zeta \rtimes 1$, (iii) $\nu^{\frac{1}{2}}\zeta_1 \times \nu^{\frac{1}{2}}\zeta_2 \rtimes 1$ and (iv) $\nu\zeta \times \zeta \rtimes 1$.

This is mainly done by using already mentioned method of Jacquet modules which allows us to compare some known subquotients of this representations, such as Langlands quotients, Steinberg or trivial representations. Before starting a case - by - case examination, we summarize reducibility points of the principal series in the following proposition, which may be proved in much the same way as Theorem 7.1. in [13]:

Proposition 3.3. *Let $\chi_1, \chi_2 \in \widetilde{F^{\times}}$. The non-unitary principal series $\chi_1 \times \chi_2 \rtimes 1$ is reducible if and only if at least one of the following conditions hold:*

(1) $\chi_1 = \nu^{\pm 1}\chi_2$,
(2) $\chi_1^{-1} = \nu^{\pm 1}\chi_2$,
(3) $\chi_1 = \nu^{\pm\frac{1}{2}}\zeta_1$, $\zeta_1^2 = 1_{F^{\times}}$,
(4) $\chi_2 = \nu^{\pm\frac{1}{2}}\zeta_2$, $\zeta_2^2 = 1_{F^{\times}}$.

All of the following equations are given in semi-simplifications.

(i) First case is analyzed in full detail, writting all of the included Jacquet modules.

$$\begin{aligned}\nu^{\frac{1}{2}}\zeta \times \nu^{\frac{1}{2}}\zeta \rtimes 1 &= \nu^{\frac{1}{2}}\zeta \times \nu^{-\frac{1}{2}}\zeta \rtimes 1 = \zeta St_{GL(2)} \rtimes 1 + \zeta 1_{GL(2)} \rtimes 1\\ &= \nu^{\frac{1}{2}}\zeta \rtimes \zeta St_{SO(3)} + \nu^{\frac{1}{2}}\zeta \rtimes \zeta 1_{SO(3)}\end{aligned}$$

To find common irreducible subquotients of these representations, we first describe their Jacquet modules.

$$\begin{aligned}\mu^*(\nu^{\frac{1}{2}}\zeta \rtimes \zeta 1_{SO(3)}) &= 1 \otimes \nu^{\frac{1}{2}}\zeta \rtimes \zeta 1_{SO(3)} + \nu^{\frac{1}{2}}\zeta \otimes \zeta 1_{SO(3)}\\ &\quad + \nu^{-\frac{1}{2}}\zeta \otimes \zeta 1_{SO(3)} + \nu^{-\frac{1}{2}}\zeta \otimes \nu^{\frac{1}{2}}\zeta \rtimes 1\\ &\quad + \nu^{\frac{1}{2}}\zeta \times \nu^{-\frac{1}{2}}\zeta \otimes 1 + \nu^{-\frac{1}{2}}\zeta \times \nu^{-\frac{1}{2}}\zeta \otimes 1,\\ s_{min}(\nu^{\frac{1}{2}}\zeta \rtimes \zeta 1_{SO(3)}) &= 2\nu^{-\frac{1}{2}}\zeta \otimes \nu^{-\frac{1}{2}}\zeta \otimes 1 + \nu^{\frac{1}{2}}\zeta \otimes \nu^{-\frac{1}{2}}\zeta \otimes 1 + \nu^{-\frac{1}{2}}\zeta \otimes \nu^{\frac{1}{2}}\zeta \otimes 1,\\ \mu^*(\nu^{\frac{1}{2}}\zeta \rtimes \zeta St_{SO(3)}) &= 1 \otimes \nu^{\frac{1}{2}}\zeta \rtimes \zeta St_{SO(3)} + \nu^{\frac{1}{2}}\zeta \otimes \zeta St_{SO(3)}\\ &\quad + \nu^{-\frac{1}{2}}\zeta \otimes \zeta St_{SO(3)} + \nu^{\frac{1}{2}}\zeta \otimes \nu^{\frac{1}{2}}\zeta \rtimes 1\\ &\quad + \nu^{\frac{1}{2}}\zeta \times \nu^{\frac{1}{2}}\zeta \otimes 1 + \nu^{\frac{1}{2}}\zeta \times \nu^{-\frac{1}{2}}\zeta \otimes 1\\ &= 1 \otimes \nu^{\frac{1}{2}}\zeta \rtimes \zeta St_{SO(3)} + 2\nu^{\frac{1}{2}}\zeta \otimes \zeta St_{SO(3)}\\ &\quad + \nu^{-\frac{1}{2}}\zeta \otimes \zeta St_{SO(3)} + \nu^{\frac{1}{2}}\zeta \otimes \zeta 1_{SO(3)}\\ &\quad + \nu^{\frac{1}{2}}\zeta \times \nu^{\frac{1}{2}}\zeta \otimes 1 + \zeta St_{GL(2)} \otimes 1 + \zeta 1_{GL(2)} \otimes 1,\\ s_{min}(\nu^{\frac{1}{2}}\zeta \rtimes \zeta St_{SO(3)}) &= 2\nu^{\frac{1}{2}}\zeta \otimes \nu^{\frac{1}{2}}\zeta \otimes 1 + \nu^{\frac{1}{2}}\zeta \otimes \nu^{-\frac{1}{2}}\zeta \otimes 1 + \nu^{-\frac{1}{2}}\zeta \otimes \nu^{\frac{1}{2}}\zeta \otimes 1,\end{aligned}$$

