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ON EXPANSIONS AND PRO-PRO-CATEGORIES

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Abstract. If $(\mathcal{C}, \mathcal{D})$ is a category pair such that $\mathcal{D} \subseteq \mathcal{C}$ is a pro-reflective subcategory, then so is $\mathcal{D} \subseteq pro\text{-}\mathcal{C}$ and, inductively, $\mathcal{D} \subseteq pro^n\mathcal{C}$ as well. The key fact is that the terms and morphisms of $\mathcal{D}$-expansions of all the terms of a $\mathcal{C}$-system can be naturally organized in a $\mathcal{D}$-expansion of the system. Therefore, in dealing with expansions, there is no need to involve the pro-pro-category technique. In particular, the shape of a $\mathcal{C}$-system, as well as of a $\mathcal{C}$-object, reduces to the isomorphism class of a $\mathcal{D}$-system. On the other hand, a pro-pro-category could be useful for some other purposes because it admits functorial expansions which are inverse limits. Some applications of the theoretical part are considered, especially, concerning the Stone-Čech compactification and Hewitt realcompactification.

1. Introduction

The notion of an expansion is essential and the most important in the development of any shape theory - standard or abstract - in terms of inverse systems ([6,9–11,13,14]). By means of expansion, an arbitrary ("ugly") object of a category $\mathcal{C}$ is represented by an inverse system of ("nice") objects of a suitable subcategory $\mathcal{D} \subseteq \mathcal{C}$, satisfying, in addition, an appropriate universal factorization property. Then we usually say that $\mathcal{D}$ is a pro-reflective subcategory of $\mathcal{C}$ ([16], originally, a dense subcategory of $\mathcal{C}$, see [9]). One should notice that a $\mathcal{D}$-expansion relates a $\mathcal{C}$-object to a $(pro\text{-}\mathcal{D})$-object. The natural question is what about the inverse systems in $\mathcal{C}$, i.e., the $(pro\text{-}\mathcal{C})$-objects. To answer it, by following the definition strictly, the pro-pro-category $pro\text{-}(pro\text{-}\mathcal{D})$ must be taken into consideration (see, for instance, the proof of Theorem 2.4

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in [15], where a pro-reflective subcategory $\mathcal{D} \subseteq \mathcal{C}$ is characterized by the existence of a left adjoint for the induced inclusion functor $pro\text{-}\mathcal{D} \hookrightarrow pro\text{-}\mathcal{C}$).

However, dealing within a pro-pro-category setting is extremely tedious and complicated, and it often leads to unexpected ambiguities when one abbreviates notation. Therefore, for a given category $\mathcal{A}$, we have first clarified in details the relationships between $\mathcal{A}$, $pro\text{-}\mathcal{A}$ and $pro\text{-}(pro\text{-}\mathcal{A})$. Several interesting facts have occurred (see Propositions 2.1–2.6 and the "pro-pro-paradox" in Section 2). For instance, if one carelessly assumes that there exists an inclusion functor of a category $\mathcal{A}$ into the pro-category $pro\text{-}\mathcal{A}$ (one often does this), then at the next level of $pro\text{-}(pro\text{-}\mathcal{A})$ a contradiction is unavoidable. Simply saying, one may not identify $\mathcal{A}$ with its rudimentary embedding $\lfloor \mathcal{A} \rfloor$ into $pro\text{-}\mathcal{A}$. Thus, given a category pair $(\mathcal{A}, \mathcal{B})$, when it is written down $\mathcal{B} \subseteq pro\text{-}\mathcal{A}$, it is always meant that $\lfloor \mathcal{B} \rfloor \subseteq pro\text{-}\mathcal{A}$ (Remark 2.8).

Fortunately, the very definition of an expansion immediately admits an extension to inverse systems in $\mathcal{C}$. Consequently, the notion of a pro-reflective subcategory naturally extends to $\mathcal{C}$-systems. Moreover, the characterization by well known conditions (appropriate analogues of) (AE1) and (AE2) ([10,11, 14]) remains valid (Lemma 3.4), and further, an expansion in the pro-setting is also the expansion in the pro-pro-setting (Lemma 3.9).

It is a well known fact that, in general, the expansions and limits differ. On the other hand, it is fairly useful when an inverse limit is (or yields) an expansion. A part of motivation for this work was the fact that every $\mathcal{C}$-system $\boldsymbol{Y}$ admits a $(tow\text{-}\mathcal{C})$-system $\underline{\boldsymbol{Y}}$ and a natural morphism $\underline{\boldsymbol{q}} : \boldsymbol{Y} \to \underline{\boldsymbol{Y}}$ of $pro\text{-}(pro\text{-}\mathcal{C})$ which is an inverse limit ([11, Lemma II.9.2]). We have proven that it is a $(tow\text{-}\mathcal{C})$-expansion as well (Theorem 3.5). Thus, the pro-pro-category setting can provide expansions which are inverse limits as well. More precisely, every category $\mathcal{C}$ is pro-reflective for $tow\text{-}\mathcal{C}$ and $pro\text{-}\mathcal{C}$, and $tow\text{-}\mathcal{C}$ is a pro-reflective subcategory of $pro\text{-}\mathcal{C}$ by means of expansions which are inverse limits (Corollary 3.6). Furthermore, the correspondence $\boldsymbol{Y} \mapsto \underline{\boldsymbol{Y}}$ extends to a fully faithful functor of $pro\text{-}\mathcal{C}$ to $pro\text{-}(tow\text{-}\mathcal{C})$ (Theorem 3.7). There is another interesting fact concerning the mentioned limit morphism $\underline{\boldsymbol{q}} : \boldsymbol{Y} \to \underline{\boldsymbol{Y}}$. Namely, a morphism $\boldsymbol{q} : Y \to \boldsymbol{Y}$ of $pro\text{-}\mathcal{C}$ is a $\mathcal{D}$-expansion of Y if and only if the composite morphism $\underline{\boldsymbol{q}}\boldsymbol{q} : Y \to \underline{\boldsymbol{Y}}$ of $pro\text{-}(pro\text{-}\mathcal{C})$ is a $(tow\text{-}\mathcal{D})$-expansion of Y (Theorem 3.11).

Nevertheless, concerning the (abstract) shape theoretical purpose, the most interesting fact is that one can avoid the use of $(tow\text{-}\mathcal{D})$- and $(pro\text{-}\mathcal{D})$-expansions of $\mathcal{C}$-systems, i.e., the $\mathcal{D}$-expansions cover all one needs. Namely, the main general result of the paper is the following one (Theorem 4.2):

If $\mathcal{D} \subseteq \mathcal{C}$ is a pro-reflective subcategory, then so is $\mathcal{D} \subseteq pro\text{-}\mathcal{C}$.

It follows by the next (main) lemma (Lemma 4.1):

Let $\mathcal{A}$ be a category and let $\mathcal{B} \subseteq \mathcal{A}$ be a subcategory. Let $\boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ be an inverse system in $\mathcal{A}$ such that, for every $\lambda \in \Lambda$, there exists a $\mathcal{B}$-expansion

$$\boldsymbol{p}_\lambda : X_\lambda \to \boldsymbol{X}_\lambda = (X_{\mu^\lambda}, p_{\mu^\lambda \mu'^\lambda}, M^\lambda)$$

of X_λ. Then the terms and morphisms of the family of expansions $(\boldsymbol{p}_\lambda)_{\lambda\in\Lambda}$ can be naturally organized in a $\mathcal{B}$-expansion $\boldsymbol{p}' : \boldsymbol{X} \to \boldsymbol{X}'$ of $\boldsymbol{X}$.

This lemma corresponds to the analogous results concerning inverse limits and resolutions ([12,17]). Theorem 4.2 and the fact that if two of $\boldsymbol{p}$, $\boldsymbol{q}$ and $\boldsymbol{qp}$ are expansions then so is the third one (Lemma 4.4), admit a wide application. Some of them are given in the last section. There we consider the relationships between various shape categories "between" $Sh_{(\mathcal{C},\mathcal{D})}$ and $Sh_{(pro\text{-}\mathcal{C},pro\text{-}\mathcal{D})}$, yielded by a pair $(\mathcal{C},\mathcal{D})$, where $\mathcal{D}$ is a pro-reflective subcategory of $\mathcal{C}$. Further, since it makes sense to consider the shape of a $\mathcal{C}$-system, we have shown that $Sh(H\boldsymbol{X}) = Sh(\lfloor \lim \boldsymbol{X} \rfloor)$ for $(pro\text{-}cT_2, cPol)$ and $(tow\text{-}cM, cPol)$ (Corollaries 5.6 and 5.7). At the end, we deal with the category pairs $(cpl\text{-}T_3, cT_2)$ and $(cpl\text{-}T_3, \mathbb{R}\text{-}cpt)$, whereas $cpl\text{-}T_3$ is the category of completely regular (Tychonoff) spaces, $\mathbb{R}\text{-}cpt$ is the category of realcompact spaces and cT_2 is the category of compact Hausdorff spaces (all as full subcategories of Top - the category of topological spaces and mappings). It is a well known fact ([11, Example I.2.1]) that the Stone-Čech compactification $(j_X : X \to \beta X)$ and the Hewitt realcompactification $(k_X : X \to \nu X)$ are appropriate (rudimentary) expansions, and thus, there exist the corresponding shape categories. Further, β and ν are functors which admit extensions, keeping to be expansions, to the corresponding pro-categories (Corollary 5.13). In light of the previous theoretical results, we consider the shapes of inverse systems in $cpl\text{-}T_3$ with respect to cT_2 and $\mathbb{R}\text{-}cpt$. The main question is about continuity of β and ν: Under what conditions $\beta(\lim \boldsymbol{X}) \approx \lim(\beta \boldsymbol{X})$ $(\nu(\lim \boldsymbol{X}) \approx \lim(\nu \boldsymbol{X}))$ holds? Some partial answers are given by Corollary 5.22, Theorems 5.18, 5.19 and 5.21 and Remark 5.23. Finally, it is noticed (Remark 5.24) that the previous theory and technique can be applied to Hausdorff reflections ([3, j-4]). Namely, every Hausdorff reflection $h_X : X \to X_H$ is a rudimentary T_2-expansion of X, and this correspondence admits a functorial extension to the appropriate (pro-)category.

2. Motivation and preliminaries

Let $(\mathcal{A},\mathcal{B})$ be a pair of categories, $\mathcal{B} \subseteq \mathcal{A}$. Recall the notion of an expansion ([11, I.2.1.]):

An *$\mathcal{A}$-expansion with respect to $\mathcal{B}$* of an $X \in Ob\mathcal{A}$ is a morphism $\boldsymbol{p} : X \to \boldsymbol{X}$ of $pro\text{-}\mathcal{A}$ (X is viewed as a rudimentary system) such that, for every $\boldsymbol{Y} \in Ob(pro\text{-}\mathcal{B})$ and every morphism $\boldsymbol{u} : X \to \boldsymbol{Y}$ of $pro\text{-}\mathcal{A}$, there exists a unique morphism $\boldsymbol{v} : \boldsymbol{X} \to \boldsymbol{Y}$ of $pro\text{-}\mathcal{A}$ such that $\boldsymbol{vp} = \boldsymbol{u}$. If $\boldsymbol{X}$ and $\boldsymbol{v}$ belong to $pro\text{-}\mathcal{B}$, then we say that $\boldsymbol{p}$ is a *$\mathcal{B}$-expansion* (*of the $\mathcal{A}$-object* X).

Observe that this definition does not admit (formally) the notion of an "$\mathcal{A}$-expansion (with respect to $\mathcal{B}$) of an *$\mathcal{A}$-system* $\boldsymbol{X}$". Therefore, for instance, the identity morphism $1_{\boldsymbol{X}} : \boldsymbol{X} \to \boldsymbol{X}$ of *pro*-$\mathcal{A}$ cannot formally be a (trivial) $\mathcal{A}$-expansion of $\boldsymbol{X}$. Namely, to obtain an expansion of an inverse system in $\mathcal{A}$, by the definition, one necessarily needs a category pair (*pro*-$\mathcal{A}, \mathcal{K}$), where $\mathcal{K}$ is a subcategory of *pro*-$\mathcal{A}$. Then the definition works in terms of the pro-pro-category *pro*-(*pro*-$\mathcal{A}$). However, in many cases it is possible (and very useful too) to "transform" a given inverse system $\boldsymbol{X}$ in $\mathcal{A}$ into an inverse system $\boldsymbol{X}'$ in $\mathcal{B}$ and obtain a morphism $\boldsymbol{p}' : \boldsymbol{X} \to \boldsymbol{X}'$ of *pro*-$\mathcal{A}$ satisfying the condition for an $\mathcal{A}$-expansion with respect to $\mathcal{B}$. In that way one at least avoids a tedious and, in some cases, confusing work (see Propositions 2.1–2.6 and the "pro-pro-paradox" below) in the pro-pro-category *pro*-(*pro*-$\mathcal{A}$).

To avoid some ambiguities concerning the manipulation with the terms of a pro-pro-category, let us first clarify the relationship between $\mathcal{A}$ and *pro*-$\mathcal{A}$. Clearly, $\mathcal{A}$ embeds (rudimentary) into *pro*-$\mathcal{A}$ as follows.

