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A QUASISTATIC FRICTIONAL CONTACT PROBLEM WITH NORMAL COMPLIANCE AND FINITE PENETRATION FOR ELASTIC MATERIALS

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Abstract. We consider a quasistatic unilateral contact problem with finite penetration between an elastic body and an obstacle, say a foundation. The constitutive law is assumed to be nonlinear and the contact is modelled with normal compliance associated to a version of Coulomb's law of dry friction. Under a smallness assumption on the contact functions, we establish the existence of a weak solution to the problem. The proofs are based on arguments of time-dicretization, compactness and lower semicontinuity.

1. Introduction

Contact problems involving deformable bodies are quite frequent in the industry as well as in daily life and play an important role in structural and mechanical systems. Contact processes involve a complicated surface phenomena, and are modeled with highly nonlinear initial boundary value problems. Taking into account various frictional contact conditions associated with behavior laws becoming more and more complex leads to the introduction of new and non standard models, expressed by the aid of evolution variational inequalities. A first attempt to study frictional contact problems within the framework of variational inequalities was made in [8]. The mathematical, mechanical and numerical state of the art can be found in [13]. In [11] we find a detailed analysis of the contact problem in elasticity with the mathematical and numerical studies. In this work we consider a quasistatic contact problem between a nonlinear elastic body and an obstacle say a foundation.

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We assume that the contact is modelled with normal compliance similar to the one in [10] where a dynamic frictionless contact problem was studied for elastic-visco-plastic materials. Under this condition the interpenetration of the body's surface into the foundation is allowed and justified by considering the interpenetration and deformation of surface asperities. The friction is modelled with a version of Coulomb's law of dry friction. We assume that the forces applied to the body vary slowly in time so that the acceleration in the system is negligible. In this case we can study a quasistatic approach of the process. For linear elastic materials the quasistatic contact problem using a normal compliance law has been studied in [1] by considering incremental problems and in [12] by another method using a time-regularization. The quasistatic unilateral contact problem with local or nonlocal friction has been solved respectively in [14] and in [5] by using a time-discretization. The same method was also used in [16] to solve the quasistatic unilateral contact problem with a modified version of Coulomb's law of dry friction for nonlinear elastic materials. In [2] the quasistatic contact problem with Coulomb friction was solved by an established shifting technique used to obtain increased regularity at the contact surface and by the aid of auxiliary problems involving regularized friction terms and a so-called normal compliance penalization technique. Signorini 's problem with friction for nonlinear elastic materials has been solved in [6] by using a fixed point method. Also the quasistatic contact problem with normal compliance and friction has been solved in [15] for nonlinear viscoelastic materials by the same fixed point arguments. In the book [9] the authors resolve the quasistatic contact problems in viscoelasticity and viscoplasticity. Carrying out the variational analysis, the authors systematically use results on elliptic and evolutionary variational inequalities, convex analysis, nonlinear equations with monotone operators, and fixed points of operators. These two latest arguments were used in [3] to solve recently two dynamic frictionless contact problems for elastic-visco-plastic materials.
In this paper we propose a variational formulation written in the form of two variational inequalities. By means of Euler's implicit scheme as in [5,16], the quasistatic contact problem leads us to solve a well-posed variational inequality at each time step. Finally under a smallness assumption on the contact functions we prove by using lower semicontinuity and compactness arguments that the limit of the discrete solution is a solution to the continuous problem.

2. Variational formulation

We consider a nonlinear elastic body that initially occupies a domain Ω in $\mathbf{R}^d$, $d = 2, 3$. Ω is supposed to be open, bounded, with a Lipschitz boundary Γ. Γ is decomposed into three measurable parts $\Gamma = \bar{\Gamma}_1 \cup \bar{\Gamma}_2 \cup \bar{\Gamma}_3$ where Γ_1, Γ_2, Γ_3 are disjoint open sets and $meas\ \Gamma_1 > 0$. The body is subjected to volume forces of density ϕ_1, prescribed zero displacements and tractions ϕ_2

on the part Γ_1 and Γ_2, respectively. On Γ_3 the body is in unilateral contact with friction with a foundation.

Under these conditions, the classical formulation of the mechanical problem of frictional contact of the nonlinear elastic body is the following.

PROBLEM P_1. *Find a displacement field* $u : \Omega \times [0,T] \to \mathbf{R}^d$ *such that*

$$\operatorname{div}\sigma + \phi_1 = 0 \ \textit{in}\ \Omega \times (0,T)\,, \tag{2.1}$$

$$\sigma = F\,(x,\varepsilon) \quad \textit{in}\ \Omega \ \times (0,T)\,, \tag{2.2}$$

$$u = 0 \quad \textit{on}\ \Gamma_1 \times (0,T)\,, \tag{2.3}$$

$$\sigma\nu = \phi_2 \quad \textit{on}\ \Gamma_2 \times (0,T)\,, \tag{2.4}$$

$$u_\nu \le g,\ \sigma_\nu + p_\nu\,(u_\nu) \le 0,\ \ (\sigma_\nu + p_\nu\,(u_\nu))\,(u_\nu - g) = 0 \quad \textit{on}\ \Gamma_3 \times (0,T)\,, \tag{2.5}$$

$$\begin{cases} |\sigma_\tau| \le p_\tau\,(u_\nu) \\ |\sigma_\tau| < p_\tau\,(u_\nu) \Longrightarrow \dot{u}_\tau = 0 \\ |\sigma_\tau| = p_\tau\,(u_\nu) \Longrightarrow \exists\lambda \ge 0 \ \textit{s.t.}\ \ \sigma_\tau = -\lambda\dot{u}_\tau \end{cases} \qquad \textit{on}\ \Gamma_3 \times (0,T)\,, \tag{2.6}$$

$$u(0) = u_0 \quad \textit{in}\ \Omega. \tag{2.7}$$

In the study of the mechanical problem P_1 we adopt the following notations and hypotheses: we use the function spaces

$$\begin{aligned} &H = \left(L^2\,(\Omega)\right)^d,\ Q = \left\{\tau = (\tau_{ij})\,;\tau_{ij} = \tau_{ji} \in L^2\,(\Omega)\right\},\\ &H_1 = \left(H^1\,(\Omega)\right)^d,\ Q_1 = \{\tau \in Q; \operatorname{div}\tau \in H\}\,. \end{aligned}$$