$$\begin{aligned}
\mu^*(\zeta St_{GL(2)} \rtimes 1) &= 1 \otimes \zeta St_{GL(2)} \rtimes 1 + \nu^{\frac{1}{2}}\zeta \otimes \nu^{-\frac{1}{2}}\zeta \rtimes 1 \\
&\quad + \nu^{\frac{1}{2}}\zeta \otimes \nu^{\frac{1}{2}}\zeta \rtimes 1 + 2\zeta St_{GL(2)} \otimes 1 + \nu^{\frac{1}{2}}\zeta \times \nu^{\frac{1}{2}}\zeta \otimes 1 \\
&= 1 \otimes \zeta St_{GL(2)} \rtimes 1 + 2\nu^{\frac{1}{2}}\zeta \otimes \zeta St_{SO(3)} \\
&\quad + 2\nu^{\frac{1}{2}}\zeta \otimes \zeta 1_{SO(3)} + 2\zeta St_{GL(2)} \otimes 1 + \nu^{\frac{1}{2}}\zeta \times \nu^{\frac{1}{2}}\zeta \otimes 1, \\
s_{min}(\zeta St_{GL(2)} \rtimes 1) &= 2\nu^{\frac{1}{2}}\zeta \otimes \nu^{-\frac{1}{2}}\zeta \otimes 1 + 2\nu^{\frac{1}{2}}\zeta \otimes \nu^{\frac{1}{2}}\zeta \otimes 1, \\
\mu^*(\zeta 1_{GL(2)} \rtimes 1) &= 1 \otimes \zeta 1_{GL(2)} \rtimes 1 + \nu^{-\frac{1}{2}}\zeta \otimes \nu^{\frac{1}{2}}\zeta \rtimes 1 + \nu^{-\frac{1}{2}}\zeta \otimes \nu^{\frac{1}{2}}\zeta \rtimes 1 \\
&\quad + \zeta 1_{GL(2)} \otimes 1 + \zeta 1_{GL(2)} \otimes 1 + \nu^{-\frac{1}{2}}\zeta \times \nu^{-\frac{1}{2}}\zeta \otimes 1 \\
&= 1 \otimes \zeta 1_{GL(2)} \rtimes 1 + 2\nu^{-\frac{1}{2}}\zeta \otimes \zeta St_{SO(3)} \\
&\quad 2\nu^{-\frac{1}{2}}\zeta \otimes \zeta 1_{SO(3)} + 2\zeta 1_{GL(2)} \otimes 1 + \nu^{-\frac{1}{2}}\zeta \times \nu^{-\frac{1}{2}}\zeta \otimes 1, \\
s_{min}(\zeta 1_{GL(2)} \rtimes 1) &= 2\nu^{-\frac{1}{2}}\zeta \otimes \nu^{\frac{1}{2}}\zeta \otimes 1 + 2\nu^{-\frac{1}{2}}\zeta \otimes \nu^{-\frac{1}{2}}\zeta \otimes 1.
\end{aligned}$$

From Jacquet modules with respect to the minimal parabolic subgroup we conclude that the representations $\nu^{\frac{1}{2}}\zeta \rtimes \zeta 1_{SO(3)}$ and $\zeta St_{GL(2)} \rtimes 1$ have an irreducible subquotient in common (as in [13, Chapter 3]), which is different from both $\nu^{\frac{1}{2}}\zeta \rtimes \zeta 1_{SO(3)}$ and $\zeta St_{GL(2)} \rtimes 1$. For simplicity of notation, let τ_1 stand for this subqoutient.

We get directly: $s_{Sieg}(\tau_1) = \zeta St_{GL(2)} \otimes 1$, $s_{min}(\tau_1) = \nu^{\frac{1}{2}}\zeta \otimes \nu^{-\frac{1}{2}}\zeta \otimes 1$. τ_1 is irreducible and tempered.

Let υ denote the irreducible subquotient which $\nu^{\frac{1}{2}}\zeta \rtimes \zeta St_{SO(3)}$ and $\zeta 1_{GL(2)} \rtimes 1$ have in common. From Jacquet modules we obtain directly: $s_{Sieg}(\upsilon) = \zeta 1_{GL(2)} \otimes 1$, $s_{min}(\upsilon) = \nu^{-\frac{1}{2}}\zeta \otimes \nu^{\frac{1}{2}}\zeta \otimes 1$.

Because of the following inclusions: $L(\nu^{\frac{1}{2}}\zeta, \zeta St_{SO(3)}) \hookrightarrow \nu^{-\frac{1}{2}}\zeta \rtimes \zeta St_{SO(3)}$ and $\nu^{-\frac{1}{2}}\zeta \rtimes \zeta St_{SO(3)} \hookrightarrow \nu^{-\frac{1}{2}}\zeta \times \nu^{\frac{1}{2}}\zeta \rtimes 1$, Frobenius reciprocity implies $s_{min}(L(\nu^{\frac{1}{2}}\zeta, \zeta St_{SO(3)})) \geq \nu^{-\frac{1}{2}}\zeta \otimes \nu^{\frac{1}{2}}\zeta \otimes 1$. Multiplicity of $\nu^{-\frac{1}{2}}\zeta \otimes \nu^{\frac{1}{2}}\zeta \otimes 1$ in $s_{min}(\nu^{\frac{1}{2}}\zeta \rtimes \zeta St_{SO(3)})$ is equal to 1, so $\upsilon = L(\nu^{\frac{1}{2}}\zeta, \zeta St_{SO(3)})$.
Since $\zeta 1_{GL(2)} \rtimes 1 \twoheadrightarrow L(\nu^{\frac{1}{2}}\zeta, \nu^{\frac{1}{2}}\zeta, 1)$ and $L(\nu^{\frac{1}{2}}\zeta, \nu^{\frac{1}{2}}\zeta, 1) \hookrightarrow \nu^{-\frac{1}{2}}\zeta \times \nu^{-\frac{1}{2}}\zeta \rtimes 1$ we conclude that $s_{min}(L(\nu^{\frac{1}{2}}\zeta, \nu^{\frac{1}{2}}\zeta, 1)) \geq \nu^{-\frac{1}{2}}\zeta \otimes \nu^{-\frac{1}{2}}\zeta \otimes 1$.