With each object X of $\mathcal{A}$ it is associated the rudimentary inverse system $\lfloor X \rfloor$ of *pro*-$\mathcal{A}$ by putting $\lfloor X \rfloor = (X_1 = X, p_{11} = 1_X, \{1\})$, and with each morphism $f : X \to Y$ of $\mathcal{A}$ it is associated the rudimentary morphism $\lfloor f \rfloor \equiv [(\varphi, f_1)] : \lfloor X \rfloor \to \lfloor Y \rfloor$ of *pro*-$\mathcal{A}$ by putting $\varphi = 1_{\{1\}}$ and $f_1 = f$. It follows that, formally,

$$\lfloor f \rfloor : \lfloor X \rfloor \to \lfloor Y \rfloor \text{ equals to } f : X \to Y.$$

Obviously, this correspondence is injective on the objects and morphisms, and it preserves the identities and composition. Therefore, it is a faithful embedding functor. Let $\lfloor \mathcal{A} \rfloor$ denote the image of $\mathcal{A}$ in *pro*-$\mathcal{A}$ by that functor. Then, of course, $\lfloor \mathcal{A} \rfloor \subseteq$ *pro*-$\mathcal{A}$ is a (full) subcategory. Since the categories $\mathcal{A}$ and $\lfloor \mathcal{A} \rfloor$ are naturally isomorphic, one usually identifies $\mathcal{A} \equiv \lfloor \mathcal{A} \rfloor$ (via $X \equiv \lfloor X \rfloor$ and $f \equiv \lfloor f \rfloor$), and assumes that there exists the "inclusion" functor $\mathcal{A} \hookrightarrow$ *pro*-$\mathcal{A}$, i.e., that $\mathcal{A} \subseteq$ *pro*-$\mathcal{A}$, and says that $\mathcal{A}$ is a subcategory of *pro*-$\mathcal{A}$. However, by assuming this, one simply forgets that $\mathcal{A}$ is *not* a subpro-category ($\lfloor \mathcal{A} \rfloor$ is one!). We will show hereby that the mentioned identification leads to a contradiction (see the "pro-pro-paradox" below). Recall, in addition, that $\mathcal{B} \subseteq \mathcal{A}$ obviously implies that *pro*-$\mathcal{B} \subseteq$ *pro*-$\mathcal{A}$.

Now a few indispensable words about a pro-pro-category. Since the pro-category *pro*-$\mathcal{A}$ is the quotient category (*inv*-$\mathcal{A}$)/($\sim$), the pro-pro-category *pro*-(*pro*-$\mathcal{A}$) is the corresponding quotient category (*inv*-(*pro*-$\mathcal{A}$))/($\sim$). An object of *pro*-(*pro*-$\mathcal{A}$), denoted by $\underline{\boldsymbol{X}}$, is an inverse system $(\boldsymbol{X}_\lambda, \boldsymbol{p}_{\lambda\lambda'}, \Lambda)$ in *pro*-$\mathcal{A}$, while a morphism of *pro*-(*pro*-$\mathcal{A}$)($\underline{\boldsymbol{X}}, \underline{\boldsymbol{Y}}$), denoted by $\underline{\boldsymbol{f}} : \underline{\boldsymbol{X}} \to \underline{\boldsymbol{Y}}$, is the equivalence class $[(f, \boldsymbol{f}_\mu)]$ of a morphism $(f, \boldsymbol{f}_\mu) : \underline{\boldsymbol{X}} \to \underline{\boldsymbol{Y}}$ of *inv*-(*pro*-$\mathcal{A}$). The category *pro*-$\mathcal{A}$ ($\equiv \mathcal{A}'$) embeds (rudimentary) into *pro*-(*pro*$\mathcal{A}$) ($\equiv$ *pro*-$\mathcal{A}'$) in the same way as $\mathcal{A}$ embeds (rudimentary) into *pro*-$\mathcal{A}$. This means that with each object $\boldsymbol{X}$ of *pro*-$\mathcal{A}$ it is associated the rudimentary object

$$\underline{\boldsymbol{X}} = \lfloor \boldsymbol{X} \rfloor = (\boldsymbol{X}_1 = \boldsymbol{X}, \boldsymbol{p}_{11} = \boldsymbol{1}_{\boldsymbol{X}}, \{1\})$$

of *pro*-(*pro*-$\mathcal{A}$). Especially, if $\boldsymbol{X}$ itself is a rudimentary object $\lfloor X \rfloor$, $X \in Ob(\mathcal{A})$, then $\underline{\boldsymbol{X}} = \lfloor\lfloor X \rfloor\rfloor$ (obviously, every rudimentary $\underline{\boldsymbol{X}} = \lfloor \boldsymbol{X} \rfloor$ is isomorphic in *pro*-(*pro*-$\mathcal{A}$) to each $\underline{\boldsymbol{X}}' = (\boldsymbol{X}'_\nu = \boldsymbol{X}, \boldsymbol{p}'_{\nu\nu'} = \mathbf{1}_{\boldsymbol{X}}, N)$, which provides different embeddings!). Further, with each morphism $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ of *pro*-$\mathcal{A}$ it is associated the rudimentary morphism

$$\underline{\boldsymbol{f}} = \lfloor \boldsymbol{f} \rfloor : \lfloor \boldsymbol{X} \rfloor \to \lfloor \boldsymbol{Y} \rfloor$$

of *pro*-(*pro*-$\mathcal{A}$), which is the equivalence class $[(1_{\{1\}}, \boldsymbol{f}_1 = \boldsymbol{f})]$ of the morphism

$$(1_{\{1\}}, \boldsymbol{f}_1 = \boldsymbol{f}) : (\boldsymbol{X}, \mathbf{1}_{\boldsymbol{X}}, \{1\}) \to (\boldsymbol{Y}, \mathbf{1}_{\boldsymbol{Y}}, \{1\})$$

of *inv*-(*pro*-$\mathcal{A}$). Especially, if $\boldsymbol{f}$ is itself a rudimentary morphism $\lfloor f \rfloor$, $f \in \mathcal{A}(X,Y)$, then $\underline{\boldsymbol{f}} = \lfloor\lfloor f \rfloor\rfloor : \lfloor\lfloor X \rfloor\rfloor \to \lfloor\lfloor Y \rfloor\rfloor$.

In that way the category *pro*-$\mathcal{A}$ embeds into the category *pro*-(*pro*-$\mathcal{A}$) having the image $\lfloor$*pro*-$\mathcal{A}\rfloor \subseteq$ *pro*-(*pro*-$\mathcal{A}$). On the other hand, since $\lfloor \mathcal{A} \rfloor \subseteq$ *pro*-$\mathcal{A}$, there exists another "similar" subcategory *pro*-$\lfloor \mathcal{A} \rfloor \subseteq$ *pro*-(*pro*-$\mathcal{A}$). Finally, $\lfloor\lfloor \mathcal{A} \rfloor\rfloor \subseteq$ *pro*-(*pro*-$\mathcal{A}$) is also a subcategory.

PROPOSITION 2.1. *For every category $\mathcal{A}$, the categories pro-$\mathcal{A}$, $\lfloor$pro-$\mathcal{A}\rfloor$ and pro-$\lfloor \mathcal{A} \rfloor$ are naturally isomorphic. Further,*

$$\lfloor \textit{pro-}\mathcal{A} \rfloor \cap (\textit{pro-}\lfloor \mathcal{A} \rfloor) = \lfloor\lfloor \mathcal{A} \rfloor\rfloor \subseteq \textit{pro-}(\textit{pro-}\mathcal{A}).$$

PROOF. The categories *pro*-$\mathcal{A}$ and $\lfloor$*pro*-$\mathcal{A}\rfloor$ are naturally isomorphic by the restriction of the above described, in general, functorial embedding $\mathcal{K} \to \lfloor \mathcal{K} \rfloor \subseteq$ *pro*-$\mathcal{K}$. Further, recall that $\mathcal{A} \cong \lfloor \mathcal{A} \rfloor$ and observe that $\mathcal{K} \cong \mathcal{K}'$ implies *pro*-$\mathcal{K} \cong$ *pro*-$\mathcal{K}'$. Consequently, by putting $\mathcal{K} = \mathcal{A}$ and $\mathcal{K}' = \lfloor \mathcal{A} \rfloor$, we obtain *pro*-$\mathcal{A} \cong$ *pro*-$\lfloor \mathcal{A} \rfloor$. Nevertheless, we will show explicitly that the categories $\lfloor$*pro*-$\mathcal{A}\rfloor$ and *pro*-$\lfloor \mathcal{A} \rfloor$ are naturally isomorphic. Let

$$\underline{\boldsymbol{X}}^* = (\lfloor X_\lambda \rfloor, \lfloor p_{\lambda\lambda'} \rfloor, \Lambda)$$

be any object of *pro*-$\lfloor \mathcal{A} \rfloor$. Since it is induced by a unique $\boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda) \in Ob(\textit{pro-}\mathcal{A})$, let us associate with it the object

$$\underline{\boldsymbol{X}}^+ = \lfloor \boldsymbol{X} \rfloor = (\boldsymbol{X}, \mathbf{1}_{\boldsymbol{X}}, \{1\})$$

of $\lfloor$*pro*-$\mathcal{A}\rfloor$. Further, let

$$\underline{\boldsymbol{f}}^* = [(f, \lfloor f_\mu \rfloor)] : \underline{\boldsymbol{X}}^* \to \underline{\boldsymbol{Y}}^* = (\lfloor Y_\mu \rfloor, \lfloor q_{\mu\mu'} \rfloor, M)$$

be a morphism of *pro*-$\lfloor \mathcal{A} \rfloor$. Then $f : M \to \Lambda$ is a function, and $\lfloor f_\mu \rfloor : \lfloor X_{f(\mu)} \rfloor \to \lfloor Y_\mu \rfloor$ are rudimentary morphisms of *pro*-$\mathcal{A}$, determined by morphisms $f_\mu : X_{f(\mu)} \to Y_\mu$ of $\mathcal{A}$, $\mu \in M$, such that

$$\boldsymbol{f} \equiv [(f, f_\mu)] : (X_\lambda, p_{\lambda\lambda'}, \Lambda) \to (Y_\mu, q_{\mu\mu'}, M) = \boldsymbol{Y}$$

is a morphism of *pro*-$\mathcal{A}$. Let us associate with $\underline{\boldsymbol{f}}^*$ the morphism

$$\underline{\boldsymbol{f}}^+ = [(f^+, \boldsymbol{f}_1^+)] : \underline{\boldsymbol{X}}^+ \to \underline{\boldsymbol{Y}}^+ = \lfloor \boldsymbol{Y} \rfloor = (\boldsymbol{Y}, \mathbf{1}_{\boldsymbol{Y}}, \{1\})$$

of $\lfloor pro\text{-}\mathcal{A} \rfloor$ by putting $f^+ = 1_{\{1\}}$ and $\boldsymbol{f}_1^+ = \boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$. It is now readily seen that the described correspondence yields a natural isomorphism of $pro\text{-}\lfloor \mathcal{A} \rfloor$ onto $\lfloor pro\text{-}\mathcal{A} \rfloor$. Finally, since the subcategories $\lfloor pro\text{-}\mathcal{A} \rfloor$ and $pro\text{-}\lfloor \mathcal{A} \rfloor$ of $pro\text{-}(pro\text{-}\mathcal{A})$ are full and share only the rudimentary systems in $pro\text{-}\mathcal{A}$ consisting of rudimentary systems in $\mathcal{A}$, the second assertion holds as well. □

Let $\boldsymbol{f} = [(f, f_\mu)] : \boldsymbol{X} \to \boldsymbol{Y}$ be a morphism of $pro\text{-}\mathcal{A}$. Then one readily sees that $\boldsymbol{f}$ induces a unique morphism

$$\underline{\boldsymbol{f}} = [(c_1, \boldsymbol{f}_\mu)] : \underline{\boldsymbol{X}}^+ \to \underline{\boldsymbol{Y}}^*$$

of $pro\text{-}(pro\text{-}\mathcal{A})$, where c_1 is the constant function $M \to \{1\}$, while $\boldsymbol{f}_\mu = [(f^\mu, f_1^\mu)] : \boldsymbol{X} \to \lfloor Y_\mu \rfloor$ of $pro\text{-}\mathcal{A}$ is defined by $f^\mu(1) = \mu$ and $f_1^\mu = f_\mu$, $\mu \in M$. An immediate consequence of this fact is that $pro\text{-}\mathcal{A}(\boldsymbol{X}, \boldsymbol{Y}) \neq \varnothing$ implies $pro\text{-}(pro\text{-}\mathcal{A})(\underline{\boldsymbol{X}}^+, \underline{\boldsymbol{Y}}^*) \neq \varnothing$.

Proposition 2.2. *Let $\boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ be an object of pro-$\mathcal{A}$ and let $\boldsymbol{1_X} = [(1_\Lambda, 1_{X_\lambda})] : \boldsymbol{X} \to \boldsymbol{X}$ be the identity morphism of pro-$\mathcal{A}$. Then the induced morphism*

$$\underline{\boldsymbol{1_X}} \equiv \underline{\boldsymbol{p}} = [(c_1, \boldsymbol{p}_\lambda)] : \underline{\boldsymbol{X}}^+ \to \underline{\boldsymbol{X}}^*$$

of pro-(pro-$\mathcal{A}$), whereas

$$\boldsymbol{p}_\lambda = [(p^\lambda = i_\lambda, p_1^\lambda = 1_{X_\lambda})] : \boldsymbol{X} \to \lfloor X_\lambda \rfloor, \qquad \lambda \in \Lambda,$$

is an $\lfloor \mathcal{A} \rfloor$-expansion of $\boldsymbol{X}$ (viewed in pro-(pro-$\mathcal{A}$) as the rudimentary $\underline{\boldsymbol{X}}^+ \equiv \lfloor \boldsymbol{X} \rfloor$).