H, Q are Hilbert spaces equipped with the respective inner products:

$$\langle u, v\rangle_H = \int_\Omega u_i v_i dx,\ \ \langle \sigma, \tau\rangle_Q = \int_\Omega \sigma_{ij}\tau_{ij} dx.$$

We denote by S_d the space of second order symmetric tensors in $\mathbf{R}^d$ $(d = 2, 3)$; $u = (u_i)$, the displacement field; $\sigma = (\sigma_{ij})$, the stress tensor;

$$\varepsilon = \varepsilon\,(u\,(x,t)) = (\varepsilon_{ij}\,(u\,(x,t))) = \frac{1}{2}\,(u_{i,j}\,(x,t) + u_{j,i}\,(x,t))\,, i, j \in \{1, ..., d\}\,,$$

where $u_{i,j}\,(x,t) = \frac{\partial u_i(x,t)}{\partial x_j}$, $(x,t) \in \Omega\times(0,T)$, the strain tensor; $\operatorname{div}\sigma = (\sigma_{ij,j})$, the divergence of σ. We denote by u_ν and u_τ the normal and the tangential components of u on Γ given by

$$u_\nu = u\cdot\nu,\ u_\tau = u\cdot\tau,$$

where ν is the outward unit normal vector to Γ. We also denote by σ_ν and σ_τ the normal and the tangential components of σ, and we note that when $\sigma \in Q_1$ is a regular function then

$$\sigma_\nu = (\sigma\nu)\cdot\nu,\ \sigma_\tau = \sigma\nu - \sigma_\nu\nu,$$

and Green's formula holds:

$$\langle \sigma, \varepsilon(v) \rangle_Q + \langle \operatorname{div} \sigma, v \rangle_H = \int_\Gamma \sigma\nu \cdot v da \ \forall v \in H_1.$$

Now, let V be the closed subspace of H_1 defined by

$$V = \{v \in H_1; v = 0 \text{ on } \Gamma_1\},$$

and the set of admissible displacements fields given by

$$K = \{v \in V; v_\nu \leq g \text{ on } \Gamma_3\},$$

where $g \geq 0$. Since *meas* $\Gamma_1 > 0$, the following Korn's inequality holds ([8]),

$$\|\varepsilon(v)\|_Q \geq c_\Omega \|v\|_{H_1} \quad \forall v \in V, \tag{2.8}$$

where $c_\Omega > 0$ is a constant which depends only on Ω and Γ_1. We equip V with the inner product given by

$$(u, v)_V = \langle \varepsilon(u), \varepsilon(v) \rangle_Q$$

and let $\|.\|_V$ be the associated norm. It follows from (2.8) that the norms $\|.\|_{H_1}$ and $\|.\|_V$ are equivalent and $(V, \|.\|_V)$ is a real Hilbert space. Moreover, by Sobolev's trace theorem, there exists a constant $d_\Omega > 0$ depending only on the domain Ω, Γ_1 and Γ_3 such that

$$\|v\|_{(L^2(\Gamma_3))^d} \leq d_\Omega \|v\|_V \quad \forall v \in V. \tag{2.9}$$

Let us equally define $H^{\frac{1}{2}}(\Gamma_3)$ by

$$H^{\frac{1}{2}}(\Gamma_3) = \left\{ w\,|_{\Gamma_3};\, w \in H^{\frac{1}{2}}(\Gamma),\, w = 0 \text{ on } \Gamma_1 \right\}.$$

$\langle ., . \rangle_{\Gamma_3}$ shall denote the duality pairing on $H^{\frac{1}{2}}(\Gamma_3)$, $H^{-\frac{1}{2}}(\Gamma_3)$. Before we start with the variational formulation of Problem P_1 let us state in which sense the duality pairing $\langle ., . \rangle_{\Gamma_3}$ is taken. Indeed, we define a subset V_0 of H_1 by

$$V_0 = \{v \in H_1; \operatorname{div} \sigma(v) \in H\},$$

and let $\varphi \in \left(L^2(\Gamma_2)\right)^d$ and $u \in V_0$ such that $\sigma(u)\nu = \varphi$ on Γ_2. Then as in [16] we define the normal stress $\sigma_\nu(u)$ on Γ_3 as follows

$$\begin{aligned} &\langle \sigma_\nu(u), v_\nu \rangle_{\Gamma_3} = \langle \sigma(u), \varepsilon(v) \rangle_Q + \langle \operatorname{div} \sigma(u), v \rangle_H - \int_{\Gamma_2} \varphi \cdot v da \\ &\forall v \in V;\ v_\tau = 0 \text{ on } \Gamma_3. \end{aligned} \tag{2.10}$$

In the study of the mechanical problem P_1, we assume that the elasticity operator $F : \Omega \times S_d \to S_d$ satisfies the following conditions

$$(2.11)\quad \begin{cases} (a) & \text{there exists } M>0 \text{ such that} \\ & |F(x,\varepsilon_1) - F(x,\varepsilon_2)| \le M\,|\varepsilon_1-\varepsilon_2|\,, \\ & \text{for all } \varepsilon_1,\varepsilon_2 \in S_d, a.e.\ x\in\Omega; \\ \\ (b) & \text{there exists } m>0 \text{ such that} \\ & (F(x,\varepsilon_1) - F(x,\varepsilon_2))\cdot(\varepsilon_1-\varepsilon_2) \ge m\,|\varepsilon_1-\varepsilon_2|^2\,, \\ & \text{for all } \varepsilon_1,\varepsilon_2 \in\ S_d,\ a.e.\ x\in\Omega; \\ \\ (c) & \text{the mapping } x\to F(x,\varepsilon) \text{ is Lebesgue measurable on } \Omega, \\ & \text{for any } \varepsilon\in S_d; \\ \\ (d) & F(x,0)=0 \text{ for } a.e.\ x\in\Omega. \end{cases}$$

For every real Banach space $(X,\ \|.\|_X)$ and $T>0$ we use the notation $C([0,T];X)$ for the space of continuous functions from $[0,T]$ to X; recall that $C([0,T];X)$ is a real Banach space with the norm

$$\|x\|_{C([0,T];X)} = \max_{t\in[0,T]} \|x(t)\|_X\,.$$

For $p\in[1,\infty]$ we use the standard notation of $L^p(0,T;V)$. We also use the Sobolev space $W^{1,\infty}(0,T;V)$ equipped with the norm

$$\|v\|_{W^{1,\infty}(0,T:V)} = \|v\|_{L^\infty(0,T;V)} + \|\dot v\|_{L^\infty(0,T;V)}\,,$$

where a dot now represents the weak derivative with respect to the time variable.