Now it is obvious that $L(\nu^{\frac{1}{2}}\zeta, \nu^{\frac{1}{2}}, 1) \subseteq \nu^{\frac{1}{2}}\zeta \rtimes \zeta 1_{SO(3)} \cap \zeta 1_{GL(2)} \rtimes 1$ and $s_{Sieg}(L(\nu^{\frac{1}{2}}\zeta, \nu^{\frac{1}{2}}\zeta, 1)) \geq \nu^{-\frac{1}{2}}\zeta \times \nu^{-\frac{1}{2}}\zeta \otimes 1$.

Representations $\zeta 1_{GL(2)} \otimes 1$ and $\zeta St_{GL(2)} \otimes 1$ are irreducible and unitary, multiplicity of $\zeta 1_{GL(2)} \otimes 1$ in $s_{Sieg}(\zeta 1_{GL(2)} \rtimes 1)$ is equal to 2, which implies that $\zeta 1_{GL(2)} \rtimes 1$ is a representation of length 2. Now we get directly:

$$\begin{aligned}
\zeta St_{GL(2)} \rtimes 1 &= \tau_1 + L(\widehat{\nu^{\frac{1}{2}}\zeta, \nu^{\frac{1}{2}}}\zeta, 1), \\
\nu^{\frac{1}{2}}\zeta \rtimes \zeta St_{SO(3)} &= L(\nu^{\frac{1}{2}}\zeta, \zeta St_{SO(3)}) + L(\widehat{\nu^{\frac{1}{2}}\zeta, \nu^{\frac{1}{2}}}\zeta, 1).
\end{aligned}$$

Again, from Jacquet modules we see that $L(\widehat{\nu^{\frac{1}{2}}\zeta, \nu^{\frac{1}{2}}}\zeta, 1)$ is tempered representation and we denote it by τ_2. We summarize the above discussion as follows:

Proposition 3.4. *Let $\zeta \in \widehat{F^\times}$ such that $\zeta^2 = 1_{F^\times}$. Then the representations $\zeta 1_{GL(2)} \rtimes 1$, $\zeta St_{GL(2)} \rtimes 1$, $\nu^{\frac{1}{2}}\zeta \rtimes \zeta 1_{SO(3)}$ and $\nu^{\frac{1}{2}}\zeta \rtimes \zeta St_{SO(3)}$ are reducible and $\nu^{\frac{1}{2}}\zeta \times \nu^{\frac{1}{2}}\zeta \rtimes 1$ is a representation of length 4. The representations $\zeta St_{GL(2)} \rtimes 1$ and $\nu^{\frac{1}{2}}\zeta \rtimes \zeta 1_{SO(3)}$ (respectively $\nu^{\frac{1}{2}}\zeta \rtimes \zeta St_{SO(3)}$) have exactly one irreducible subquotient in common. That subquotient is tempered, and is denoted by τ_1 (respectively τ_2). In $R(S)$ we have:*

$$\nu^{\frac{1}{2}}\zeta \times \nu^{\frac{1}{2}}\zeta \rtimes 1 = \zeta 1_{GL(2)} \rtimes 1 + \zeta St_{GL(2)} \rtimes 1 = \nu^{\frac{1}{2}}\zeta \rtimes \zeta 1_{SO(3)} + \nu^{\frac{1}{2}}\zeta \rtimes \zeta St_{SO(3)}$$

and

$$\begin{aligned} \zeta 1_{GL(2)} \rtimes 1 &= L(\nu^{\frac{1}{2}}\zeta, \nu^{\frac{1}{2}}, \rtimes 1) + L(\nu^{\frac{1}{2}}\zeta, \zeta St_{SO(3)}), \\ \zeta St_{GL(2)} \rtimes 1 &= \tau_1 + \tau_2, \\ \nu^{\frac{1}{2}}\zeta \rtimes \zeta 1_{SO(3)} &= L(\nu^{\frac{1}{2}}\zeta, \nu^{\frac{1}{2}}\zeta, 1) + \tau_1, \\ \nu^{\frac{1}{2}}\zeta \rtimes \zeta St_{SO(3)} &= L(\nu^{\frac{1}{2}}\zeta, \zeta St_{SO(3)}) + \tau_2. \end{aligned}$$

(ii) In this case some older results of Casselman are used. We have already observed that $\nu^{\frac{3}{2}}\zeta \times \nu^{\frac{1}{2}}\zeta \rtimes 1 \cong \zeta(\nu^{\frac{3}{2}} \times \nu^{\frac{1}{2}} \rtimes 1)$. Since $St_{SO(5)} \hookrightarrow \nu^{\frac{3}{2}} \times \nu^{\frac{1}{2}} \rtimes 1$, [2] implies that $\nu^{\frac{3}{2}} \times \nu^{\frac{1}{2}} \rtimes 1$ is the representation of the length $2^2 = 4$, so as $\nu^{\frac{3}{2}}\zeta \times \nu^{\frac{1}{2}}\zeta \rtimes 1$. Irreducible subquotients of the representation $\nu^{\frac{3}{2}}\zeta \times \nu^{\frac{1}{2}}\zeta \rtimes 1$ are $\zeta St_{SO(5)}$ (which is square - integrable), $\zeta 1_{SO(5)}$, $L(\nu\zeta St_{GL(2)}, 1)$ and $L(\nu^{\frac{3}{2}}\zeta, \zeta St_{SO(3)})$. Using Jacquet modules we easily get the following proposition:

Proposition 3.5. *Let $\zeta \in \widehat{F^\times}$ such that $\zeta^2 = 1_{F^\times}$. Then the representations $\nu^{\frac{3}{2}}\zeta \rtimes \zeta 1_{SO(3)}$, $\nu^{\frac{3}{2}}\zeta \rtimes \zeta St_{SO(3)}$, $\nu\zeta 1_{GL(2)} \rtimes 1$ and $\nu\zeta St_{GL(2)} \rtimes 1$ are reducible and $\nu^{\frac{3}{2}}\zeta \times \nu^{\frac{1}{2}}\zeta \rtimes 1$ is a representation of length 4. In $R(S)$ we have:*

$$\nu^{\frac{3}{2}}\zeta \times \nu^{\frac{1}{2}}\zeta \rtimes 1 = \nu^{\frac{3}{2}}\zeta \rtimes \zeta 1_{SO(3)} + \nu^{\frac{3}{2}}\zeta \rtimes \zeta St_{SO(3)} = \nu\zeta 1_{GL(2)} \rtimes 1 + \nu\zeta St_{GL(2)} \rtimes 1$$

and

$$\begin{aligned} \nu^{\frac{3}{2}}\zeta \rtimes \zeta 1_{SO(3)} &= \zeta 1_{SO(5)} + L(\nu\zeta St_{GL(2)}, 1), \\ \nu^{\frac{3}{2}}\zeta \rtimes \zeta St_{SO(3)} &= \zeta St_{SO(5)} + L(\nu^{\frac{3}{2}}\zeta, \zeta St_{SO(3)}), \\ \nu\zeta 1_{GL(2)} \rtimes 1 &= \zeta 1_{SO(5)} + L(\nu^{\frac{3}{2}}\zeta, \zeta St_{SO(3)}), \\ \nu\zeta St_{GL(2)} \rtimes 1 &= \zeta St_{SO(5)} + L(\nu\zeta St_{GL(2)}, 1). \end{aligned}$$

(iii) Let $\zeta_1, \zeta_2 \in \widehat{F^\times}$ such that $\zeta_i^2 = 1_{F^\times}$, $i = 1, 2$ ($\zeta_1 \neq \zeta_2$)

$$\begin{aligned}\nu^{\frac{1}{2}}\zeta_1 \times \nu^{\frac{1}{2}}\zeta_2 \rtimes 1 \simeq \nu^{\frac{1}{2}}\zeta_2 \times \nu^{\frac{1}{2}}\zeta_1 \rtimes 1 &= \nu^{\frac{1}{2}}\zeta_1 \rtimes \zeta_2 St_{SO(3)} + \nu^{\frac{1}{2}}\zeta_1 \rtimes \zeta_2 1_{SO(3)} \\ &= \nu^{\frac{1}{2}}\zeta_2 \rtimes \zeta_1 St_{SO(3)} + \nu^{\frac{1}{2}}\zeta_2 \rtimes \zeta_1 1_{SO(3)}.\end{aligned}$$

From $s_{Sieg}(\nu^{\frac{1}{2}}\zeta_1 \rtimes \zeta_2 St_{SO(3)}) = \nu^{\frac{1}{2}}\zeta_1 \times \nu^{\frac{1}{2}}\zeta_2 \otimes 1 + \nu^{\frac{1}{2}}\zeta_2 \times \nu^{-\frac{1}{2}}\zeta_1 \otimes 1$ we conclude that $\nu^{\frac{1}{2}}\zeta_1 \rtimes \zeta_2 St_{SO(3)}$ is a representation of length less then or equal 2. In the same way we can conclude that all the above representations are of the length less then or equal 2.

We take a look at the following sequence of the short intertwining operators:

$$\begin{aligned}\nu^{\frac{1}{2}}\zeta_1 \times \nu^{\frac{1}{2}}\zeta_2 \rtimes 1 \xrightarrow{A_1} \nu^{\frac{1}{2}}\zeta_1 \times \nu^{-\frac{1}{2}}\zeta_2 \rtimes 1 &\xrightarrow{A_2} \nu^{-\frac{1}{2}}\zeta_2 \times \nu^{\frac{1}{2}}\zeta_1 \rtimes 1 \\ \xrightarrow{A_3} \nu^{-\frac{1}{2}}\zeta_2 \times \nu^{-\frac{1}{2}}\zeta_1 \rtimes 1 &\xrightarrow{A_4} \nu^{-\frac{1}{2}}\zeta_1 \times \nu^{-\frac{1}{2}}\zeta_2 \rtimes 1\end{aligned}$$

Notice that A_2 and A_4 in the above sequence are isomorphisms.

Of course, $Im(A_4 \circ A_3 \circ A_2 \circ A_1)$ is equal to $L(\nu^{\frac{1}{2}}\zeta_1, \nu^{\frac{1}{2}}\zeta_2, 1)$. Since A_4 is an isomorphism, this implies that $ImA_3|_{Im(A_2 \circ A_1)} = L(\nu^{\frac{1}{2}}\zeta_1, \nu^{\frac{1}{2}}\zeta_2, 1)$.