Proof. In order to prove the statement, we are to consider the category pair $(\mathcal{C}, \mathcal{D}) = (pro\text{-}\mathcal{A}, \lfloor \mathcal{A} \rfloor)$. Let an object $\underline{\boldsymbol{Y}}^* = (\lfloor Y_\mu \rfloor, \lfloor q_{\mu\mu'} \rfloor, M)$ of $pro\text{-}\lfloor \mathcal{A} \rfloor \subseteq pro\text{-}(pro\text{-}\mathcal{A})$ and a morphism $\underline{\boldsymbol{u}} = [(c_1, \boldsymbol{u}_\mu)] : \lfloor \boldsymbol{X} \rfloor \to \underline{\boldsymbol{Y}}^*$ of $pro\text{-}(pro\text{-}\mathcal{A})$ be given. Then, for every $\mu \in M$, $\boldsymbol{u}_\mu : \boldsymbol{X} \to \lfloor Y_\mu \rfloor$ is a morphism of $pro\text{-}\mathcal{A}$ such that $\boldsymbol{u}_\mu = \lfloor q_{\mu\mu'} \rfloor \boldsymbol{u}_{\mu'}$ whenever $\mu \leq \mu'$. Further, $\boldsymbol{u}_\mu = [(u^\mu, u_1^\mu)]$, where $u^\mu : \{1\} \to \Lambda$ is a function and $u_1^\mu : X_{u^\mu(1)} \to Y_\mu$ is a morphism of $\mathcal{A}$ such that, for every related pair $\mu \leq \mu'$, there exists a $\lambda \in \Lambda$ $\lambda \geq u^\mu(1), u^{\mu'}(1)$, so that

$$u_1^\mu p_{u^\mu(1)\lambda} = q_{\mu\mu'} u_1^{\mu'} p_{u^{\mu'}(1)\lambda}$$

(in $\mathcal{A}$). Let us define $v : M \to \Lambda$ by putting $v(\mu) = u^\mu(1)$. Then, for each $\mu \in M$, put $\boldsymbol{v}_\mu : \lfloor X_{v(\mu)} \rfloor \to \lfloor Y_\mu \rfloor$ to be the rudimentary morphism $\lfloor u_1^\mu \rfloor$ of $\lfloor \mathcal{A} \rfloor \subseteq pro\text{-}\mathcal{A}$. It is now straightforward to see that

$$\underline{\boldsymbol{v}} = [(v, \boldsymbol{v}_\mu)] : \underline{\boldsymbol{X}}^* \to \underline{\boldsymbol{Y}}^*$$

is a morphism of $pro\text{-}\lfloor \mathcal{A} \rfloor \subseteq pro\text{-}(pro\text{-}\mathcal{A})$, which is the unique morphism of $pro\text{-}(pro\text{-}\mathcal{A})$ satisfying $\underline{\boldsymbol{v}}\ \underline{\boldsymbol{p}} = \underline{\boldsymbol{u}}$. Indeed, the factorization $\underline{\boldsymbol{v}}\ \underline{\boldsymbol{p}} = \underline{\boldsymbol{u}}$ holds trivially. To prove the uniqueness, let

$$\underline{\boldsymbol{w}} = [(w, \boldsymbol{w}_\mu)] : \underline{\boldsymbol{X}}^* \to \underline{\boldsymbol{Y}}^*$$

be any morphism of *pro*-(*pro*$\mathcal{A}$) such that

$$\underline{\boldsymbol{w}}\underline{\boldsymbol{p}} = \underline{\boldsymbol{v}}\underline{\boldsymbol{p}} : \lfloor \boldsymbol{X} \rfloor \to \underline{\boldsymbol{Y}}^*.$$

Therefore, since $\lfloor \boldsymbol{X} \rfloor$ is a rudimentary object,

$$(\forall \mu \in M)\ \boldsymbol{w}_\mu \boldsymbol{p}_{w(\mu)} = \boldsymbol{v}_\mu \boldsymbol{p}_{v(\mu)} : \boldsymbol{X} \to \lfloor Y_\mu \rfloor$$

in *pro*-$\mathcal{A}$. Since *pro*-$\lfloor\mathcal{A}\rfloor \subseteq$ *pro*-(*pro*$\mathcal{A}$) is a full subcategory, any representative $(w, \boldsymbol{w}_\mu)$ of $\underline{\boldsymbol{w}}$ consists of a function $w : M \to \Lambda$ and of morphisms

$$\boldsymbol{w}_\mu = [(1_{\{1\}}, w_1^\mu)] = \lfloor w_\mu \rfloor : \lfloor X_{w(\mu)} \rfloor \to \lfloor Y_\mu \rfloor, \qquad \mu \in M,$$

of $\lfloor\mathcal{A}\rfloor \subseteq$ *pro*-$\mathcal{A}$ (rudimentary morphisms of *pro*-$\mathcal{A}$). Consequently (in *inv*-$\mathcal{A}$), for every $\mu \in M$,

$$(1_{\{1\}}, w_1^\mu = w_\mu)(i_\lambda, 1_{X_\lambda}) \sim (1_{\{1\}}, v_1^\mu = v_\mu)(i_\lambda, 1_{X_\lambda}) : \boldsymbol{X} \to \lfloor Y_\mu \rfloor.$$

Hence, for every $\mu \in M$,

$$(i_{w(\mu)}, w_\mu) \sim (i_{v(\mu)}, v_\mu) : \boldsymbol{X} \to \lfloor Y_\mu \rfloor.$$

This means that, for every $\mu \in M$, there exists a $\lambda \in \Lambda$, $\lambda \geq w(\mu), v(\mu)$, such that (in $\mathcal{A}$)

$$w_\mu p_{w(\mu)\lambda} = v_\mu p_{v(\mu)\lambda}.$$

Consequently, for every $\mu \in M$,

$$\lfloor w_\mu \rfloor \lfloor p_{w(\mu)\lambda} \rfloor = \lfloor v_\mu \rfloor \lfloor p_{v(\mu)\lambda} \rfloor,$$

i.e.,

$$\boldsymbol{w}_\mu \lfloor p_{w(\mu)\lambda} \rfloor = \boldsymbol{v}_\mu \lfloor p_{v(\mu)\lambda} \rfloor$$

in $\lfloor\mathcal{A}\rfloor \subseteq$ *pro*-$\mathcal{A}$, which proves that $\underline{\boldsymbol{w}} = \underline{\boldsymbol{v}}$. Therefore, the morphism $\underline{\boldsymbol{p}} : \boldsymbol{X}(\equiv \underline{\boldsymbol{X}}^+) \to \underline{\boldsymbol{X}}^*$ of *pro*-(*pro*-$\mathcal{A}$) is an $\lfloor\mathcal{A}\rfloor$-expansion of $\boldsymbol{X}$ of *pro*-$\mathcal{A}$ (i.e., of $\underline{\boldsymbol{X}}^+$ of $\lfloor$*pro*-$\mathcal{A}\rfloor$). ∎

PROPOSITION 2.3. *There exist a category $\mathcal{A}$ and an inverse sequence $\boldsymbol{Y}$ in $\mathcal{A}$ such that, for every inverse system $\boldsymbol{X}$ in $\mathcal{A}$, there is no morphism of $\underline{\boldsymbol{X}}^*$ to $\underline{\boldsymbol{Y}}^+$ of* pro-(pro-$\mathcal{A}$). *Thus, in general, $\underline{\boldsymbol{X}}^+$ and $\underline{\boldsymbol{X}}^+$ cannot be isomorphic objects of* pro-(pro-$\mathcal{A}$).

PROOF. Put $\mathcal{A}$ to be the category *Set* (or *Top* or *HTop*). Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be the inverse systems in $\mathcal{A}$, and let us assume that there exists a morphism $\underline{\boldsymbol{g}} : \underline{\boldsymbol{X}}^* \to \underline{\boldsymbol{Y}}^+$ of *pro*-(*pro*-$\mathcal{A}$). Then, $\underline{\boldsymbol{g}}$ is the equivalence class of a $(g, \boldsymbol{g}_1)$, where $g : \{1\} \to \Lambda$ is the index function and $\boldsymbol{g}_1 : \lfloor X_{g(1)} \rfloor \to \boldsymbol{Y}$ is a morphism of *pro*-$\mathcal{A}$. This implies that $\boldsymbol{X}$ and $\boldsymbol{Y}$ must have the following (nontrivial) property.

There exists a $\lambda_* \equiv g(1) \in \Lambda$ such that, for every $\mu \in M$, there exists a morphism

$$g_\mu \equiv g_\lambda^{\lambda_*} : X_{\lambda_*} \to Y_\mu$$

of $\mathcal{A}$ satisfying the following commutativity condition:

$$(\forall \mu \leq \mu')\ g_\mu = q_{\mu\mu'} g_{\mu'}.$$

Notice that the converse also holds, i.e., the existence of a $\underline{\boldsymbol{g}} : \underline{\boldsymbol{X}}^* \to \underline{\boldsymbol{Y}}^+$ is equivalent to the exhibited condition for $\boldsymbol{X}$ and $\boldsymbol{Y}$.

Now, the proof follows by Example 2.4 below. □

EXAMPLE 2.4. Let $\mathcal{A} = Set$ (or $Pol \subseteq Top$, or $HPol \subseteq HTop$) and let $\boldsymbol{Y} = (Y_j, q_{jj'}, \mathbb{N})$ be the inverse sequence in $\mathcal{A}$ defined by

$$Y_1 = \mathbb{N}, \quad \text{and} \quad Y_{j+1} = Y_j \setminus \{j\} = \mathbb{N} \setminus [1, j]_{\mathbb{N}}, \quad j \in \mathbb{N},$$

(discrete spaces) with the inclusion bonding functions

$$q_{jj'} : Y_{j'} \hookrightarrow Y_j, \qquad j \leq j'$$

(mappings or homotopy classes $[q_{jj'}] = \{q_{jj'}\}$). Then, for every inverse system $\boldsymbol{X}$ in $\mathcal{A}$, there exists no morphism of $\underline{\boldsymbol{X}}^*$ to $\underline{\boldsymbol{Y}}^+$ of *pro*-(*pro*-$\mathcal{A}$). Indeed, if it were

$$pro\text{-}(pro\text{-}\mathcal{A})(\underline{\boldsymbol{X}}^*, \underline{\boldsymbol{Y}}^+) \neq \varnothing,$$

then it would exist a $\underline{\boldsymbol{g}} : \underline{\boldsymbol{X}}^* \to \underline{\boldsymbol{Y}}^+$, and thus, it would exist a $\lambda_* = g(1) \in \Lambda$ such that, for every $j \in \mathbb{N}$, there exists a function $g_j : X_{\lambda_*} \to Y_j$ satisfying $g_j = q_{jj'} g_{j'}$, whenever $j \leq j'$. Given a $j \in \mathbb{N}$, choose an arbitrary $x_0 \in X_{\lambda_*}$, and consider the value $g_j(x_0) \equiv k \in Y_j$. Then, by construction, $k \geq j$. Choose a $j' > k$, and denote $k' \equiv g_{j'}(x_0)$. Then, $k' \geq j' > k$. On the other hand, $k = g_j(x_0) = q_{jj'} g_{j'}(x_0) = k'$ - a contradiction. Therefore, it must be $pro\text{-}(pro\text{-}\mathcal{A})(\underline{\boldsymbol{X}}^*, \underline{\boldsymbol{Y}}^+) = \varnothing$.

The previous example shows that it is possible $pro\text{-}(pro\text{-}\mathcal{A})(\underline{\boldsymbol{X}}^*, \underline{\boldsymbol{X}}^+) = \varnothing$ though, for every $\mathcal{A}$ and every $\boldsymbol{X}$, $pro\text{-}\mathcal{A}(\boldsymbol{X}, \boldsymbol{X}) \neq \varnothing$. A deeper view into the relationship between $\underline{\boldsymbol{X}}^+$ and $\underline{\boldsymbol{X}}^*$ (and $\boldsymbol{X}$ as well) is exhibited by the following three propositions.