The forces are assumed to satisfy

$$(2.12)\qquad \phi_1\in W^{1,\infty}(0,T;H)\,,\ \phi_2\in W^{1,\infty}\left(0,T;\left(L^2(\Gamma_2)\right)^d\right).$$

Let $f:[0,T]\to V$ given by

$$(f(t),v)_V = \int_\Omega \phi_1\cdot v dx + \int_{\Gamma_2}\phi_2\cdot v da \quad \forall\ v\in V,\ t\in[0,T]\,.$$

The assumption (2.12) implies that

$$f\in W^{1,\infty}(0,T;V)\,.$$

We assume that the contact functions p_r $(r=\nu,\tau)$ satisfy

$$(2.13)\qquad \begin{cases} (a) & p_r :]-\infty,g]\to\mathbf{R}_+; \\ \\ (b) & \text{there exists } L_r>0 \text{ such that} \\ & |p_r(u)-p_r(v)| \le L_r\,|u-v|\,,\ \text{ for all } u,\ v\le g; \\ \\ (c) & p_r(v)=0 \text{ for all } v<0. \end{cases}$$

When $u_\nu < 0$, i.e., when there is separation between the body and the obstacle then the condition (2.5) combined with hypothesis (2.13) shows that the reaction of the foundation vanishes (since $\sigma_\nu = 0$). When $0 \leq u_\nu < g$ then $-\sigma_\nu = p_\nu(u_\nu)$ which means that the reaction of the foundation is uniquely determined by the normal displacement. When $u_\nu = g$ then $-\sigma_\nu \geq p_\nu(g)$ and σ_ν is not uniquely determined. We note then when $g = 0$ and $p_\nu = 0$ then the condition (2.5) becomes the classical Signorini contact condition without a gap

$$u_\nu \leq 0,\ \sigma_\nu \leq 0,\ \sigma_\nu u_\nu = 0,$$

and when $g > 0$ and $p_\nu = 0$, condition (2.5) becomes the classical Signorini contact condition with a gap and without friction

$$u_\nu \leq g,\ \sigma_\nu \leq 0,\ \sigma_\nu (u_\nu - g) = 0.$$

The last two conditions are used to model the unilateral conditions with a rigid foundation.
Conditions (2.6) represent a version of Coulomb's law of dry friction in which p_τ is a prescribed nonnegative function, the so-called friction bound and $\dot{u}_\tau$ the tangential velocity on the boundary. The tangential shear cannot exceed the maximal frictional resistance $p_\tau(u_\nu)$. Then, if the strict inequality is satisfied, the surface adheres to the foundation and is in the so-called stick state, and when equality is satisfied there is relative sliding, the so-called slip state. Examples of normal compliance functions can be found in [1, 3, 10, 15].
Next, we define the friction functional $j : V \times V \to \mathbf{R}$ by

$$j(v, w) = \int_{\Gamma_3} p_\nu(v_\nu)\, w_\nu da + \int_{\Gamma_3} p_\tau(v_\nu)\, |w_\tau|\, da,\ \forall v, w \in V,$$

and we assume that the initial data satisfies

(2.14)
$$u_0 \in K,\ \langle F\varepsilon(u_0), \varepsilon(v - u_0)\rangle_Q + j(u_0, v - u_0) \geq (f(0), v - u_0)_V \quad \forall v \in K.$$

Now we turn to the weak formulation of Problem P_1. As in [5], assume that u is a smooth function satisfying (2.1)–(2.7). Let $v \in V$ and multiply the equilibrium of forces (2.1) by $v - \dot{u}(t)$, integrate the result over Ω and use Green's formula to obtain

$$\int_\Omega \sigma(t)(\varepsilon(v) - \varepsilon(\dot{u}(t)))\, dx = \int_\Omega \phi_1(t)\cdot(v - \dot{u}(t))dx + \int_\Gamma \sigma(t)\,\nu\cdot(v - \dot{u}(t))\, da.$$

Taking into account the boundary condition (2.4) and $v = 0$ on Γ_1, we see that

$$\int_\Gamma \sigma(t)\,\nu\cdot(v - \dot{u}(t))\, da = \int_{\Gamma_2} \phi_2(t)\cdot(v - \dot{u}(t))da + \int_{\Gamma_3} \sigma(t)\,\nu\cdot(v - \dot{u}(t))\, da.$$

Moreover we have

$$\int_{\Gamma_3} \sigma(t)\,\nu\cdot(v - \dot{u}(t))\, da = \int_{\Gamma_3} \sigma_\nu(t)(v_\nu - \dot{u}_\nu(t))\, da + \int_{\Gamma_3} \sigma_\tau(t)(v_\tau - \dot{u}_\tau(t))\, da,$$

and

$$\int_{\Gamma_3} \sigma_\nu(t)(v_\nu - \dot{u}_\nu(t))\,da = \int_{\Gamma_3} (\sigma_\nu(t) + p_\nu(u_\nu(t)))(v_\nu - \dot{u}_\nu(t))\,da - \int_{\Gamma_3} p_\nu(u_\nu(t))(v_\nu - \dot{u}_\nu(t))\,da.$$

Next, we need to the following formulation (2.15) obtained from the contact condition (2.5) and the law of friction (2.6).

$$\text{(2.15)} \qquad \begin{cases} (a)\ u_\nu \leq g,\ (\sigma_\nu + p_\nu(u_\nu))(v_\nu - u_\nu) \geq 0 \ \ \forall v_\nu \leq g, \\ (b)\ \sigma_\tau(v_\tau - \dot{u}_\tau) + p_\tau(u_\nu)(|v_\tau| - |\dot{u}_\tau|) \geq 0 \ \ \forall v_\tau. \end{cases}$$

Indeed, it follows from (2.15)(b) and (2.15)(a) that the function u satisfies respectively the inequalities

$$\text{(2.16)} \qquad \begin{aligned} &\int_\Omega (F\varepsilon(u(t)))(\varepsilon(v) - \varepsilon(\dot{u}(t))\,dx + j(u(t), v) - j(u(t), \dot{u}(t)) \\ &\quad \geq (f(t), v - \dot{u}(t))_V \\ &\qquad + \int_{\Gamma_3} (\sigma_\nu(u(t)) + p_\nu(u_\nu(t)))(v_\nu - \dot{u}_\nu(t))\,da \quad \forall v \in V, \end{aligned}$$

and

$$\text{(2.17)} \qquad \int_{\Gamma_3} (\sigma_\nu(u(t)) + p_\nu(u_\nu(t)))(z_\nu - u_\nu(t)) \geq 0 \quad \forall z \in K.$$

Finally we combine (2.7), (2.16) and (2.17) to derive the variational formulation of Problem P_1.