Also, $KerA_1 = \nu^{\frac{1}{2}}\zeta_1 \rtimes \zeta_2 St_{SO(3)}$, $ImA_1 = \nu^{\frac{1}{2}}\zeta_1 \rtimes \zeta_2 1_{SO(3)}$ and $KerA_3 = \nu^{-\frac{1}{2}}\zeta_2 \rtimes \zeta_1 St_{SO(3)}$. This leaves us two possibilities:

- $KerA_3 \cap ImA_2|_{ImA_1} = 0$.

  We see at once that ImA_3 is equal to $L(\nu^{\frac{1}{2}}\zeta_1, \nu^{\frac{1}{2}}\zeta_2, 1)$. But, $ImA_3 = \nu^{-\frac{1}{2}}\zeta_2 \rtimes \zeta_1 1_{SO(3)}$ also. Obviously, $\nu^{\frac{1}{2}}\zeta_2 \rtimes \zeta_1 1_{SO(3)}$ is then an irreducible representation, while Aubert duality implies that $\nu^{\frac{1}{2}}\zeta_2 \rtimes \zeta_1 St_{SO(3)}$ is also irreducible and is equal to its Langlands quotient.

  This gives $\nu^{\frac{1}{2}}\zeta_1 \times \nu^{\frac{1}{2}}\zeta_2 \rtimes 1 = L(\nu^{\frac{1}{2}}\zeta_1, \nu^{\frac{1}{2}}\zeta_2, 1) + L(\nu^{\frac{1}{2}}\zeta_2, \zeta_1 St_{SO(3)})$.

  But, the representation $L(\nu^{\frac{1}{2}}\zeta_1, \zeta_2 St_{SO(3)})$ (the Langlands quotient of $\nu^{\frac{1}{2}}\zeta_1 \rtimes \zeta_2 St_{SO(3)}$) is also a composition factor of $\nu^{\frac{1}{2}}\zeta_1 \times \nu^{\frac{1}{2}}\zeta_2 \rtimes 1$, different from both $L(\nu^{\frac{1}{2}}\zeta_1, \nu^{\frac{1}{2}}\zeta_2, 1)$ and $L(\nu^{\frac{1}{2}}\zeta_2, \zeta_1 St_{SO(3)})$.

  Therefore it follows that:
- $KerA_3 \cap ImA_2|_{ImA_1} \neq 0$.

  Clearly, $\nu^{-\frac{1}{2}}\zeta_2 \rtimes \zeta_1 St_{SO(3)} \cap \nu^{\frac{1}{2}}\zeta_1 \rtimes \zeta_2 1_{SO(3)} \neq 0$.

  Since $L(\nu^{\frac{1}{2}}\zeta_2, \zeta_1 St_{SO(3)}) \hookrightarrow \nu^{-\frac{1}{2}}\zeta_2 \rtimes \zeta_1 St_{SO(3)}$, it follows easily that $\nu^{-\frac{1}{2}}\zeta_2 \rtimes \zeta_1 St_{SO(3)} \cap \nu^{\frac{1}{2}}\zeta_1 \rtimes \zeta_2 1_{SO(3)} = L(\nu^{\frac{1}{2}}\zeta_2, \zeta_1 St_{SO(3)})$ and $\nu^{\frac{1}{2}}\zeta_1 \rtimes \zeta_2 1_{SO(3)} = L(\nu^{\frac{1}{2}}\zeta_1, \nu^{\frac{1}{2}}\zeta_2, 1) + L(\nu^{\frac{1}{2}}\zeta_2, \zeta_1 St_{SO(3)})$.

  Also, since

$$L(\nu^{\frac{1}{2}}\zeta_1, \nu^{\frac{1}{2}}\zeta_2, 1) \hookrightarrow \nu^{-\frac{1}{2}}\zeta_1 \times \nu^{-\frac{1}{2}}\zeta_2 \rtimes 1,$$
$$L(\nu^{\frac{1}{2}}\zeta_2, \zeta_1 St_{SO(3)}) \hookrightarrow \nu^{-\frac{1}{2}}\zeta_2 \times \nu^{\frac{1}{2}}\zeta_1 \rtimes 1,$$

Frobenius reciprocity implies

$$s_{min}(L(\nu^{\frac{1}{2}}\zeta_1, \nu^{\frac{1}{2}}\zeta_2, 1)) \geq \nu^{-\frac{1}{2}}\zeta_1 \otimes \nu^{-\frac{1}{2}}\zeta_2 \otimes 1 + \nu^{-\frac{1}{2}}\zeta_2 \otimes \nu^{-\frac{1}{2}}\zeta_1 \otimes 1$$
$$s_{min}(L(\nu^{\frac{1}{2}}\zeta_2, \zeta_1 St_{SO(3)})) \geq \nu^{-\frac{1}{2}}\zeta_2 \otimes \nu^{\frac{1}{2}}\zeta_1 \otimes 1 + \nu^{\frac{1}{2}}\zeta_1 \otimes \nu^{-\frac{1}{2}}\zeta_2 \otimes 1.$$