PROPOSITION 2.5. *If an inverse system $\boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ in a category $\mathcal{A}$ is uniformly movable, then there exists a morphism $\underline{\boldsymbol{g}} : \underline{\boldsymbol{X}}^* \to \underline{\boldsymbol{X}}^+$ of pro-(pro-$\mathcal{A}$), but not conversely.*

PROOF. Let $\boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ be uniformly movable. Then each λ admits a $\lambda' \geq \lambda$ and a morphism

$$\boldsymbol{r}^{\lambda'} = [(r^{\lambda'} = c_1, r_\lambda^{\lambda'})] : \lfloor X_{\lambda'} \rfloor = (X_{\lambda'}, 1_{X_{\lambda'}}, \{1\}) \to \boldsymbol{X}$$

of *pro*-$\mathcal{A}$ such that

$$\boldsymbol{p}_\lambda \boldsymbol{r}^{\lambda'} = \lfloor p_{\lambda\lambda'} \rfloor,$$

where $\boldsymbol{p}_\lambda : \boldsymbol{X} \to \lfloor X_\lambda \rfloor$ is the equivalence class of $(i_\lambda, 1_{X_\lambda})$. It follows that

$$(\forall \lambda_1 \leq \lambda)\ r_{\lambda_1}^{\lambda'} = p_{\lambda_1 \lambda'}$$

and

$$(\forall \lambda_1 \leq \lambda_2)\ r_{\lambda_1}^{\lambda'} = p_{\lambda_1 \lambda_2} r_{\lambda_2}^{\lambda'}.$$

Fix a $\lambda_0 \in \Lambda$ and denote a corresponding $\lambda_0' \equiv \lambda_*$. Then, according to the appropriate part (and note) of the proof of Proposition 2.3, the latter property of $\boldsymbol{r}^{\lambda_*}$ assures that the morphism $\boldsymbol{r}^{\lambda_*} : \lfloor X_{\lambda_*} \rfloor \to \boldsymbol{X}$ yields the morphism

$$\underline{\boldsymbol{r}} = [(r, \boldsymbol{r}_1 = \boldsymbol{r}^{\lambda_*})] : \underline{\boldsymbol{X}}^* \to \underline{\boldsymbol{X}}^+,$$

of *pro*-(*pro*-$\mathcal{A}$), whereas $r : \{1\} \to \Lambda$, $r(1) = \lambda_*$ (actually, that property of $\boldsymbol{r}^{\lambda_*}$ is equivalent to the existence of $\underline{\boldsymbol{r}}$!). The converse does not hold because the latter property of $\boldsymbol{r}^{\lambda_*}$ does not, in general, imply the former one. Thus, the existence of an $\boldsymbol{r}^{\lambda_*} : \lfloor X_{\lambda_*} \rfloor \to \boldsymbol{X}$ cannot imply that $\boldsymbol{X}$ is uniformly movable. ∎

PROPOSITION 2.6. *For every inverse system $\boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ in a category $\mathcal{A}$, the following assertions are equivalent:*

(i) *$\underline{\boldsymbol{X}}^+$ and $\underline{\boldsymbol{X}}^*$ are isomorphic objects of pro-(pro-$\mathcal{A}$);*

(ii) *there exist a $\lambda_* \in \Lambda$ and a morphism $\boldsymbol{g}^{\lambda_*} : \lfloor X_{\lambda_*} \rfloor \to \boldsymbol{X}$ of pro-$\mathcal{A}$ such that $\boldsymbol{g}^{\lambda_*}\boldsymbol{i}_{\lambda_*} : \boldsymbol{X} \to \boldsymbol{X}$ is an isomorphism of pro-$\mathcal{A}$, where $\boldsymbol{i}_{\lambda_*} = [(i^{\lambda_*}(1) = \lambda_*, 1_{X_{\lambda_*}})] : \boldsymbol{X} \to \lfloor X_{\lambda_*} \rfloor$.*

PROOF. Assume that $\underline{\boldsymbol{X}}^+$ and $\underline{\boldsymbol{X}}^*$ are isomorphic objects of *pro*-(*pro*-$\mathcal{A}$). Then there exist morphisms $\underline{\boldsymbol{f}} : \underline{\boldsymbol{X}}^+ \to \underline{\boldsymbol{X}}^*$ and $\underline{\boldsymbol{g}} : \underline{\boldsymbol{X}}^* \to \underline{\boldsymbol{X}}^+$ of *pro*-(*pro*-$\mathcal{A}$) such that $\underline{\boldsymbol{g}}\ \underline{\boldsymbol{f}} = \underline{\boldsymbol{1}}_{\underline{\boldsymbol{X}}^+}$ and $\underline{\boldsymbol{f}}\ \underline{\boldsymbol{g}} = \underline{\boldsymbol{1}}_{\underline{\boldsymbol{X}}^*}$. As we showed before, $\underline{\boldsymbol{f}}$ is given by the constant index function $f : \Lambda \to \{1\}$ and by morphisms $\boldsymbol{f}_\lambda : \boldsymbol{X} \to \lfloor X_\lambda \rfloor$ of *pro*-$\mathcal{A}$, $\lambda \in \Lambda$, such that $\boldsymbol{f}_\lambda = \lfloor p_{\lambda\lambda'} \rfloor \boldsymbol{f}_{\lambda'}$, whenever $\lambda \leq \lambda'$. Recall that $\boldsymbol{f}_\lambda = [(f^\lambda, f_1^\lambda)]$, where $f^\lambda : \{1\} \to \Lambda$ and $f_1^\lambda \equiv f_\lambda : X_{f^\lambda(1)} \to X_\lambda$ are morphisms of $\mathcal{A}$, $\lambda \in \Lambda$. Then, for every related pair $\lambda \leq \lambda'$ in Λ, there exists a $\lambda'' \geq g^\lambda(\lambda), f^{\lambda'}(\lambda')$ such that

$$f_\lambda p_{f^\lambda(\lambda)\lambda''} = p_{\lambda\lambda'} f_{\lambda'} p_{f^{\lambda'}(\lambda')\lambda''}.$$

Observe that, by putting

$$h : \Lambda \to \Lambda, \quad h(\lambda) = f^\lambda(1),$$

and

$$h_\lambda = f_\lambda : X_{h(\lambda)} \to X_\lambda, \qquad \lambda \in \Lambda,$$

we obtain a morphism $\boldsymbol{h} = [(h, h_\lambda)] : \boldsymbol{X} \to \boldsymbol{X}$ of *pro*-$\mathcal{A}$. On the other side, $\underline{\boldsymbol{g}}$ is given by a unique $\lambda_* \in \Lambda$ and a unique morphism $\boldsymbol{g}^{\lambda_*} : \lfloor X_{\lambda_*} \rfloor \to \boldsymbol{X}$ of *pro*-$\mathcal{A}$. Then, further, $\boldsymbol{g}^{\lambda_*} = [(g^{\lambda_*}, g_\lambda^{\lambda_*})]$, where $g^{\lambda_*} = c_1 : \Lambda \to \{1\}$ and $g_\lambda^{\lambda_*} : X_{\lambda_*} \to X_\lambda$ are morphisms of $\mathcal{A}$, $\lambda \in \Lambda$, such that

$$g_\lambda^{\lambda_*} = p_{\lambda\lambda'} g_{\lambda'}^{\lambda_*}$$

whenever $\lambda \leq \lambda'$. Denote

$$\boldsymbol{i}_{\lambda_*} = [(i^{\lambda_*}(1) = \lambda_*, 1_{X_{\lambda_*}})] : \boldsymbol{X} \to \lfloor X_{\lambda_*} \rfloor.$$

We claim that (in *pro*-$\mathcal{A}$)

$$\boldsymbol{h}\boldsymbol{g}^{\lambda_*}\boldsymbol{i}_{\lambda_*} = \boldsymbol{1}_{\boldsymbol{X}} \quad \text{and} \quad \boldsymbol{g}^{\lambda_*}\boldsymbol{i}_{\lambda_*}\boldsymbol{h} = \boldsymbol{1}_{\boldsymbol{X}}.$$

Since $\underline{\boldsymbol{f}}\ \underline{\boldsymbol{g}} = \underline{\mathbf{1}}_{\underline{\boldsymbol{X}}^*}$, for each $\lambda \in \Lambda$, there exists a $\lambda' \geq \lambda, \lambda_*$ such that

$$\boldsymbol{f}_\lambda \boldsymbol{g}^{\lambda_*} \lfloor p_{\lambda_* \lambda'} \rfloor = \lfloor p_{\lambda\lambda'} \rfloor .$$

Then,

$$f_\lambda g^{\lambda_*}_{f^\lambda(1)} p_{\lambda_*\lambda'} = h_\lambda g^{\lambda_*}_{h(\lambda)} 1_{X_{\lambda_*}} p_{\lambda_*\lambda'} = p_{\lambda\lambda'},$$

which shows that $\boldsymbol{h}\boldsymbol{g}^{\boldsymbol{\lambda_*}}\boldsymbol{i}_{\lambda_*} = \mathbf{1}_{\boldsymbol{X}}$. On the other hand, since

$$\underline{\boldsymbol{g}}\ \underline{\boldsymbol{f}} = \underline{\mathbf{1}}_{\underline{\boldsymbol{X}}^+} = \mathbf{1}_{\lfloor \boldsymbol{X} \rfloor} = \lfloor \mathbf{1}_{\boldsymbol{X}} \rfloor ,$$

it follows that

$$\lfloor \boldsymbol{g}^{\lambda_*} \boldsymbol{f}_{\lambda_*} \rfloor = \lfloor \boldsymbol{g}^{\lambda_*} \rfloor \lfloor \boldsymbol{f}_{\lambda_*} \rfloor = \lfloor \mathbf{1}_{\boldsymbol{X}} \rfloor ,$$

and consequently, that

$$\boldsymbol{g}^{\boldsymbol{\lambda_*}} \boldsymbol{f}_{\lambda_*} = \mathbf{1}_{\boldsymbol{X}}.$$

Then, for every $\lambda \in \Lambda$, there exists a $\lambda' \geq \lambda, f^{\lambda_*}(1)$ such that

$$g^{\lambda_*}_\lambda f_{\lambda_*} p_{f^{\lambda_*}(1)\lambda'} = g^{\lambda_*}_\lambda 1_{X_{\lambda_*}} f_{\lambda_*} p_{f^{\lambda_*}(1)\lambda'} = g^{\lambda_*}_\lambda 1_{X_{\lambda_*}} h_{\lambda_*} p_{h(\lambda_*)\lambda'} = p_{\lambda\lambda'},$$

which shows that $\boldsymbol{g}^{\lambda_*}\boldsymbol{i}_{\lambda_*}\boldsymbol{h} = \mathbf{1}_{\boldsymbol{X}}$, and the claim is proved. Thus, (i) implies (ii).

Conversely, let us assume that there exist a $\lambda_* \in \Lambda$ and a morphism $\boldsymbol{g}^{\boldsymbol{\lambda_*}} = [(c_1, g^{\lambda_*}_\lambda)] : \lfloor X_{\lambda_*} \rfloor \to \boldsymbol{X}$ of *pro*-$\mathcal{A}$ such that $\boldsymbol{g}^{\boldsymbol{\lambda_*}}\boldsymbol{i}_{\lambda_*} : \boldsymbol{X} \to \boldsymbol{X}$ is an isomorphism of *pro*-$\mathcal{A}$, whereas $\boldsymbol{i}_{\lambda_*} = [(i^{\lambda_*}(1) = \lambda_*, 1_{X_{\lambda_*}})] : \boldsymbol{X} \to \lfloor X_{\lambda_*} \rfloor$. Notice that $\boldsymbol{g}^{\boldsymbol{\lambda_*}}\boldsymbol{i}_{\lambda_*} = [(i^{\lambda_*}c_1, g^{\lambda_*}_\lambda 1_{X_{\lambda_*}})] = [(c_{\lambda_*}, g^{\lambda_*}_\lambda)]$. Let $\boldsymbol{h} = [(h, h_\lambda)] : \boldsymbol{X} \to \boldsymbol{X}$ be the inverse of $\boldsymbol{g}^{\boldsymbol{\lambda_*}}\boldsymbol{i}_{\lambda_*}$ in *pro*-$\mathcal{A}$, i.e., let $\boldsymbol{g}^{\boldsymbol{\lambda_*}}\boldsymbol{i}_{\lambda_*}\boldsymbol{h} = \mathbf{1}_{\boldsymbol{X}}$ and $\boldsymbol{h}\boldsymbol{g}^{\boldsymbol{\lambda_*}}\boldsymbol{i}_{\lambda_*} = \mathbf{1}_{\boldsymbol{X}}$. For every $\lambda \in \Lambda$, let $\boldsymbol{f}_\lambda : \boldsymbol{X} \to \lfloor X_\lambda \rfloor$ be a (unique) morphism of *pro*-$\mathcal{A}$ determined by

$$(f^\lambda(1) = h(\lambda), f^\lambda_1 = h_\lambda) : X_{h(\lambda)} \to X_\lambda$$

of *inv*-$\mathcal{A}$. Then, $\boldsymbol{f}_\lambda = \lfloor p_{\lambda\lambda'} \rfloor \boldsymbol{f}_{\lambda'}$ in *pro*-$\mathcal{A}$, whenever $\lambda \leq \lambda'$, and thus, we have obtained the morphism

$$\underline{\boldsymbol{f}} = [(f, \boldsymbol{f}_\lambda)] : \underline{\boldsymbol{X}}^+ \to \underline{\boldsymbol{X}}^*$$

of *pro*-(*pro*-$\mathcal{A}$), where $f : \Lambda \to \{1\}$ is the constant function. Further, let

$$\underline{\boldsymbol{g}} = [(g, \boldsymbol{g}_1)] : \underline{\boldsymbol{X}}^* \to \underline{\boldsymbol{X}}^+$$

be the equivalence class of the morphism

$$(g, \boldsymbol{g}_1) : \underline{\boldsymbol{X}}^* \to \underline{\boldsymbol{X}}^+$$

of *inv*-(*pro*-$\mathcal{A}$), where

$$g : \{1\} \to \Lambda, \quad g(1) = \lambda_*,$$

and

$$\boldsymbol{g}_1 = \boldsymbol{g}^{\lambda_*} : \lfloor X_{\lambda_*} \rfloor \to \boldsymbol{X}.$$

Then, $\underline{\boldsymbol{g}}\ \underline{\boldsymbol{f}} = \underline{\boldsymbol{1}}_{\underline{\boldsymbol{X}}^+}$. Indeed, since $\boldsymbol{g}^{\lambda_*}\boldsymbol{i}_{\lambda_*}\boldsymbol{h} = \boldsymbol{1}_{\boldsymbol{X}}$, for every $\lambda \in \Lambda$, there exists a $\lambda' \geq \lambda, h(\lambda_*)$ such that

$$g_\lambda^{\lambda_*} h_{\lambda_*} p_{h(\lambda_*)\lambda'} = p_{\lambda\lambda'}.$$

Therefore (in $\mathcal{A}$),

$$g_\lambda^{\lambda_*} f_1^{\lambda_*} p_{h(\lambda_*)\lambda'} = p_{\lambda\lambda'},$$

which shows that (in *pro*-$\mathcal{A}$)

$$\boldsymbol{g}_1^{\lambda_*} \boldsymbol{f}_{\lambda_*} = \boldsymbol{1}_{\boldsymbol{X}},$$

and thus (in *inv*-(*pro*-$\mathcal{A}$)),

$$(g, \boldsymbol{g}_1)(f, \boldsymbol{f}_\lambda) = (1_{\{1\}}, \boldsymbol{g}_1^{\lambda_*} \boldsymbol{f}_{\lambda_*}) = \boldsymbol{1}_{\boldsymbol{X}}.$$

Hence, $\underline{\boldsymbol{g}}\ \underline{\boldsymbol{f}} = \lfloor \boldsymbol{1}_{\boldsymbol{X}} \rfloor = \underline{\boldsymbol{1}}_{\underline{\boldsymbol{X}}^+}$ in *pro*-(*pro*-$\mathcal{A}$).