Problem P_2. *Find a displacement field $u \in W^{1,\infty}(0,T;V)$ such that $u(0) = u_0$ in Ω and such that for all $t \in [0,T]$, $u(t) \in K \cap V_0$, and for almost all $t \in [0,T]$,*

$$\text{(2.18)} \qquad \begin{aligned} &\langle F\varepsilon(u(t)), \varepsilon(v) - \varepsilon(\dot{u}(t))\rangle_Q + j(u(t), v) - j(u(t), \dot{u}(t)) \\ &\quad \geq (f(t), v - \dot{u}(t))_V \\ &\qquad + \langle \sigma_\nu(u(t)) + p_\nu(u_\nu(t)), v_\nu - \dot{u}_\nu(t)\rangle_{\Gamma_3} \quad \forall v \in V, \end{aligned}$$

$$\text{(2.19)} \qquad \langle \sigma_\nu(u(t)) + p_\nu(u_\nu(t)), z_\nu - u_\nu(t)\rangle_{\Gamma_3} \geq 0 \quad \forall z \in K.$$

One has the following theorem:

Theorem 2.1. *Let $T > 0$ and assume that (2.11), (2.12), (2.13), and (2.14) hold. Then Problem P_2 has at least one solution if*

$$L_\nu + L_\tau < \frac{m}{d_\Omega^2}.$$

3. A time-discretization

For the proof of Theorem 2.1, we carry a time-discretization of Problem P_2. We need a partition of the time interval $[0,T]$, with $0=t_0<t_1<\ldots<t_n=T$, where $t_i=i\Delta t$, $i=0,\ldots,n$, with step size $\Delta t=\dfrac{T}{n}$. We denote by u^i the approximation of u at time t_i and $\Delta u^i=u^{i+1}-u^i$. For a continuous function $w(t)$ we use the notation $w^i=w(t_i)$. By using an implicit scheme, we obtain a sequence of incremental Problems P_n^i defined for $u^0=u_0$ by

Problem P_n^i. *Find $u^{i+1}\in K\cap V_0$ such that*

(3.1)
$$\begin{cases} \left\langle F\varepsilon\left(u^{i+1}\right),\varepsilon(w)-\varepsilon\left(u^{i+1}\right)\right\rangle_Q+j\left(u^{i+1},w-u^i\right)-j\left(u^{i+1},\Delta u^i\right) \\ \quad\geq\left(f^{i+1},w-u^{i+1}\right)_V+\left\langle\sigma_\nu\left(u^{i+1}\right)+p_\nu\left(u_\nu^{i+1}\right),w_\nu-u_\nu^{i+1}\right\rangle_{\Gamma_3}\ \forall w\in V, \\ \left\langle\sigma_\nu\left(u^{i+1}\right)+p_\nu\left(u_\nu^{i+1}\right),w_\nu-u_\nu^{i+1}\right\rangle_{\Gamma_3}\geq 0\ \ \forall w\in K. \end{cases}$$

As in [5] Problem P_n^i is equivalent to Problem Q_n^i defined as follows:

Problem Q_n^i. *Find $u^{i+1}\in K\cap V_0$ such that*

$$\text{(3.2)}\quad\begin{cases} \left\langle F\varepsilon\left(u^{i+1}\right),\varepsilon(w)-\varepsilon\left(u^{i+1}\right)\right\rangle_Q+j\left(u^{i+1},w-u^i\right)-j\left(u^{i+1},\Delta u^i\right) \\ \qquad\geq\left(f^{i+1},w-u^{i+1}\right)_V\quad\forall\ w\in K. \end{cases}$$

We have the following result.

Proposition 3.1. *Problem Q_n^i has a unique solution if*
$$L_\nu+L_\tau<\frac{m}{d_\Omega^2}.$$

To prove this proposition, let us introduce the following intermediate problem.

Problem $Q_{n\eta}^i$. *For $\eta\in K$, find $u_\eta^{i+1}\in K$ such that*

$$\text{(3.3)}\quad\begin{cases} \left\langle F\varepsilon\left(u_\eta^{i+1}\right),\varepsilon(w)-\varepsilon\left(u_\eta^{i+1}\right)\right\rangle_Q+j\left(\eta,w-u^i\right)-j\left(\eta,u_\eta^{i+1}-u^i\right) \\ \qquad\geq\left(f^{i+1},w-u_\eta^{i+1}\right)_V\quad\forall\ w\in K. \end{cases}$$

We have the following lemma.

Lemma 3.2. *Problem $Q_{n\eta}^i$ has a unique solution.*

Proof. Using Riesz's representation theorem, we define the nonlinear operator $A:V\to V$ by
$$(Av,w)_V=\langle F\varepsilon(v),\varepsilon(w)\rangle_Q\,.$$
The hypotheses (2.11)(a) and (2.11)(b) on F imply that the operator A is strongly monotone and Lipschitz continuous; on the other hand the functional

j_η defined on K by $j_\eta(w) = j\left(\eta, w - u^i\right)$ is proper convexe and lower semicontinuous. From the theory of elliptic variational inequalities ([4]), it follows that the inequality (3.3) has a unique solution. □

Now to prove Proposition 3.1, we define the following mapping $\Phi : K \to K$, by

$$\eta \to \Phi(\eta) = u_\eta.$$

The following lemma holds.

Lemma 3.3. *Φ has a unique fixed point η^* if*

$$L_\nu + L_\tau < \frac{m}{d_\Omega^2},$$

and u_{η^*} is a unique solution of Problem Q_n^i.