This implies $\nu^{\frac{1}{2}}\zeta_2 \rtimes \zeta_1 1_{SO(3)} = L(\nu^{\frac{1}{2}}\zeta_1, \nu^{\frac{1}{2}}\zeta_2, 1) + L(\nu^{\frac{1}{2}}\zeta_1, \zeta_2 St_{SO(3)})$. Let $\sigma \leq \nu^{\frac{1}{2}}\zeta_2 \rtimes \zeta_1 St_{SO(3)}$ such that $\nu^{\frac{1}{2}}\zeta_2 \times \nu^{\frac{1}{2}}\zeta_1 \otimes 1 = s_{Sieg}(\sigma)$ (this is not contained in the Jacquet module of $L(\nu^{\frac{1}{2}}\zeta_2, \zeta_1 St_{SO(3)})$). Clearly, σ is irreducible and square-integrable, while $\nu^{\frac{1}{2}}\zeta_2 \rtimes \zeta_1 St_{SO(3)} = L(\nu^{\frac{1}{2}}\zeta_2, \zeta_1 St_{SO(3)}) + \sigma$. Using Jacquet modules we easily obtain that $\sigma \leq \nu^{\frac{1}{2}}\zeta_1 \rtimes \zeta_2 St_{SO(3)}$, $\sigma \neq L(\nu^{\frac{1}{2}}\zeta_2, \zeta_1 St_{SO(3)})$.

This analysis leads to the following:

Proposition 3.6. *Let $\zeta_1, \zeta_2 \in \widehat{F^\times}$ such that $\zeta_i^2 = 1_{F^\times}$, $i = 1, 2$ ($\zeta_1 \neq \zeta_2$). Than the representations $\nu^{\frac{1}{2}}\zeta_2 \rtimes \zeta_1 1_{SO(3)}$, $\nu^{\frac{1}{2}}\zeta_2 \rtimes \zeta_1 St_{SO(3)}$, $\nu^{\frac{1}{2}}\zeta_1 \rtimes \zeta_2 1_{SO(3)}$ and $\nu^{\frac{1}{2}}\zeta_1 \rtimes \zeta_2 St_{SO(3)}$ are reducible and $\nu^{\frac{1}{2}}\zeta_1 \times \nu^{\frac{1}{2}}\zeta_2 \rtimes 1$ is a representation of length 4. $\nu^{\frac{1}{2}}\zeta_1 \rtimes \zeta_2 St_{SO(3)}$ and $\nu^{\frac{1}{2}}\zeta_2 \rtimes \zeta_1 St_{SO(3)}$ have exactly one irreducible subquotient in common. That subquotient is square-integrable, we denote it by σ. In $R(S)$ we have:*

$$\begin{aligned} \nu^{\frac{1}{2}}\zeta_1 \times \nu^{\frac{1}{2}}\zeta_2 \rtimes 1 &= \nu^{\frac{1}{2}}\zeta_1 \rtimes \zeta_2 St_{SO(3)} + \nu^{\frac{1}{2}}\zeta_1 \rtimes \zeta_2 1_{SO(3)} \\ &= \nu^{\frac{1}{2}}\zeta_2 \rtimes \zeta_1 St_{SO(3)} + \nu^{\frac{1}{2}}\zeta_2 \rtimes \zeta_1 1_{SO(3)} \end{aligned}$$

and

$$\begin{aligned} \nu^{\frac{1}{2}}\zeta_2 \rtimes \zeta_1 1_{SO(3)} &= L(\nu^{\frac{1}{2}}\zeta_1, \zeta_2 St_{SO(3)}) + L(\nu^{\frac{1}{2}}\zeta_1, \nu^{\frac{1}{2}}\zeta_2, 1), \\ \nu^{\frac{1}{2}}\zeta_2 \rtimes \zeta_1 St_{SO(3)} &= L(\nu^{\frac{1}{2}}\zeta_2, \zeta_1 St_{SO(3)}) + \sigma, \\ \nu^{\frac{1}{2}}\zeta_1 \rtimes \zeta_2 1_{SO(3)} &= L(\nu^{\frac{1}{2}}\zeta_2, \zeta_1 St_{SO(3)}) + L(\nu^{\frac{1}{2}}\zeta_1, \nu^{\frac{1}{2}}\zeta_2, 1), \\ \nu^{\frac{1}{2}}\zeta_1 \rtimes \zeta_2 St_{SO(3)} &= L(\nu^{\frac{1}{2}}\zeta_1, \zeta_2 St_{SO(3)}) + \sigma. \end{aligned}$$

(iv) This happens to be the case that can be solved directly, without using Jacquet modules of $SO(5, F)$. In $R(S)$ we have: $\nu\zeta \times \zeta \rtimes 1 = \nu^{\frac{1}{2}}\zeta St_{GL(2)} \rtimes 1 + \nu^{\frac{1}{2}}\zeta 1_{GL(2)} \rtimes 1$. From [14], Proposition 6.3. and Corollary 6.4., we get that both $\nu^{\frac{1}{2}}\zeta St_{GL(2)} \rtimes 1$ and $\nu^{\frac{1}{2}}\zeta 1_{GL(2)} \rtimes 1$ are irreducible.