On the other hand, since $\boldsymbol{h}\boldsymbol{g}^{\lambda_*}\boldsymbol{i}_{\lambda_*} = \boldsymbol{1}_{\boldsymbol{X}}$, for every $\lambda \in \Lambda$, there exists a $\lambda' \geq \lambda, \lambda_*$ such that

$$h_\lambda g_{h(\lambda)}^{\lambda_*} p_{\lambda_*\lambda'} = p_{\lambda\lambda'}.$$

Thus,

$$f_1^\lambda g_{h(\lambda)}^{\lambda_*} p_{\lambda_*\lambda'} = p_{\lambda\lambda'},$$

which shows that (in *pro*-$\mathcal{A}$)

$$\boldsymbol{f}_\lambda \boldsymbol{g}_1^{\lambda_*} \lfloor p_{\lambda_*\lambda'} \rfloor = \lfloor 1_{X_\lambda} \rfloor \lfloor p_{\lambda\lambda'} \rfloor,$$

and thus (in *inv*-(*pro*-$\mathcal{A}$)),

$$(f, \boldsymbol{f}_\lambda)(g, \boldsymbol{g}_1) = (c_{\lambda_*}, \boldsymbol{f}_\lambda \boldsymbol{g}_1^{\lambda_*}) \sim (1_\Lambda, 1_{\lfloor X_\lambda \rfloor}).$$

Hence, $\underline{\boldsymbol{f}}\ \underline{\boldsymbol{g}} = \underline{\boldsymbol{1}}_{\underline{\boldsymbol{X}}^*}$ in *pro*-(*pro*-$\mathcal{A}$), which proves that (ii) implies (i). ∎

The characterization of Proposition 2.6 admits the following one in terms of domination by a rudimentary system, which further admits to relate it to some of the well known nice properties of inverse systems.

PROPOSITION 2.7. *Let $\mathcal{A}$ be a category and let $\boldsymbol{X} \in Ob(pro\text{-}\mathcal{A})$. Then the following assertions are equivalent:*

(i) $\underline{\boldsymbol{X}}^+ \cong \underline{\boldsymbol{X}}^*$ *in pro-(pro-$\mathcal{A}$);*
(ii) $\boldsymbol{X}$ *is dominated in pro-$\mathcal{A}$ by a rudimentary system.*

Consequently, for an inverse system $\boldsymbol{X}$ in $\mathcal{A}$, if $\boldsymbol{X}$ is stable then (i) and (ii) hold, and if (i) or (ii) holds then $\boldsymbol{X}$ is strongly movable. If $\boldsymbol{X}$ is an inverse sequence, then its strong movability is equivalent to (i), (ii).

Further, in the special case of $\mathcal{A} = HPol_*$, for every inverse systems $(\boldsymbol{X}, *)$ with connected terms, the following assertions are equivalent:

(i) ′ $\underline{(\boldsymbol{X}, *)}^+ \cong \underline{(\boldsymbol{X}, *)}^*$ *in pro-(pro-$HPol_*$);*
(ii) ′ $(\boldsymbol{X}, *)$ *is dominated in pro-$HPol_*$ by a rudimentary (pointed) system;*
(iii) $(\boldsymbol{X}, *)$ *is strongly movable;*
(iv) $(\boldsymbol{X}, *)$ *is stable;*

(v) $\boldsymbol{X}$ *is stable (in pro-HPol).*

PROOF. Let $\boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ be an inverse system in a category $\mathcal{A}$. Suppose that $\underline{\boldsymbol{X}}^+ \cong \underline{\boldsymbol{X}}^*$ in *pro*-(*pro*-$\mathcal{A}$). Then, by Proposition 2.6, there exist a $\lambda_* \in \Lambda$ and a morphism $\boldsymbol{g}^{\lambda_*} : \lfloor X_{\lambda_*} \rfloor \to \boldsymbol{X}$ of *pro*-$\mathcal{A}$ such that $\boldsymbol{g}^{\lambda_*}\boldsymbol{i}_{\lambda_*} : \boldsymbol{X} \to \boldsymbol{X}$ is an isomorphism of *pro*-$\mathcal{A}$, where $\boldsymbol{i}_{\lambda_*} = [(i^{\lambda_*}(1) = \lambda_*, 1_{X_{\lambda_*}})] : \boldsymbol{X} \to \lfloor X_{\lambda_*} \rfloor$. Let $\boldsymbol{h} : \boldsymbol{X} \to \boldsymbol{X}$ be the inverse of $\boldsymbol{g}^{\lambda_*}\boldsymbol{i}_{\lambda_*}$. Put $\boldsymbol{d} = \boldsymbol{h}\boldsymbol{g}^{\lambda_*} : \lfloor X_{\lambda_*} \rfloor \to \boldsymbol{X}$ and $\boldsymbol{u} = \boldsymbol{i}_{\lambda_*} : \boldsymbol{X} \to \lfloor X_{\lambda_*} \rfloor$. Then, $\boldsymbol{d}\boldsymbol{u} = \boldsymbol{h}\boldsymbol{g}^{\lambda_*}\boldsymbol{i}_{\lambda_*} = \boldsymbol{1}_{\boldsymbol{X}}$, which implies that $\boldsymbol{X} \leq \lfloor X_{\lambda_*} \rfloor$ in *pro*-$\mathcal{A}$. Thus, (i) implies (ii). Conversely, let there exist a $Y \in Ob\mathcal{A}$ such that $\boldsymbol{X} \leq \lfloor Y \rfloor$ in *pro*-$\mathcal{A}$. This means that there exist a $\boldsymbol{d} : \lfloor Y \rfloor \to \boldsymbol{X}$ and a $\boldsymbol{u} : \boldsymbol{X} \to \lfloor Y \rfloor$ of *pro*-$\mathcal{A}$ such that $\boldsymbol{d}\boldsymbol{u} = \boldsymbol{1}_{\boldsymbol{X}}$. Then there exist an index $\lambda_* \in \Lambda$ and a morphism $u^{\lambda_*} : X_{\lambda_*} \to Y$ of $\mathcal{A}$ such that $\boldsymbol{u} = [(u(1) = \lambda_*, u^{\lambda_*})]$. On the other hand, $\boldsymbol{d}$ is the equivalence class of an appropriate (*inv*-$\mathcal{A}$)-morphism $(c_1, d_\lambda) : \lfloor Y \rfloor \to \boldsymbol{X}$. Thus, $\boldsymbol{d}\boldsymbol{u} = [(c_{\lambda_*}, d_\lambda u^{\lambda_*})]$. Notice that

$$(c_1, d_\lambda u^{\lambda_*}) : \lfloor X_{\lambda_*} \rfloor \to \boldsymbol{X}$$

is a morphism of *inv*-$\mathcal{A}$. Put $\boldsymbol{g}^{\lambda_*} = [(c_1, d_\lambda u^{\lambda_*})]$.

Then,

$$\boldsymbol{g}^{\lambda_*}\boldsymbol{i}_{\lambda_*} = [(i^{\lambda_*}c_1, d_\lambda u_{\lambda_*} 1_{X_{\lambda_*}})] = [(c_{\lambda_*}, d_\lambda u^{\lambda_*}))] = \boldsymbol{d}\boldsymbol{u} = \boldsymbol{1}_{\boldsymbol{X}}.$$

Hence, $\boldsymbol{g}^{\lambda_*}\boldsymbol{i}_{\lambda_*}$ is an isomorphism of *pro*-$\mathcal{A}$. By Proposition 2.6, it follows that $\underline{\boldsymbol{X}}^+ \cong \underline{\boldsymbol{X}}^*$ in *pro*-(*pro*-$\mathcal{A}$), which proves that (ii) implies (i).

Assume now that $\boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ is stable, i.e., let there exist an object Y of $\mathcal{A}$ and an isomorphism

$$\boldsymbol{g} = [(\boldsymbol{c}_1, g_\lambda)] : \lfloor Y \rfloor \to \boldsymbol{X} \text{ of } \textit{pro-}\mathcal{A}.$$

Then, especially, $\boldsymbol{X} \leq \lfloor Y \rfloor$ in *pro*-$\mathcal{A}$ and, as we have proven, $\underline{\boldsymbol{X}}^+ \cong \underline{\boldsymbol{X}}^*$ in *pro*-(*pro*-$\mathcal{A}$). Further, by Corollary II.9.1 of [11], $\boldsymbol{X} \leq \lfloor Y \rfloor$ implies that $\boldsymbol{X}$ is strongly movable. If, in addition, $\boldsymbol{X}$ is an inverse sequence, then the conclusion follows by [11, Theorem II.9.6] and the proven facts.

Consider now the special case of inverse systems consisting of pointed connected polyhedra bonded by the pointed homotopy classes of pointed mappings, i.e. the pro-category *pro*-$HPol_*$. Let $(\boldsymbol{X}, *) = ((X_\lambda, *), p_{\lambda\lambda'}, \Lambda) \in Ob(\textit{pro-}HPol_*)$. Then

$$\underline{(\boldsymbol{X}, *)}^+ = ((\boldsymbol{X}, *), \boldsymbol{1}, \{1\}) \text{ and } \underline{(\boldsymbol{X}, *)}^* = (\lfloor (X_\lambda, *) \rfloor, \lfloor p_{\lambda\lambda'} \rfloor, \Lambda)$$

are objects of *pro*-(*pro*-$HPol_*$). By [11, Theorem II.9.7], $(\boldsymbol{X}, *)$ is stable if and only if it is strongly movable (the both properties regarding to *pro*-$HPol_*$). Therefore, by the previously proven statements, it follows that assertions (i)′, (ii)′, (iii) and (iv) are mutually equivalent. Finally, by [11, Theorem II.9.2], the stability of an $(\boldsymbol{X}, *) \in Ob(\textit{pro-}HPol_*)$ is equivalent to stability of $\boldsymbol{X} \in Ob(\textit{pro-}HPol)$. ∎

Observe that the property $\underline{\boldsymbol{X}}^+ \cong \underline{\boldsymbol{X}}^*$ in *pro*-(*pro*-$\mathcal{A}$) is an isomorphism invariant of inverse systems in any *pro*-$\mathcal{A}$ (namely, it is readily seen that "+" and "$*$" admit extensions to functors!). Therefore, it is a shape (standard and abstract) invariant (defined via appropriate expansions). Moreover, according to Proposition 2.7 and some relevant results of [11, II.9], this invariant, generally, lies strictly between the stability and strong movability.

The "pro-pro-paradox". Let $\boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ be an inverse system in $\mathcal{A}$. Then the identity morphism $\mathbf{1}_{\boldsymbol{X}}$ of *pro*-$\mathcal{A}$ yields the (rudimentary) "identity"

$$\underline{\boldsymbol{p}}^+ = \lfloor \mathbf{1}_{\boldsymbol{X}} \rfloor : \boldsymbol{X} \to \underline{\boldsymbol{X}}^+ = \lfloor \boldsymbol{X} \rfloor$$

of $\lfloor$*pro*-$\mathcal{A}\rfloor \subseteq$ *pro*-(*pro*-$\mathcal{A}$), which is the trivial $\lfloor$*pro*-$\mathcal{A}\rfloor$-expansion of $\boldsymbol{X}$. On the other hand, by Proposition 2.2, the identity morphism $\mathbf{1}_{\boldsymbol{X}} = [(1_\Lambda, 1_{X_\lambda})]$: $\boldsymbol{X} \to \boldsymbol{X}$ induces the morphism

$$\underline{\mathbf{1}_{\boldsymbol{X}}} \equiv \underline{\boldsymbol{p}}^* = [(c_1, \boldsymbol{p}_\lambda)] : \underline{\boldsymbol{X}}^+ \to \underline{\boldsymbol{X}}^*$$