Proof. Let's set $v = u_{\eta_2}$ in the inequality of Problem $\mathrm{Q}_{n\eta_1}^i$ and $v = u_{\eta_1}$ in the inequality of Problem $\mathrm{Q}_{n\eta_2}^i$. Adding the resulting inequalities, we obtain the following inequality

$$\begin{aligned}&\langle F\varepsilon(u_{\eta_1}) - F\varepsilon(u_{\eta_2}), \varepsilon(u_{\eta_1} - u_{\eta_2})\rangle_Q \\ &\quad\le j\left(\eta_1, u_{\eta_2} - u^i\right) - j\left(\eta_1, u_{\eta_1} - u^i\right) + j\left(\eta_2, u_{\eta_1} - u^i\right) - j\left(\eta_2, u_{\eta_2} - u^i\right)\end{aligned}$$

Using (2.11)(b), (2.9) and (2.13)(b), we obtain

$$\|\Phi(u_{\eta_1}) - \Phi(u_{\eta_2})\|_V \le \frac{d_\Omega^2}{m}(L_\nu + L_\tau)\|\eta_1 - \eta_2\|_V.$$

Then when $L_\nu + L_\tau < \frac{m}{d_\Omega^2}$, Φ is contractive; thus it admits a unique fixed point $_{\eta^*}$ and u_{η^*} is a unique solution of Problem Q_n^i. □

4. Existence of a solution

The main result of this section is to show the existence of a solution obtained as a limit of the interpolate function of the discrete solution. For this it is necessary at first to establish the following lemma.

Lemma 4.1. *For $L_\nu + L_\tau < \frac{m}{d_\Omega^2}$, there exists positive constants c_1, c_2 such that*

$$\left\|u^{i+1}\right\|_V \le c_1 \left\|f^{i+1}\right\|_V, \quad \left\|\Delta u^i\right\|_V \le c_2 \left\|\Delta f^i\right\|_V. \tag{4.1}$$

Proof. Set $w = 0$ in inequality (3.2); then, using the assumption (2.11)(b) on F, (2.13) and the relation (2.9), we obtain by a standard reasoning that for $L_\nu + L_\tau < \frac{m}{d_\Omega^2}$, there exists a constant $c_1 > 0$ such that the first inequality hods.

To prove the second inequality, set $v = u^i$ in inequality (3.2) and set $v = u^{i+1}$ in the same inequality satisfied by u^i and adding them up, we obtain

$$-\left\langle F\varepsilon\left(u^{i+1}\right) - F\varepsilon\left(u^i\right), \varepsilon\left(\Delta u^i\right)\right\rangle_Q + j\left(u^i, u^{i+1} - u^{i-1}\right)$$
$$-j\left(u^i, u^i - u^{i-1}\right) - j\left(u^{i+1}, \Delta u^i\right) \geq (-\Delta f^i, \Delta u^i)_V.$$

On the other hand

$$j\left(u^i, u^{i+1} - u^{i-1}\right) - j\left(u^i, u^i - u^{i-1}\right) - j\left(u^{i+1}, u^{i+1} - u^i\right)$$
$$= \int_{\Gamma_3}\left(p_\nu\left(u_\nu^i\right)\right)\left(u_\nu^{i+1} - u_\nu^i\right) da + \int_{\Gamma_3} p_\tau\left(u_\nu^i\right)\left(\left|u_\tau^{i+1} - u_\tau^{i-1}\right| - \left|u_\tau^{i-1} - u_\tau^i\right|\right) da$$
$$- \int_{\Gamma_3}\left(p_\nu\left(u_\nu^{i+1}\right)\right)\left(u_\nu^{i+1} - u_\nu^i\right) da - \int_{\Gamma_3} p_\tau\left(u_\nu^{i+1}\right)\left|u_\tau^{i+1} - u_\tau^i\right| da.$$

Then using the relation

$$\left|\left|u_\tau^{i+1} - u_\tau^{i-1}\right| - \left|u_\tau^{i-1} - u_\tau^i\right|\right| \leq \left|\Delta u_\tau^i\right|,$$

we find

$$\left\langle F\varepsilon\left(u^{i+1}\right) - F\varepsilon\left(u^i\right), \varepsilon\left(\Delta u^i\right)\right\rangle_Q$$
$$\leq \int_{\Gamma_3}\left|p_\nu\left(u_\nu^{i+1}\right) - p_\nu\left(u_\nu^i\right)\right|\left|\Delta u_\nu^i\right| da$$
$$+ \int_{\Gamma_3}\left|p_\tau\left(u_\nu^{i+1}\right) - p_\tau\left(u_\nu^i\right)\right|\left|\Delta u_\tau^i\right| da + (\Delta f^i, \Delta u^i)_V.$$

Therefore the assumption (2.13)(b) on the contact functions p_r $(r = \tau, \nu)$ leads to

$$\left\langle F\varepsilon\left(u^{i+1}\right) - F\varepsilon\left(u^i\right), \varepsilon\left(\Delta u^i\right)\right\rangle_Q$$
$$\leq L_\nu \int_{\Gamma_3}\left|\Delta u_\nu^i\right|^2 da + L_\tau \int_{\Gamma_3}\left|\Delta u_\nu^i\right|\left|\Delta u_\tau^i\right| da + (\Delta f^i, \Delta u^i)_V.$$

Moreover we use (2.9) and (2.11)(b) and obtain

$$m\left\|\Delta u^i\right\|_V^2 \leq \left\langle F\varepsilon\left(u^i\right) - F\varepsilon\left(u^i\right), \varepsilon\left(\Delta u^i\right)\right\rangle_Q$$
$$\leq d_\Omega^2\left(L_\nu + L_\tau\right)\left\|\Delta u^i\right\|_V^2 + \left\|\Delta f^i\right\|_V\left\|\Delta u^i\right\|_V.$$

So it follows that for $L_\nu + L_\tau < \frac{m}{d_\Omega^2}$, there exists a constant $c_2 > 0$ such that

$$\left\|\Delta u^i\right\|_V \leq c_2\left\|\Delta f^i\right\|_V.$$

$\square$

Next, as in [5] we define the continuous function u^n in $[0, T] \to V$ by

$$u^n(t) = u^i + \frac{t - t_i}{\Delta t}\Delta u^i,\ \forall t \in\ \left[t_i, t_{i+1}\right],\ i = 0, ..., n-1.$$

Then as in [16] we have the following lemma.