Proposition 3.7. *Let $\zeta \in \widehat{F^\times}$ such that $\zeta^2 = 1_{F^\times}$. Then the representations $\nu^{\frac{1}{2}}\zeta St_{GL(2)} \rtimes 1$ and $\nu^{\frac{1}{2}}\zeta 1_{GL(2)} \rtimes 1$ are irreducible and in $R(S)$ we have:*

$$\nu\zeta \times \zeta \rtimes 1 = \nu^{\frac{1}{2}}\zeta St_{GL(2)} \rtimes 1 + \nu^{\frac{1}{2}}\zeta 1_{GL(2)} \rtimes 1$$

and

$$\nu^{\frac{1}{2}}\zeta St_{GL(2)} \rtimes 1 = L(\nu^{\frac{1}{2}}\zeta St_{GL(2)}, 1),$$
$$\nu^{\frac{1}{2}}\zeta 1_{GL(2)} \rtimes 1 = L(\nu\zeta, \zeta \rtimes 1).$$

We still haven't covered all the cases, because we have started from the representations $\nu^s\chi St_{GL(2)} \rtimes 1$ and $\nu^s\chi \rtimes \zeta St_{SO(3)}$, for $s > 0$. We have to see what happens when $s = 0$ (in the case of the so-called generalized unitary principal series), i.e., we have to determine composition series of the representations $\nu^{\frac{1}{2}}\chi \times \nu^{-\frac{1}{2}}\chi \rtimes 1$ and $\chi \times \nu^{\frac{1}{2}}\zeta \rtimes 1$. These composition series are obtained in the following two propositions:

Proposition 3.8. *Let $\chi \in \widehat{F^\times}$, such that $\chi^2 \neq 1_{F^\times}$. Then the both representations $\chi St_{GL(2)} \rtimes 1$ and $\chi 1_{GL(2)} \rtimes 1$ are irreducible. In $R(S)$ we have $\nu^{\frac{1}{2}}\chi \times \nu^{-\frac{1}{2}}\chi \rtimes 1 = \chi St_{GL(2)} \rtimes 1 + \chi 1_{GL(2)} \rtimes 1$. For Langlands parameters we have $\chi St_{GL(2)} \rtimes 1 = L(\chi St_{GL(2)} \rtimes 1)$ and $\chi 1_{GL(2)} \rtimes 1 = L(\nu^{\frac{1}{2}}\chi, \nu^{\frac{1}{2}}\chi^{-1}, 1)$.*

Proof. Similarly as before, we have:

$$\begin{aligned}\mu^*(\chi St_{GL(2)} \rtimes 1) = {} & 1 \otimes \chi St_{GL(2)} \rtimes 1 + \nu^{\frac{1}{2}}\chi \otimes \nu^{-\frac{1}{2}}\chi \rtimes 1 \\ & + \nu^{\frac{1}{2}}\chi^{-1} \otimes \nu^{\frac{1}{2}}\chi \rtimes 1 + \chi St_{GL(2)} \otimes 1 \\ & + \chi^{-1} St_{GL(2)} \otimes 1 + \nu^{\frac{1}{2}}\chi \times \nu^{\frac{1}{2}}\chi^{-1} \otimes 1\end{aligned}$$

Since $\chi \neq \chi^{-1}$ ($\chi^2 \neq 1_{F^\times}$), all the summands in the previous relation are irreducible. Also, since $\chi St_{GL(2)} \rtimes 1$ is an unitary representation and multiplicity of $\chi St_{GL(2)} \otimes 1$ in $s_{Sieg}(\chi St_{GL(2)} \rtimes 1)$ is equal to 1, $\chi St_{GL(2)} \rtimes 1$ is irreducible. ∎

If $\chi = \chi^{-1}$, we just put ζ instead of χ and get $\zeta St_{GL(2)} \rtimes 1 \hookrightarrow \nu^{\frac{1}{2}}\zeta \times \nu^{-\frac{1}{2}}\zeta \rtimes 1 = \nu^{\frac{1}{2}}\zeta \times \nu^{\frac{1}{2}}\zeta \rtimes 1$ which has been solved in (i).

Proposition 3.9. *Let $\chi \in \widehat{F^\times}$. Then the both representations $\chi \rtimes \zeta St_{SO(3)}$ and $\chi \rtimes \zeta 1_{SO(3)}$ are irreducible. In $R(S)$ we have $\chi \times \nu^{\frac{1}{2}}\zeta \rtimes 1 = \chi \rtimes \zeta St_{SO(3)} + \chi \rtimes \zeta 1_{SO(3)}$. In terms of the Langlands parameters we have $\chi \rtimes \zeta St_{SO(3)} = L(\chi \rtimes \zeta St_{SO(3)})$ and $\chi \rtimes \zeta 1_{SO(3)} = L(\nu^{\frac{1}{2}}\zeta, \chi \rtimes 1)$.*

Proof. Observe that for $\chi \in \widehat{F^\times}$ we have:

$$\begin{aligned}\mu^*(\chi \rtimes \zeta St_{SO(3)}) = {} & 1 \otimes \chi \rtimes \zeta St_{SO(3)} + \chi \otimes \zeta St_{SO(3)} + \chi^{-1} \otimes \zeta St_{SO(3)} \\ & + \nu^{\frac{1}{2}}\zeta \otimes \chi \rtimes 1 + \chi \times \nu^{\frac{1}{2}}\zeta \otimes 1 + \nu^{\frac{1}{2}}\zeta \times \chi^{-1} \otimes 1\end{aligned}$$

Let π be an irreducible subquotient of $\chi \rtimes \zeta St_{SO(3)}$ such that $\nu^{\frac{1}{2}}\zeta \otimes \chi \rtimes 1 \leq s_{(1)}(\pi)$. Then $\nu^{\frac{1}{2}}\zeta \otimes \chi \otimes 1 + \nu^{\frac{1}{2}}\zeta \otimes \chi^{-1} \otimes 1 \leq s_{min}(\pi)$ and $\chi \times \nu^{\frac{1}{2}}\zeta \otimes 1 + \nu^{\frac{1}{2}}\zeta \times \chi^{-1} \otimes 1 \leq s_{Sieg}(\pi)$. This implies $\pi \simeq \chi \rtimes \zeta St_{SO(3)}$ and $\chi \rtimes \zeta St_{SO(3)}$ is irreducible. ∎

4. Representations with support in maximal parabolic subgroups

First we consider the case of the representations which have cuspidal support in P_{Sieg}.