of *pro*-(*pro*-$\mathcal{A}$), that is an $\lfloor\mathcal{A}\rfloor$-expansion of $\lfloor \boldsymbol{X} \rfloor$ ($= \underline{\boldsymbol{X}}^+$). Then, it is readily seen that $\underline{\boldsymbol{p}}^*$ is a (rudimentary) (*pro*-$\lfloor\mathcal{A}\rfloor$)-expansion of $\boldsymbol{X}$ as well (see also Lemma 3.9 below). Now, if omitting the brackets "$\lfloor\cdot\rfloor$" would be allowed, then both $\underline{\boldsymbol{p}}^+$ and $\underline{\boldsymbol{p}}^*$ would be the (*pro*-$\mathcal{A}$)-expansions $\underline{\boldsymbol{p}}^+ : \boldsymbol{X} \to \underline{\boldsymbol{X}}^+$, $\underline{\boldsymbol{p}}^* : \boldsymbol{X} \to \underline{\boldsymbol{X}}^*$ of $\boldsymbol{X}$. Since we just have proven (Propositions 2.6 and 2.7) that, in general, $\underline{\boldsymbol{X}}^+$ and $\underline{\boldsymbol{X}}^*$ are not isomorphic, it would contradict Remark I.2.2 of [11]. The explanation of this "pro–pro-paradox" is rather simple. Indeed, first, *pro*-$\mathcal{A}$ is *not* a subcategory of *pro*-(*pro*-$\mathcal{A}$), and, second, although naturally isomorphic, the categories $\lfloor$*pro*-$\mathcal{A}\rfloor$ and *pro*-$\lfloor\mathcal{A}\rfloor$ are *different* subcategories of *pro*-(*pro*-$\mathcal{A}$). Thus, $\underline{\boldsymbol{p}}^+$ and $\underline{\boldsymbol{p}}^*$ are expansions (morphisms) in different category pairs -

$$(pro\text{-}(pro\text{-}\mathcal{A}), \lfloor pro\text{-}\mathcal{A}\rfloor) \quad \text{and} \quad (pro\text{-}(pro\text{-}\mathcal{A}), pro\text{-}\lfloor\mathcal{A}\rfloor),$$

respectively. In other words, $\lfloor$*pro*-$\mathcal{A}\rfloor$ and *pro*-$\lfloor\mathcal{A}\rfloor$ are quite different isomorphic pro-reflective subcategories of *pro*-(*pro*-$\mathcal{A}$). There is another "odd" fact in this setting. Namely, though $\lfloor$*pro*-$\mathcal{A}\rfloor$ and *pro*-$\lfloor\mathcal{A}\rfloor$ are isomorphic subcategories of *pro*-(*pro*$\mathcal{A}$), their corresponding (induced) objects $\underline{\boldsymbol{X}}^+$ and $\underline{\boldsymbol{X}}^*$ respectively, in general, are not isomorphic (unless $\boldsymbol{X}$ is dominated by a rudimentary system, Proposition 2.7).

Remark 2.8. As a conclusion, by the identification $\mathcal{A} \equiv \lfloor\mathcal{A}\rfloor$, i.e., by assuming that there exists the inclusion functor $\mathcal{A} \hookrightarrow$ *pro*-$\mathcal{A}$, one allows omitting the brackets "$\lfloor\cdot\rfloor$" in the notation. This further implies the identification of some inverse systems and some morphisms of inverse systems. Especially, in that case, every $\boldsymbol{X}$ (of *pro*-$\mathcal{A}$) identifies to the induced systems $\underline{\boldsymbol{X}}^*$ and $\underline{\boldsymbol{X}}^+$ (of *pro*-(*pro*$\mathcal{A}$)). However, as we have shown, it leads to a contradiction. To simplify our writing and to avoid possible ambiguities, let the **convention** be as follows:

If $\mathcal{K}$ is a category and if $\mathcal{L} \subseteq \mathcal{K}$ is a subcategory, then **whenever we write** $\mathcal{L} \subseteq pro\text{-}\mathcal{K}$, **we mean** $\lfloor \mathcal{L} \rfloor \subseteq pro\text{-}\mathcal{K}$.

3. Expansions of systems

Let us now slightly extend the notion of an expansion $\boldsymbol{q} : Y \to \boldsymbol{Y}$ ($Y \equiv \lfloor Y \rfloor$, see [11], I.2.1) to the "relative case" as well as to any system $\boldsymbol{X}$ in $\mathcal{A}$.

Definition 3.1. *Let $\mathcal{A}$ be a category and let $\mathcal{B}, \mathcal{B}' \subseteq \mathcal{A}$ be a pair of its subcategories. A* ***$\mathcal{B}$-expansion with respect to $\mathcal{B}'$ of an inverse system*** *$\boldsymbol{X} \in Ob(pro\text{-}\mathcal{A})$ is a morphism $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ of pro-$\mathcal{A}$, with $\boldsymbol{Y} \in Ob(pro\text{-}\mathcal{B})$, having the following universal property:*

For every $\boldsymbol{P} \in Ob(pro\text{-}\mathcal{B}')$ and every morphism $\boldsymbol{u} : \boldsymbol{X} \to \boldsymbol{P}$ of pro-$\mathcal{A}$ there exists a unique morphism $\boldsymbol{v} : \boldsymbol{Y} \to \boldsymbol{P}$ of pro-$\mathcal{A}$ such that $\boldsymbol{vf} = \boldsymbol{u}$.

If $\mathcal{B}' = \mathcal{B}$ and $\boldsymbol{v}$ belongs to pro-$\mathcal{B}$, we simply say that $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ is a ***$\mathcal{B}$-expansion*** *(of $\boldsymbol{X}$).*

If $\mathcal{B} = \mathcal{A}$ and $\boldsymbol{X} = \lfloor Y \rfloor$, the notion reduces to the usual one, i.e., to an $\mathcal{A}$-expansion (with respect to $\mathcal{B}'$) of Y. Further, if $\mathcal{B}' = \mathcal{B}$ and $\boldsymbol{X} = \lfloor Y \rfloor$, the notion again reduces to the usual one, i.e., to a $\mathcal{B}$-expansion of Y. Finally, if $\mathcal{B}, \mathcal{B}' \subseteq \mathcal{A}$ are full subcategories and $\boldsymbol{X} = \lfloor Y \rfloor$, then a $\mathcal{B}$-expansion with respect to $\mathcal{B}'$ of a $Y \in Ob\mathcal{A}$, i.e., $\boldsymbol{f} \equiv \boldsymbol{q} : Y \to \boldsymbol{Y}$, is just a usual $\mathcal{A}$-expansion of Y with $\boldsymbol{Y} \in Ob(pro\text{-}\mathcal{B})$ and $\boldsymbol{P} \in Ob(pro\text{-}\mathcal{B}')$. Thus, our definition of an expansion extends the original one in all prospects. Notice that, by the above definition, each isomorphism $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ of *pro*-$\mathcal{A}$ is an $\mathcal{A}$-expansion with respect to every $\mathcal{B}' \subseteq \mathcal{A}$. of $\boldsymbol{X}$.

Recall that a subcategory $\mathcal{D} \subseteq \mathcal{C}$ is said to be *pro-reflective* ([16, Section 3.3]; originally, *dense in* $\mathcal{C}$, see [11, I.2.2]) provided every $\mathcal{C}$-object X admits a $\mathcal{D}$-expansion $\boldsymbol{p} : X \to \boldsymbol{X}$ (with respect to $\mathcal{D}$ itself!). Especially, if $\mathcal{B} \subseteq \mathcal{A}$, then $\lfloor \mathcal{B} \rfloor \equiv \mathcal{D} \subseteq \mathcal{C} \equiv pro\text{-}\mathcal{A}$ is a pro-reflective subcategory provided, for every $\boldsymbol{X} \in Ob(pro\text{-}\mathcal{A})$, there exist a $\underline{\boldsymbol{Y}}^* \in Ob(pro\text{-}\lfloor \mathcal{B} \rfloor)$ and a morphism $\underline{\boldsymbol{f}} : \lfloor \boldsymbol{X} \rfloor \equiv \underline{\boldsymbol{X}}^+ \to \underline{\boldsymbol{Y}}^*$ of *pro*-(*pro*-$\mathcal{A}$) such that, for every $\underline{\boldsymbol{P}}^* \in Ob(pro\text{-}\lfloor \mathcal{B} \rfloor)$ and every morphism $\underline{\boldsymbol{u}} : \underline{\boldsymbol{X}}^+ \to \underline{\boldsymbol{P}}^*$ of *pro*-(*pro*-$\mathcal{A}$), there exists a unique morphism $\underline{\boldsymbol{v}} : \underline{\boldsymbol{Y}}^* \to \underline{\boldsymbol{P}}^*$ of *pro*-(*pro*-$\mathcal{A}$) such that $\underline{\boldsymbol{v}}\ \underline{\boldsymbol{f}} = \underline{\boldsymbol{u}}$, i.e., the corresponding diagram

$$\begin{array}{ccc} \underline{\boldsymbol{Y}}^* & \overset{\underline{\boldsymbol{f}}}{\leftarrow} & \lfloor \boldsymbol{X} \rfloor \equiv \underline{\boldsymbol{X}}^+ \\ \underline{\boldsymbol{v}} & \searrow & \downarrow \underline{\boldsymbol{u}} \\ & & \underline{\boldsymbol{P}}^* \end{array}$$

in *pro*-(*pro*-$\mathcal{A}$) commutes. Observe that the morphisms $\underline{\boldsymbol{f}}$, $\underline{\boldsymbol{u}}$ and $\underline{\boldsymbol{v}}$ are not rudimentary ones. Further, by Proposition 2.2, $\lfloor \mathcal{A} \rfloor \subseteq pro\text{-}\mathcal{A}$ is a pro-reflective subcategory (see also Corollary 3.6 below).

We now extend the notion of a pro-reflective subcategory, according to Definition 3.1, in the following way.

DEFINITION 3.2. *Let* ($\mathcal{A}$,$\mathcal{B}$) *be a category pair. Then* $\mathcal{B}$ *is said to be a* ***system pro-reflective*** *subcategory of* $\mathcal{A}$ *provided every* $\mathcal{A}$*-system* $\boldsymbol{X}$ *admits a* $\mathcal{B}$*-expansion* $\boldsymbol{f}:\boldsymbol{X}\to\boldsymbol{Y}$.

Notice that Remark I.2.2 of [11] obviously generalizes to this setting, i.e., all ("absolute") $\mathcal{B}$-expansions of an $\mathcal{A}$-system are naturally isomorphic (as objects) in *pro*-$\mathcal{A}$ (in *pro*-$\mathcal{B}$, whenever $\mathcal{B}\subseteq\mathcal{A}$ is full). However, it is *not* true in the "relative case", i.e., for $\mathcal{B}$-expansions (even of an $\mathcal{A}$-object) with respect to a $\mathcal{B}'\subsetneq\mathcal{B}$, as the next example shows.

EXAMPLE 3.3. Let $\mathcal{A}\subseteq Top$ (the category of topological spaces and mappings) be the full subcategory determined by all completely regular spaces (i.e., Tychonoff spaces), let $\mathcal{B}\subseteq\mathcal{A}$ be the full subcategory determined by all realcompact spaces, and let $\mathcal{B}'\subseteq\mathcal{A}$ be the full subcategory determined by all compact Hausdorff spaces. Clearly, $\mathcal{B}'\subseteq\mathcal{B}$ is also a full subcategory. By [11, Example I.2.1], $\mathcal{B}$ and $\mathcal{B}'$ are pro-reflective subcategories of $\mathcal{A}$ via rudimentary expansions (i.e., via the Hewitt realcompactification, $X\mapsto\nu X$, and the Stone-Čech compactification, $X\mapsto\beta X$, respectively). Let $X\in Ob(\mathcal{B})\subseteq Ob(\mathcal{A})$. Then, νX is homeomorphic to X, and thus, $\lfloor 1_X\rfloor:\lfloor X\rfloor\to\lfloor X\rfloor$ is a $\mathcal{B}$-expansion (with respect to $\mathcal{B}$) of $\lfloor X\rfloor\in Ob\lfloor\mathcal{B}\rfloor\subseteq Ob(pro\text{-}\mathcal{B})$. Since $\mathcal{B}'\subseteq\mathcal{B}$, $\lfloor 1_X\rfloor:\lfloor X\rfloor\to\lfloor X\rfloor$ is a $\mathcal{B}$-expansion with respect to $\mathcal{B}'$ of $\lfloor X\rfloor$ as well. Further, $\lfloor j_X\rfloor:\lfloor X\rfloor\to\lfloor\beta X\rfloor$ is a $\mathcal{B}'$-expansion (with respect to $\mathcal{B}'$) of $\lfloor X\rfloor$. Thus, $\lfloor j_X\rfloor:\lfloor X\rfloor\to\lfloor\beta X\rfloor$ is also a $\mathcal{B}$-expansion with respect to $\mathcal{B}'$ of $\lfloor X\rfloor$. Now, if it were $\lfloor X\rfloor\cong\lfloor\beta X\rfloor$ in *pro*-$\mathcal{B}$, then $\lfloor X\rfloor$ would be isomorphic to $\lfloor\beta X\rfloor$ in $\lfloor\mathcal{B}\rfloor\subseteq pro\text{-}\mathcal{B}$, and thus, X would be homeomorphic, to βX in $\mathcal{B}\subseteq Top$. However, if X is not compact, X and βX cannot be homeomorphic spaces.

Let us show that the characterization of an expansion obtained in Theorem I.2.1 of [11] (see also [10] and [14]) remains valid in this extended setting, i.e., for inverse systems.