LEMMA 4.2. *There exists a function $u \in W^{1,\infty}(0,T;V)$, such that passing to a subsequence still denoted by (u^n) we have*

$$u^n \to u \text{ weak } * \text{ in } W^{1,\infty}(0,T;V).$$

On the other hand as in [5,16] let's introduce the following piecewise constant functions

$$\widetilde{u}^n : [0,T] \to V,\ \widetilde{f}^n : [0,T] \to V,$$

defined by

$$\widetilde{u}^n(t) = u^{i+1},\ \widetilde{f}^n(t) = f(t_{i+1}),\ \forall\, t \in (t_i, t_{i+1}],\ i = 0, ..., n-1.$$

As in [9] the following result holds.

LEMMA 4.3. *There exists a subsequence still denoted $(\widetilde{u}^n)$ such that*

$$\widetilde{u}^n \to u \text{ weak } * \text{ in } L^\infty(0,T;V),$$
$$\widetilde{u}^n(t) \to u(t) \text{ weakly in } V \text{ a.e. } t \in [0,T].$$

REMARK 4.4. Also as in [16] we have the following results

$$\widetilde{f}^n \to f \text{ strongly in } L^2(0,T;V), \tag{4.2}$$

$$u(t) \in K \cap V_0 \text{ for all } t \in [0,T].$$

Now we have all the ingredients to prove the following result.

PROPOSITION 4.5. *The sequence $(\widetilde{u}^n)$ converges strongly to the function u in $L^2(0,T;V)$ and u is a solution to Problem P_2.*

PROOF. From (3.2) we deduce the inequality

$$\begin{aligned} &\langle F\varepsilon(\tilde{u}^n(t)), \varepsilon(w) - \varepsilon(\tilde{u}^n(t))\rangle_Q + j(\tilde{u}^n(t), w - \tilde{u}^n(t)) \\ &\quad \geq \left(\tilde{f}^n(t), w - \tilde{u}^n(t)\right)_V \quad \forall\, w \in K,\ a.e.\ t \in [0,T]. \end{aligned} \tag{4.3}$$

To show the strong convergence, we take $w = \tilde{u}^{n+m}(t)$ in (4.3) and $w = \tilde{u}^n(t)$ in the same inequality satisfied by $\tilde{u}^{n+m}(t)$ and adding the resulting inequalities, it follows by using (2.11)(b), the relation (2.9) and $L_\nu + L_\tau < \frac{m}{d_\Omega^2}$, that there exists a constant $C_1 > 0$ such that

$$\begin{aligned} &\left\|\tilde{u}^{n+m}(t) - \tilde{u}^n(t)\right\|_V^2 \\ &\quad \leq C_1 \left(\int_{\Gamma_3} \left(|p_\tau(\tilde{u}_\nu^n(t))| + \left|p_\tau\left(\tilde{u}_\nu^{n+m}(t)\right)\right|\right) \left|\tilde{u}_\tau^{n+m}(t) - \tilde{u}_\tau^n(t)\right| da \right. \\ &\qquad \left. + \left\|\tilde{f}^{n+m}(t) - \tilde{f}^n(t)\right\|_V^2 \right). \end{aligned}$$

Moreover using (2.13)(b) we show that there exists a constant $C_2 > 0$ such that

$$\int_{\Gamma_3} \left(\left|p_\tau\left(\tilde{u}_\nu^n(t)\right)\right| + \left|p_\tau\left(\tilde{u}_\nu^{n+m}(t)\right)\right|\right)\left|\tilde{u}_\tau^{n+m}(t) - \tilde{u}_\tau^n(t)\right| da$$
$$\leq C_2\left(\left\|\tilde{u}_\tau^{n+m}\right\|_{L^2(\Gamma_3)^d} + \left\|\tilde{u}_\tau^n\right\|_{L^2(\Gamma_3)^d}\right)\left\|\tilde{u}_\tau^{n+m}(t) - \tilde{u}_\tau^n(t)\right\|_{L^2(\Gamma_3)^d}.$$

Using the relation (2.9) and that $(\tilde{u}_n)$ is bounded in $L^\infty(0,T;V)$ we deduce

$$\int_{\Gamma_3} \left(\left|p_\tau\left(\tilde{u}_\nu^{n+m}(t)\right)\right| + \left|p_\tau\left(\tilde{u}_\nu^n(t)\right)\right|\right)\left|\tilde{u}_\tau^{n+m}(t) - \tilde{u}_\tau^n(t)\right| da$$
$$\leq C_3 \|f\|_{L^\infty(0,T;V)}\left\|\tilde{u}_\tau^{n+m}(t) - \tilde{u}_\tau^n(t)\right\|_{L^2(\Gamma_3)^d},$$

where $C_3 > 0$. To complete the rest of the proof we refer the reader to see [16, Lemma 7] and conclude that

$$\tilde{u}^n \to u \text{ strongly in } L^2(0,T;V). \tag{4.4}$$

Now to show that u is a solution of Problem P_2; in inequality (3.1) set, for $z \in V$, $w = u^i + z\Delta t$ and divide by Δt, we get

$$\left\langle F\varepsilon(u^{i+1}), \varepsilon(z) - \varepsilon\left(\frac{\Delta u^i}{\Delta t}\right)\right\rangle_Q + j\left(u^{i+1}, z\right) - j\left(u^{i+1}, \frac{\Delta u^i}{\Delta t}\right)$$
$$\geq \left(f^{i+1}, z - \frac{\Delta u^i}{\Delta t}\right)_V + \left\langle \sigma_\nu\left(u^{i+1}\right) + p_\nu\left(u_\nu^{i+1}\right), z_\nu - \frac{\Delta u_\nu^i}{\Delta t}\right\rangle_{\Gamma_3} \forall z \in V,$$

from which we deduce for any $z \in L^2(0,T;V)$ the inequality

$$\begin{cases} \left\langle F\varepsilon\left(\widetilde{u}^n(t)\right), \varepsilon(z(t)) - \varepsilon\left(\dot{u}^n(t)\right)\right\rangle_Q + j\left(\widetilde{u}^n(t), z(t)\right) - j\left(\widetilde{u}^n(t), \dot{u}^n(t)\right) \\ \geq \left(\widetilde{f}^n(t), z(t) - \dot{u}^n(t)\right)_V + \left\langle \sigma_\nu\left(\tilde{u}^n(t)\right) + p_\nu\left(\tilde{u}_\nu^n(t)\right), z_\nu - \dot{u}_\nu^n(t)\right\rangle_{\Gamma_3}, \\ \quad \text{for } a.a.\ t \in [0,T]. \end{cases}$$