Proposition 4.1. *Let ρ be an irreducible unitarizable supercuspidal representation of $GL(2,F)$. There is at most one $s \geq 0$ such that $\nu^s\rho \rtimes 1$ reduces.*

(i) *If $\rho \neq \widetilde{\rho}$ then $\rho \rtimes 1$ is irreducible. Also, the representations $\nu^s\rho \rtimes 1$, $s > 0$ are irreducible.*

(ii) *If $\rho = \widetilde{\rho}$ and $\rho \rtimes 1$ reduces (that is the case when $\omega_\rho = 1$, where ω_ρ is the central character of ρ), all of the representations $\nu^s\rho \rtimes 1$, $s > 0$ are irreducible.*

(iii) *If $\rho = \widetilde{\rho}$ and $\rho \rtimes 1$ is irreducible (that is the case when $\omega_\rho \neq 1$), then unique $s > 0$ such that the representation $\nu^s\rho \rtimes 1$ reduces is equal to $\frac{1}{2}$.*

Proof. The unique $s > 0$ such that representation $\nu^s\rho \rtimes 1$ reduces is obtained by determining the poles of the Plancherel measure, which in this case coincide with the poles of

$$\frac{L(1-2s,\rho,Sym^2\rho_2)L(1+2s,\rho,Sym^2\rho_2)}{L(2s,\rho,Sym^2\rho_2)L(-2s,\rho,Sym^2\rho_2)}$$

(this quotient is equal to the Plancherel measure $\mu(s,\rho)$ up to a monomial in q^s). $\square$

Now we consider the case of the representations which have cuspidal support in $P_{(1)}$.

Proposition 4.2. *Let $\chi \in \widehat{F^\times}$ and let σ be an irreducible unitarizable supercuspidal representation of $SO(3,F) \simeq PGL(2,F)$ (observe that σ is generic). There is at most one $s \geq 0$ such that $\nu^s\chi \rtimes \sigma$ reduces.*

(i) *If $\chi \neq \chi^{-1}$ then $\chi \rtimes \sigma$ is irreducible. Also, the representations $\nu^s\chi \rtimes \sigma$ are irreducible for $s > 0$.*

(ii) *If $\chi = \chi^{-1}$, then $\nu^s\chi \rtimes \sigma$ reduces only for $s = \frac{1}{2}$.*

Proof. This unique point of reducibility $s = \frac{1}{2}$ is obtained by determining the poles of

$$\frac{L(1-2s,\chi,Sym^2\rho_1)L(1+2s,\chi,Sym^2\rho_1)}{L(2s,\chi,Sym^2\rho_1)L(-2s,\chi,Sym^2\rho_1)} \cdot \frac{L(1-s,\chi\times\widetilde{\sigma})L(1+s,\chi\times\sigma)}{L(s,\chi\times\sigma)L(-s,\chi\times\widetilde{\sigma})}$$

(this is equal to the Plancherel measure $\mu(s,\chi\otimes\sigma)$ up to a monomial in q^s). The L-function $L(1-2s,\chi,Sym^2\rho_1)$ has a pole for $s=\frac{1}{2}$, while all the other

L-functions which are the factors of $\mu(s, \chi \otimes \sigma)$ are holomorphic and non-zero for $s = \frac{1}{2}$ ([10, Proposition 7.3]). $\square$

References

[1] A.-M. Aubert, *Dualité dans le groupe de Grothendieck de la catégorie des représentations lisses de longueur finie d'un groupe réductif p-adique*, Trans. Amer. Math. Soc. **347** (1995), 2179–2189.

[2] W. Casselman, *Introduction to the theory of admissible representations of p-adic reductive groups*, preprint.

[3] M. Hanzer, *The unitary dual of the Hermitian quaternionic group of split rank 2*, Pacific J. Math. **226** (2006), 353–388.

[4] M. Hanzer and G. Muić, *On an algebraic approach to the Zelevinsky classification for classical p-adic groups*, J. Algebra **320** (2008), 3206–3231.

[5] C. D. Keys, *On the decomposition of reducible principal series representations of p-adic Chevalley groups*, Pacific J. Math. **101** (1982), 351–388.

[6] I. Matić, *The unitary dual of p-adic* SO*(5,F)*, Proc. Amer. Math. Soc. **138** (2010), 759–767.

[7] G. Muić, *The unitary dual of p-adic G_2*, Duke Math. J. **90** (1997), 465–493.

[8] G. Muić, *A geometric construction of intertwining operators for reductive p-adic groups*, Manuscripta Math. **125** (2008), 241–272.

[9] P. J. Sally, Jr. and M. Tadić, *Induced representations and classifications for $GSp(2,F)$ and $Sp(2,F)$*, Mém. Soc. Math. France (N.S.) **52** (1993), 75–133.

[10] F. Shahidi, *A proof of Langlands' conjecture on Plancherel measures; complementary series for p-adic groups*, Ann. of Math. (2) **132** (1990), 273–330.

[11] F. Shahidi, *Twisted endoscopy and reducibility of induced representations for p-adic groups*, Duke Math. J. **66** (1992), 1–41.

[12] T. A. Springer, *Reductive groups*, in Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1, Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., Providence, 1979, 3–27.

[13] M. Tadić, *Representations of p-adic symplectic groups*, Compositio Math. **90** (1994), 123–181.

[14] M. Tadić, *On reducibility of parabolic induction*, Israel J. Math. **107** (1998), 29–91.

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