LEMMA 3.4. *Let* $\mathcal{A}$ *be a category and let* $\mathcal{B},\mathcal{B}'\subseteq\mathcal{A}$ *be a pair of its subcategories. Further, let* $\boldsymbol{X}=(X_\lambda,p_{\lambda\lambda'},\Lambda)$ *be an inverse system in* $\mathcal{A}$ *and let* $\boldsymbol{Y}=(Y_\mu,q_{\mu\mu'},M)$ *be an inverse system in* $\mathcal{B}$. *A morphism* $\boldsymbol{f}:\boldsymbol{X}\to\boldsymbol{Y}$ *of pro-*$\mathcal{A}$ *is a* $\mathcal{B}$*-expansion with respect to* $\mathcal{B}'$ *of* $\boldsymbol{X}$ *if and only if the following two conditions are fulfilled:*

(AE1) *For every* $P\in Ob\mathcal{B}'$ *and every* $\boldsymbol{u}:\boldsymbol{X}\to\lfloor P\rfloor$ *of pro-*$\mathcal{A}$*, there exist a* $\mu\in M$ *and* $v^\mu:Y_\mu\to P$ *of* $\mathcal{A}$ *such that* $\lfloor v^\mu\rfloor\boldsymbol{j}_\mu\boldsymbol{f}=\boldsymbol{u}$ *in pro-*$\mathcal{A}$*, where* $\boldsymbol{j}_\mu=[(j^\mu,1_{Y_\mu})]:\boldsymbol{Y}\to\lfloor Y_\mu\rfloor$;

(AE2) *If* $v_1^\mu,v_2^\mu:Y_\mu\to P$ *of* $\mathcal{A}$ *satisfy* $\lfloor v_1^\mu\rfloor\boldsymbol{j}_\mu\boldsymbol{f}=\lfloor v_2^\mu\rfloor\boldsymbol{j}_\mu\boldsymbol{f}$ *in pro-*$\mathcal{A}$*, then there exists a* $\mu'\geq\mu$ *such that* $v_1^\mu q_{\mu\mu'}=v_2^\mu q_{\mu\mu'}$ *in* $\mathcal{A}$.

PROOF. The necessity part is obtained by considering the rudimentary case $\boldsymbol{P}=\lfloor P\rfloor$. Indeed, given a

$$\boldsymbol{u}=[(u,u_1)]:\boldsymbol{X}\to\lfloor P\rfloor,$$

then the (unique) existing

$$\boldsymbol{v} = [(v, v_1)] : \boldsymbol{Y} \to \lfloor P \rfloor$$

provides the desired $\mu = v(1) \in M$ and $v^\mu = v_1 : Y_\mu \to P$ for (AE1), while the uniqueness of $\boldsymbol{v}$ implies that (AE2) holds.

Let us prove the sufficiency. Let a $\boldsymbol{P} = (P_\nu, r_{\nu\nu'}, N) \in Ob(pro\text{-}\mathcal{B}')$ and a $\boldsymbol{u} = [(u, u_\nu)] : \boldsymbol{X} \to \boldsymbol{P}$ of pro-$\mathcal{A}$ be given. Let (f, f_μ) be a representative of $\boldsymbol{f}$. For every $\nu \in N$, denote by $\boldsymbol{u}^\nu = [(u^\nu, u_1^\nu = u_\nu)] : \boldsymbol{X} \to \lfloor P_\nu \rfloor$, $u^\nu(1) = u(\nu)$, the morphism of pro-$\mathcal{A}$ induced by u_ν of $\boldsymbol{u}$. Clearly, $\boldsymbol{u}^\nu$ reduces to the (rudimentary) morphism $u_\nu : X_{u(\nu)} \to P_\nu$. By (AE1), there exist a $\mu_\nu \in M$ and a

$$v_\nu^\mu : Y_{\mu_\nu} \to P_\nu$$

of $\mathcal{A}$ such that $\lfloor v_\nu^\mu \rfloor \boldsymbol{j}_{\mu_\nu} \boldsymbol{f} = \boldsymbol{u}^\nu$. This means that there exists a $\lambda \in \Lambda$, $\lambda \geq u(\nu), f(\mu_\nu)$ such that

$$v_\nu^\mu f_{\mu_\nu} p_{f(\mu_\nu)\lambda} = u_\nu p_{u(\nu)\lambda}.$$

Denote by $v : N \to M$ the function determined by $\nu \mapsto v(\nu) = \mu_\nu$, and denote by $v_\nu : Y_{v(\nu)} \to P_\nu$ the morphism v_ν^μ of $\mathcal{A}$. Let us show that

$$(v, v_\nu) : \boldsymbol{Y} \to \boldsymbol{P}$$

is a morphism of inv-$\mathcal{A}$. Let $\nu \leq \nu'$ in N. Since Λ is directed, there exist $\lambda_1, \lambda_2 \in \Lambda$, $\lambda_1 \geq u(\nu), fv(\nu)$ and $\lambda_2 \geq u(\nu'), fv(\nu')$, such that

$$v_\nu f_{v(\nu)} p_{fv(\nu)\lambda_1} = u_\nu p_{u(\nu)\lambda_1}$$

and

$$v_{\nu'} f_{v(\nu')} p_{fv(\nu')\lambda_2} = u_{\nu'} p_{u(\nu')\lambda_2}.$$

Since M is directed, there exists a $\mu \in M$, $\mu \geq v(\nu), v(\nu')$. Further, since (u, u_ν) is a morphism of inv-$\mathcal{A}$, there exists a $\lambda \in \Lambda$, $\lambda \geq \lambda_1, \lambda_2, f(\mu)$ such that

$$u_\nu p_{u(\nu)\lambda} = r_{\nu\nu'} u_{\nu'} p_{u(\nu')\lambda}.$$

Further, since (f, f_μ) is a morphism of inv-$\mathcal{A}$, the above relations imply that (in $\mathcal{A}$)

$$v_\nu q_{v(\nu)\mu} f_\mu p_{f(\mu)\lambda} = r_{\nu\nu'} v_{\nu'} q_{v(\nu')\mu} f_\mu p_{f(\mu)\lambda}.$$

This means that the $\mathcal{A}$-morphisms

$$v_\nu q_{v(\nu)\mu}, r_{\nu\nu'} v_{\nu'} q_{v(\nu')\mu} : Y_\mu \to P_\nu$$

satisfy (in pro-$\mathcal{A}$)

$$\lfloor v_\nu q_{v(\nu)\mu} \rfloor \boldsymbol{j}_\mu \boldsymbol{f} = \lfloor r_{\nu\nu'} v_{\nu'} q_{v(\nu')\mu} \rfloor \boldsymbol{j}_\mu \boldsymbol{f}.$$

By (AE2), there exists a $\mu' \in M$, $\mu' \geq \mu$, such that (in $\mathcal{A}$)

$$(v_\nu q_{v(\nu)\mu}) q_{\mu\mu'} = (r_{\nu\nu'} v_{\nu'} q_{v(\nu')\mu}) q_{\mu\mu'}.$$

Therefore,

$$v_\nu q_{v(\nu)\mu'} = r_{\nu\nu'} v_{\nu'} q_{v(\nu')\mu'},$$

which proves that $(v, v_\nu) : \boldsymbol{Y} \to \boldsymbol{P}$ is a morphism of *inv*-$\mathcal{A}$. Put $\boldsymbol{v} = [(v, v_\nu)] : \boldsymbol{Y} \to \boldsymbol{P}$. Then the very construction implies that $\boldsymbol{vf} = \boldsymbol{u}$ in *pro*-$\mathcal{A}$. It remains to verify the uniqueness of $\boldsymbol{v}$. Suppose that there exists a pair $\boldsymbol{v}, \boldsymbol{w} : \boldsymbol{Y} \to \boldsymbol{P}$ such that $\boldsymbol{vf} = \boldsymbol{wf}$. Let (v, v_ν) and (w, w_ν) be representatives of $\boldsymbol{v}$ and $\boldsymbol{w}$ respectively. Then, for every $\nu \in N$, there exists a $\lambda \in \Lambda$, $\lambda \geq fv(\nu), fw(\nu)$, such that

$$v_\nu f_{v(\nu)} p_{fv(\nu)\lambda} = w_\nu f_{w(\nu)} p_{fw(\nu)\lambda}.$$

Choose a $\mu \in M$ such that $\mu \geq v(\nu), w(\nu)$. Then there exists a $\lambda' \in \Lambda$, $\lambda' \geq \lambda, f(\mu)$, such that

$$v_\nu q_{v(\nu)\mu} f_\mu p_{f(\mu)\lambda'} = w_\nu q_{w(\nu)\mu} f_\mu p_{f(\mu)\lambda'}.$$

This means that the $\mathcal{A}$-morphisms

$$v_\nu q_{v(\nu)\mu}, w_\nu q_{w(\nu)\mu} : Y_\mu \to P_\nu$$

satisfy (in *pro*-$\mathcal{A}$)

$$\lfloor v_\nu q_{v(\nu)\mu} \rfloor \boldsymbol{j}_\mu \boldsymbol{f} = \lfloor w_\nu q_{w(\nu)\mu} \rfloor \boldsymbol{j}_\mu \boldsymbol{f}.$$

By (AE2), there exists a $\mu' \in M$, $\mu' \geq \mu$, such that (in $\mathcal{A}$)

$$(v_\nu q_{v(\nu)\mu}) q_{\mu\mu'} = (w_\nu q_{w(\nu)\mu}) q_{\mu\mu'}.$$

Therefore,

$$v_\nu q_{v(\nu)\mu'} = w_\nu q_{w(\nu)\mu'},$$

which shows that $(v, v_\nu) \sim (w, w_\nu)$ in *inv*-$\mathcal{A}$, i.e., $\boldsymbol{v} = \boldsymbol{w}$. □

Recall that *tow*-$\mathcal{A}$ denotes the full subcategory of *pro*-$\mathcal{A}$ determined by all the objects $\boldsymbol{X} = (X_i, p_{ii'}, I) \in Ob(\textit{pro-}\mathcal{A})$, where $I \subseteq \mathbb{N}$ carries the inherited order. The most important objects of *tow*-$\mathcal{A}$ are the inverse sequences in $\mathcal{A}$. Recall Lemma lII.9.2 of [11]. Let $\boldsymbol{Y} = (Y_\mu, q_{\mu\mu'}, M)$ be an inverse system in an arbitrary category $\mathcal{C}$. Denote by

$$\underline{\boldsymbol{Y}} = (\boldsymbol{Y}_{\underline{\mu}}, \boldsymbol{q}_{\underline{\mu}\,\underline{\mu}'}, \underline{M})$$

the inverse system in *tow*-$\mathcal{C} \subseteq$ *pro*-$\mathcal{C}$ (an object of *pro*-(*tow*-$\mathcal{C}$) $\subseteq$ *pro*-(*pro*-$\mathcal{C}$)) indexed by all increasing sequences $\underline{\mu} = (\mu_j)$ in M, whereas

- $\boldsymbol{Y}_{\underline{\mu}} = (Y_{\mu_j}, q_{\mu_j \mu_{j'}}, \mathbb{N})$, $\underline{\mu} \in \underline{M}$, is the corresponding inverse sequence in $\boldsymbol{Y}$;
- $(\underline{M}, \leq)$ is ordered coordinatewise;
- $\boldsymbol{q}_{\underline{\mu}\,\underline{\mu}'} : \boldsymbol{Y}_{\underline{\mu}'} \to \boldsymbol{Y}_{\underline{\mu}}$, $\underline{\mu} \leq \underline{\mu}'$ in $\underline{M}$, is the level morphism (of *tow*-$\mathcal{C} \subseteq$ *pro*-$\mathcal{C}$) induced by the bonding morphisms $q_{\mu_j \mu_{j'}} : Y_{\mu_{j'}} \to Y_{\mu_j}$ of $\boldsymbol{Y}$.

Let, for every $\underline{\mu} = (\mu_j) \in \underline{M}$, $i_{\underline{\mu}} : \mathbb{N} \to M$ be the function defined by $i_{\underline{\mu}}(j) = \mu_j$, $j \in \mathbb{N}$, and let $\boldsymbol{q}_{\underline{\mu}} = [(i_{\underline{\mu}}, 1_{Y_{\mu_j}})] : \boldsymbol{Y} \to \boldsymbol{Y}_{\underline{\mu}}$ be the corresponding morphism of *pro*-$\mathcal{C}$.

Then $\underline{\boldsymbol{q}} = (\boldsymbol{q}_{\underline{\mu}}) : \boldsymbol{Y} \to \underline{\boldsymbol{Y}}$, $\underline{\mu} \in \underline{M}$, is a morphism of *pro*-(*pro*-$\mathcal{C}$) and, moreover, it is an *inverse limit* of $\underline{\boldsymbol{Y}}$ in *pro*-(*pro*-$\mathcal{C}$).

The following fact is somewhat surprising: The (limit) morphism $\underline{\boldsymbol{q}} = (\boldsymbol{q}_{\underline{\mu}}) : \boldsymbol{Y} \to \underline{\boldsymbol{Y}}$ is also an *expansion*. More precisely, the next theorem holds.

THEOREM 3.5. *For every category $\mathcal{C}$ and every inverse system $\boldsymbol{Y} = (Y_\mu, q_{\mu\mu'}, M)$ in $\mathcal{C}$, the morphism*

$$\underline{\boldsymbol{q}} = (\boldsymbol{q}_{\underline{\mu}}) : \boldsymbol{Y} \to \underline{\boldsymbol{Y}} = (\boldsymbol{Y}_{\underline{\mu}}, \boldsymbol{q}_{\underline{\mu}\,\underline{\mu}'}, \underline{M})$$

of pro-(pro-$\mathcal{C}$) is a (tow-$\mathcal{C}$)-expansion of $\boldsymbol{Y}$.