Integrating both sides of the previous inequality on $(0,T)$ we obtain the inequality

$$\begin{aligned} &\int_0^T \left\langle F\varepsilon\left(\widetilde{u}^n(t)\right), \varepsilon(z(t)) - \varepsilon\left(\dot{u}^n(t)\right)\right\rangle_Q dt + \int_0^T j\left(\widetilde{u}^n(t), z(t)\right) dt \\ &\quad - \int_0^T j\left(\widetilde{u}^n(t), \dot{u}^n(t)\right) dt \\ &\geq \int_0^T \left(\widetilde{f}^n(t), z(t) - \dot{u}^n(t)\right)_V dt \\ &\quad + \int_0^T \left\langle \sigma_\nu\left(\tilde{u}^n(t)\right) + p_\nu\left(\tilde{u}_\nu^n(t)\right), z_\nu(t) - \dot{u}_\nu^n(t)\right\rangle_{\Gamma_3} dt \end{aligned} \tag{4.5}$$

Now before passing to the limit in the previous inequality we start with the proof of the following lemma.

LEMMA 4.6. *For all $z \in L^2(0,T;V)$ we have:*

$$\begin{aligned}\lim_{n\to\infty}\int_0^T &\left\langle F\left(\varepsilon\left(\widetilde{u}^n\right)\right),\varepsilon\left(z\left(t\right)\right)-\varepsilon\left(\dot{u}^n\left(t\right)\right)\right\rangle_Q dt\\ &=\int_0^T \left\langle F\varepsilon\left(u\left(t\right)\right),\varepsilon\left(z\left(t\right)\right)-\varepsilon\left(\dot{u}\left(t\right)\right)\right\rangle_Q dt,\end{aligned} \tag{4.6}$$

$$\lim_{n\to\infty}\int_0^T j\left(\widetilde{u}^n\left(t\right),z\left(t\right)\right)dt=\int_0^T j\left(u\left(t\right),z\left(t\right)\right)dt, \tag{4.7}$$

$$\lim_{n\to\infty}\int_0^T \left(\widetilde{f}^n\left(t\right),z\left(t\right)-\dot{u}^n\left(t\right)\right)_V dt=\int_0^T \left(f\left(t\right),z\left(t\right)-\dot{u}\left(t\right)\right)_V dt, \tag{4.8}$$

$$\begin{aligned}\lim_{n\to\infty}\int_0^T &\left\langle \sigma_\nu\left(\tilde{u}^n\left(t\right)\right)+p_\nu\left(\tilde{u}^n_\nu\left(t\right)\right),z_\nu\left(t\right)-\dot{u}^n_\nu\left(t\right)\right\rangle_{\Gamma_3} dt\\ &=\int_0^T \left\langle \sigma_\nu\left(u\left(t\right)\right)+p_\nu\left(u_\nu\left(t\right)\right),z_\nu\left(t\right)-\dot{u}_\nu\left(t\right)\right\rangle_{\Gamma_3} dt.\end{aligned} \tag{4.9}$$

PROOF. For the proof of (4.6) we refer the reader to [16, Lemma 10]. To prove (4.7) we write the term $j\left(\widetilde{u}^n(t),z(t)\right)$ as

$$j\left(\widetilde{u}^n(t),z(t)\right)=\left(j\left(\widetilde{u}^n(t),z(t)\right)-j\left(u(t),z(t)\right)\right)+j\left(u(t),z(t)\right).$$

We have

$$\begin{aligned}&j\left(\widetilde{u}^n(t),z(t)\right)-j\left(u(t),z(t)\right)\\ &=\int_{\Gamma_3}\left(p_\nu\left(\tilde{u}^n_\nu(t)\right)-p_\nu\left(u_\nu(t)\right)\right)z_\nu(t)da+\int_{\Gamma_3}\left(p_\tau\left(\tilde{u}^n_\nu(t)\right)-p_\tau\left(u_\nu(t)\right)\right)\left|z_\tau(t)\right|da.\end{aligned}$$

So using (2.13)(b) and the relation (2.9), we get

$$\left|\int_0^T\left(j\left(\widetilde{u}^n\left(t\right),z\left(t\right)\right)-j\left(u\left(t\right),z\left(t\right)\right)\right)dt\right|\le C_4\left\|\widetilde{u}_n-u\right\|_{L^2(0,T;V)}\left\|z\right\|_{L^2(0,T;V)},$$

where $C_4>0$. Then we deduce from (4.4) that

$$\lim_{n\to\infty}\int_0^T\left(j\left(\widetilde{u}^n\left(t\right),z\left(t\right)\right)-j\left(u\left(t\right),z\left(t\right)\right)\right)dt=0.$$

Finally to prove (4.8) it suffices to use (4.2) and to prove (4.9) it suffices also to use (2.10), (2.13), (4.2) and (4.4). □

LEMMA 4.7. *We have:*

$$\liminf_{n\to\infty}\int_0^T j\left(\widetilde{u}^n\left(t\right),\dot{u}^n\left(t\right)\right)dt\ge\int_0^T j\left(u\left(t\right),\dot{u}\left(t\right)\right)dt. \tag{4.10}$$

Proof. First, we have

$$\int_0^T j\left(\widetilde{u}^n(t),\dot{u}^n(t)\right)dt=\int_0^T \left(j\left(\widetilde{u}^n(t),\dot{u}^n(t)\right)-j\left(u(t),\dot{u}^n(t)\right)\right)dt + \int_0^T j\left(u(t),\dot{u}^n(t)\right)dt.$$

There exists a constant $C_5>0$ such that the first term of the second side of the equality can be estimated as

$$\left|\int_0^T \left(j\left(\widetilde{u}^n(t),\dot{u}^n(t)\right)-j\left(u(t),\dot{u}^n(t)\right)\right)dt\right| \leq C_5\left\|\widetilde{u}^n-u\right\|_{L^2(0,T;V)}\left\|\dot{u}^n\right\|_{L^2(0,T:V)}.$$

Since $\dot{u}^n$ is bounded in $L^2(0,T;V)$ it follows from (4.4) that

$$\lim_{n\to\infty}\int_0^T \left(j\left(\widetilde{u}^n(t),\dot{u}^n(t)\right)-j\left(u(t),\dot{u}^n(t)\right)\right)dt=0.$$

For the convergence of the other term, set

$$p_r\left(u_\nu(t)\right)=k_r(t),\ r=\nu,\tau,\ k_r(t)\geq 0\ \forall\ t\in[0,T].$$

Setting $k=(k_\nu,k_\tau)$, and keeping in mind the assumptions on the functions p_r, it follows that

$$k\in C\left([0,T];\left(L^2(\Gamma_3)\right)^2\right).$$

Moreover, if we define the function ϕ_k by

$$\phi_k(z)=\int_{\Gamma_3}k_\nu z_\nu da+\int_{\Gamma_3}k_\tau\left|z_\tau\right|da,$$

then ϕ_k is lower semicontinuous and we have

$$\liminf_{n\to\infty}\int_0^T\phi_k\left(\dot{u}^n(t)\right)dt\geq\int_0^T\phi_k\left(\dot{u}(t)\right)dt,$$

whence we deduce (4.10). ∎

Now using Lemma 4.6 and passing to the limit in (4.5) we deduce the inequality

(4.11)
$$\begin{cases}\displaystyle\int_0^T\Big(\left\langle F\varepsilon(u(t)),\varepsilon(z(t))-\varepsilon\left(\dot{u}(t)\right)\right\rangle_Q+j\left(u(t),z(t)\right)-j\left(u(t),\dot{u}(t)\right)\Big)dt\\ \displaystyle\geq\int_0^T\left(f(t),z(t)-\dot{u}(t)\right)_V dt\\ \displaystyle\quad+\int_0^T\left\langle\sigma_\nu(u(t))+p_\nu\left(u_\nu(t)\right),z_\nu(t)-\dot{u}_\nu(t)\right\rangle_{\Gamma_3}dt,\qquad\forall z\in L^2(0,T;V).\end{cases}$$

As in [5] from inequality (4.11) we deduce the inequality (2.18). Also integrating the inequality (4.3) with respect to time and passing to the limit we obtain the following inequality from which we deduce the inequality (2.19).

$$\begin{cases} \displaystyle\int_0^T \Big(\langle F\varepsilon\left(u\left(t\right)\right), \varepsilon\left(z\left(t\right)\right) - \varepsilon\left(u\left(t\right)\right)\rangle_Q + j\left(u\left(t\right), z\left(t\right) - u\left(t\right)\right)\Big)\, dt \\ \quad \geq \displaystyle\int_0^T \left(f\left(t\right), z\left(t\right) - u\left(t\right)\right)_V dt, \forall z \in L^2(0,T;V) \text{ such that } z\left(t\right) \in K, \\ \quad a.e.\ t \in \left[0,T\right]. \end{cases}$$

Therefore we conclude that u is a solution of Problem P_2.

5. Conclusion

Our main result in this paper concerns the existence of the weak solution in the study of a quasistatic frictional contact problem with finite penetration for nonlinear elastic materials. The contact is modelled with normal compliance law associated to a version of Coulomb's law of dry friction. Under a smallness assumption on the contact functions we show the existence of a weak solution. Finally, we note that the important question of uniqueness of the solution is not resolved here, and remains still open.

References

[1] L.-E. Andersson, *A quasistatic frictional problem with normal compliance*, Nonlinear Anal. **16** (1991), 347–369.

[2] L.-E. Andersson, *Existence results for quasistatic contact problems with Coulomb friction*, Appl. Math. Optim. **42** (2000), 169–202.

[3] Y. Ayyad, M. Sofonea, *Analysis of two dynamic frictionless contact problems for elastic-visco-plastic materials*, Electron. J. Differential Equations **2007**, No. 55, pp. 1–17.

[4] H. Brezis, *Equations et inéquations non linéaires dans les espaces vectoriels en dualité*, Ann. Inst. Fourier (Grenoble) **18** (1968), 115–175.

[5] M. Cocou, E. Pratt and M. Raous, *Formulation and approximation of quasistatic frictional contact*, Internat. J. Engrg. Sci. **34** (1996), 783–798.

[6] S. Drabla and M Sofonea, *Analysis of a Signorini problem with friction*, IMA J. Appl. Math. **63** (1999), 113–130.

[7] G. Duvaut, *Equilibre d'un solide élastique avec contact unilatéral et frottement de Coulomb*, C. R. Acad. Sci. Paris Sér. A **290** (1980), 263–265.

[8] G. Duvaut and J. L. Lions, Les inéquations en mécanique et en physique, Dunod, Paris, 1972.

[9] W. Han and M. Sofonea, Quasistatic contact problems in viscoelasticity and viscoplasticity. Studies in advanced Mathematics 30, Americal Mathematical Society and International Press, Somerville, 2002.

[10] J. Jarusěk and M. Sofonea, *On the solvability of dynamic elastic-visco-plastic contact problems*, ZAMM Z. Angew. Math. Mech. **88** (2008), 3–22.

[11] N. Kikuchi and T. J. Oden, Contact problems in elasticity: a study of variational inequalities and finite element methods, SIAM, Philadelphia, 1988.

[12] A. Klarbring, A. Mikelić and M. Shillor, *A global existence result for the quasistatic problem with normal compliance*, Internat. Ser. Numer. Math. **101**, Birkhäuser Verlag, Basel, 85–111, 1991.

[13] M. Raous, M. Jean and J. J. Moreau, eds., Contact Mechanics, Plenum Press, New York, 1995.

[14] R. Rocca, *Existence of a solution for a quasistatic problem of unilateral contact with local friction*, C. R. Acad. Sci. Paris Sér. I Math. **328** (1999), 1253–1258.

[15] M. Rochdi, M. Schillor and M. Sofonea, *Quasistatic viscoelastic contact with normal compliance and friction*, J. Elasticity **51** (1998), 105–126.

[16] A. Touzaline and D. J. Teniou, *A quasistatic unilateral contact problem problem with friction for nonlinear elastic materials*, Math. Model. Anal. **12** (2007), 497–514.

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