PROOF. We have to prove that, for every $\underline{\boldsymbol{P}}$ of *pro*-(*tow*-$\mathcal{C}$) and every $\underline{\boldsymbol{u}} : \lfloor \boldsymbol{Y} \rfloor \to \underline{\boldsymbol{P}}$ of *pro*-(*pro*-$\mathcal{C}$), there exists a unique $\underline{\boldsymbol{v}} : \underline{\boldsymbol{Y}} \to \underline{\boldsymbol{P}}$ of *pro*-(*pro*-$\mathcal{C}$) (actually, of *pro*-(*tow*-$\mathcal{C}$)) such that $\underline{\boldsymbol{v}}\underline{\boldsymbol{q}} = \underline{\boldsymbol{u}}$. It suffices to verify conditions (AE1) and (AE2) for $\underline{\boldsymbol{q}}$. Let $\boldsymbol{P} = (P_i, p_{ii'}, \mathbb{N})$ be any inverse sequence in $\mathcal{C}$, and let $\boldsymbol{u} : \boldsymbol{Y} \to \boldsymbol{P}$ be a morphism of *pro*-$\mathcal{C}$. Choose a special ([11, Lemma I.1.2]; $\mathbb{N}$ is cofinite) representative (u, u_i) of $\boldsymbol{u}$ in $\mathcal{C}^{\mathbb{N}} \subseteq$ *inv*-$\mathcal{C}$. Then the index function $u : \mathbb{N} \to M$ is strictly increasing. Thus, it yields $u[\mathbb{N}] \equiv \underline{\mu} = (\mu_i) \in \underline{M}$, $\mu_i \equiv u(i)$, $i \in \mathbb{N}$. Put

$$v = 1_{\mathbb{N}} : \mathbb{N} \to \mathbb{N} \quad \text{ and } \quad v_i = u_i : Y_{\mu_i} \to P_i, i \in \mathbb{N}.$$

Then, $(v, v_i) : \boldsymbol{Y}_{\underline{\mu}} \to \boldsymbol{P}$ is a morphism of $\mathcal{C}^{\mathbb{N}}$. Its equivalence class $[(v, v_i)]$ is a morphism of *tow*-$\mathcal{C} \subseteq$ *pro*-$\mathcal{C}$, denoted by $\boldsymbol{v}^{\underline{\mu}} : \boldsymbol{Y}_{\underline{\mu}} \to \boldsymbol{P}$. Observe that

$$\boldsymbol{v}^{\underline{\mu}} \boldsymbol{q}_{\underline{\mu}} = \boldsymbol{u} : \boldsymbol{Y} \to \boldsymbol{P}$$

holds in *pro*-$\mathcal{C}$ trivially by construction. This verifies condition (AE1) for $\underline{\boldsymbol{q}}$ with respect to *tow*-$\mathcal{C}$.

In order to verify condition (AE2) for $\underline{\boldsymbol{q}}$ with respect to *tow*-$\mathcal{C}$, let $\boldsymbol{v}^{\underline{\mu}}, \boldsymbol{w}^{\underline{\mu}} : \boldsymbol{Y}_{\underline{\mu}} \to \boldsymbol{P}$, $\underline{\mu} = (\mu_i) \in \underline{M}$, be a pair of morphisms of *pro*-$\mathcal{C}$ such that

$$\boldsymbol{v}^{\underline{\mu}} \boldsymbol{q}_{\underline{\mu}} = \boldsymbol{w}^{\underline{\mu}} \boldsymbol{q}_{\underline{\mu}} : \boldsymbol{Y} \to \boldsymbol{P}$$

in *pro*-$\mathcal{C}$ holds. Notice that the morphisms $\boldsymbol{v}^{\underline{\mu}}$ and $\boldsymbol{w}^{\underline{\mu}}$ belong to *tow*-$\mathcal{C}$. Choose a pair of representatives (v, v_i), (w, v_i) of $\boldsymbol{v}^{\underline{\mu}}$, $\boldsymbol{w}^{\underline{\mu}}$ in $\mathcal{C}^{\mathbb{N}}$ respectively. Then,

$$(\forall i \in \mathbb{N})(\exists \mu'_i \in M, \ \mu'_i \geq v(i), w(i)) \ v_i q_{v(i)\mu'_i} = w_i q_{w(i)\mu'_i}.$$

Now, by induction on $i \in \mathbb{N}$, one can construct an increasing sequence $\underline{\mu}'' = (\mu''_i) \in \underline{M}$ such that, for every $i \in \mathbb{N}$, $\mu''_i \geq \mu_i, \mu'_i$. Then, $\underline{\mu}'' \geq \underline{\mu}$ and

$$(\forall i \in \mathbb{N}) v_i q_{v(i)\mu''_i} = w_i q_{w(i)\mu''_i},$$

which means that

$$\boldsymbol{v}^{\underline{\mu}} \boldsymbol{q}_{\underline{\mu}\underline{\mu}''} = \boldsymbol{w}^{\underline{\mu}} \boldsymbol{q}_{\underline{\mu}\underline{\mu}''} : \boldsymbol{Y}_{\underline{\mu}''} \to \boldsymbol{P}$$

in *tow*-$\mathcal{C} \subseteq$ *pro*-$\mathcal{C}$ holds. This verifies condition (AE2) for $\underline{\boldsymbol{q}}$ with respect to *tow*-$\mathcal{C}$, and completes the proof of the theorem. □

Corollary 3.6. *Every category $\mathcal{C}$ is pro-reflective for tow-$\mathcal{C}$ and pro-$\mathcal{C}$, and tow-$\mathcal{C}$ is pro-reflective for pro-$\mathcal{C}$. Moreover, every object $\boldsymbol{Y} = (Y_\mu, q_{\mu\mu'}, M)$ of pro-$\mathcal{C}$ admits a (tow-$\mathcal{C}$)-expansion $\underline{\boldsymbol{q}} : \boldsymbol{Y} \to \underline{\boldsymbol{Y}}$ which is an inverse limit as well. More precisely, one can put $\underline{\boldsymbol{Y}}$ to be be the inverse system of all increasing inverse sequences $\boldsymbol{Y}_{\underline{\mu}}$ in $\boldsymbol{Y}$.*

Proof. By Proposition 2.2, $\mathcal{C}$ is pro-reflective for *pro*-$\mathcal{C}$, i.e., $\lfloor \mathcal{C} \rfloor \subseteq$ *pro*-$\mathcal{C}$ is a pro-reflective subcategory (Remark 2.8). Then, especially, $\lfloor \mathcal{C} \rfloor \subseteq$ *tow*-$\mathcal{C}$ is a pro-reflective subcategory. The rest follows by Theorem 3.5 and [11, Lemma II.9.2]. ◻

Let us show that the above (object) correspondence $\boldsymbol{Y} \mapsto \underline{\boldsymbol{Y}}$ admits a functorial extension (see also Corollary 4.5(ii) below).

Theorem 3.7. *For every category $\mathcal{C}$, there exists a fully faithful functor*

$$\underline{E} : pro - \mathcal{C} \to pro - (tow - \mathcal{C}).$$

Proof. Let $\boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ and $\boldsymbol{Y} = (Y_\mu, q_{\mu\mu'}, M)$ be inverse systems in $\mathcal{C}$. Let $\underline{\boldsymbol{p}} = (\boldsymbol{p}_{\underline{\lambda}}) : \boldsymbol{X} \to \underline{\boldsymbol{X}} = (\boldsymbol{X}_{\underline{\lambda}}, \boldsymbol{p}_{\underline{\lambda}\, \underline{\lambda}'}, \underline{\Lambda})$ and $\underline{\boldsymbol{q}} = (\boldsymbol{q}_{\underline{\mu}}) : \boldsymbol{Y} \to \underline{\boldsymbol{Y}} = (\boldsymbol{Y}_{\underline{\mu}}, \boldsymbol{q}_{\underline{\mu}\, \underline{\mu}'}, \underline{M})$ be *a* pair of (*tow*-$\mathcal{C}$)-expansions which are inverse limits as well - according to Corollary 3.6. Put $\underline{E}(\boldsymbol{X}) \equiv \underline{\boldsymbol{X}}$ and $\underline{E}(\boldsymbol{Y}) \equiv \underline{\boldsymbol{Y}}$. Let $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ be a morphism of *pro*-$\mathcal{C}$. Since $\underline{\boldsymbol{p}} : \boldsymbol{X} \to \underline{\boldsymbol{X}} = \underline{E}(\boldsymbol{X})$ is a (*tow*-$\mathcal{C}$)-expansion with respect to *pro*-$\mathcal{C}$ (Theorem 3.5), there exists a unique morphism $\underline{\boldsymbol{f}} : \underline{\boldsymbol{X}} \to \underline{\boldsymbol{Y}}$ of *pro*-(*pro*-$\mathcal{C}$) (actually, of *pro*-(*tow*-$\mathcal{C}$)) such that $\underline{\boldsymbol{f}}\, \underline{\boldsymbol{p}} = \underline{\boldsymbol{q}} \lfloor \boldsymbol{f} \rfloor$ in *pro*-(*pro*-$\mathcal{C}$). Put

$$\underline{E}(\boldsymbol{f}) \equiv \underline{\boldsymbol{f}} : \underline{E}(\boldsymbol{X}) \equiv \underline{\boldsymbol{X}} \to \underline{\boldsymbol{Y}} \equiv \underline{E}(\boldsymbol{Y}).$$

Observe that the mentioned uniqueness implies that $\underline{E}(\boldsymbol{1_X}) = \boldsymbol{1}_{\underline{\boldsymbol{X}}}$ and $\underline{E}(\boldsymbol{gf}) = \underline{E}(\boldsymbol{g})\underline{E}(\boldsymbol{f})$, and that $\underline{E}(\boldsymbol{f}) = \underline{E}(\boldsymbol{f}')$ implies $\boldsymbol{f} = \boldsymbol{f}'$. Thus,

$$\underline{E} : pro - \mathcal{C} \to pro - (tow - \mathcal{C}), \quad \underline{E}(\boldsymbol{X}) \equiv \underline{\boldsymbol{X}},\ \underline{E}(\boldsymbol{f}) \equiv \underline{\boldsymbol{f}},$$

is a faithful functor. Let $\underline{\boldsymbol{f}} : \underline{E}(\boldsymbol{X}) \to \underline{E}(\boldsymbol{Y})$ be a morphism of *pro*-(*tow*-$\mathcal{C}$). Then $\underline{\boldsymbol{f}}\, \underline{\boldsymbol{p}} : \boldsymbol{X} \to \underline{E}(\boldsymbol{Y}) = \underline{\boldsymbol{Y}}$ is a morphism of *pro*-(*tow*-$\mathcal{C}$). Since $\underline{\boldsymbol{q}} : \boldsymbol{Y} \to \underline{\boldsymbol{Y}}$ is an inverse limit, there exists a unique morphism $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ of *pro*-$\mathcal{C}$ such that $\underline{\boldsymbol{q}} \lfloor \boldsymbol{f} \rfloor = \underline{\boldsymbol{f}}\, \underline{\boldsymbol{p}}$. This means that $\underline{\boldsymbol{f}} = \underline{E}(\boldsymbol{f})$, which shows that the functor $\underline{E}$ is full. ◻

In addition to the proof of Theorem 3.7, it is very useful (for certain applications) to provide also an *explicit construction* of the functor $\underline{E}$ on the morphisms. Let $\underline{\boldsymbol{X}}$ and $\underline{\boldsymbol{Y}}$ be the inverse systems in *tow*-$\mathcal{C}$ obtained by all the increasing inverse sequences in inverse systems $\boldsymbol{X}$ and $\boldsymbol{Y}$ in $\mathcal{C}$ respectively. Assume first that M is cofinite. Let $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ be a morphism of *pro*-$\mathcal{C}$. Then there exists a special representative (f, f_μ) of $\boldsymbol{f}$ ([11], Lemma I.1.2). This allows to define a function $\underline{f} : \underline{M} \to \underline{\Lambda}$ by putting $\underline{f}(\underline{\mu}) = (f(\mu_j))$, where $\underline{\mu} = (\mu_j)$. Further, it admits to define, for every $\underline{\mu} \in \underline{M}$, a morphism $\boldsymbol{f}_{\underline{\mu}} : \boldsymbol{X}_{\underline{f}(\underline{\mu})} \to \boldsymbol{Y}_{\underline{\mu}}$ of *tow*-$\mathcal{C}$ by putting $\boldsymbol{f}_{\underline{\mu}} = [(\underline{f}|\underline{\mu}, f_{\mu_j})]$. Let us show that

$$(\underline{f}, \boldsymbol{f}_{\underline{\mu}}) : \underline{\boldsymbol{X}} \to \underline{\boldsymbol{Y}}$$

is a special morphism of *inv*-(*tow*-$\mathcal{C}$). Let $\underline{\mu} \leq \underline{\mu}'$ in $\underline{M}$. Notice that the function $\underline{f}$ is increasing. Thus, $\underline{f}(\underline{\mu}) \leq \underline{f}(\underline{\mu}')$. Moreover, since, for every $j \in \mathbb{N}$,

$$f_{\mu_j} p_{f(\mu_j) f(\mu'_j)} = q_{\mu_j \mu'_j} f_{\mu'_j}$$

in $\mathcal{C}$ ((f, f_{μ_j}) is special!), it follows that

$$\boldsymbol{f}_{\underline{\mu}} \boldsymbol{p}_{\underline{f}(\underline{\mu}) \underline{f}(\underline{\mu}')} = \boldsymbol{q}_{\underline{\mu}\, \underline{\mu}'} \boldsymbol{f}_{\underline{\mu}'} : \boldsymbol{X}_{\underline{f}(\underline{\mu}')} \to \boldsymbol{Y}_{\underline{\mu}}$$

in *tow*-$\mathcal{C}$. Then

$$\underline{\boldsymbol{f}} = [(\underline{f}, \boldsymbol{f}_{\underline{\mu}})] : \underline{\boldsymbol{X}} \to \underline{\boldsymbol{Y}}$$

is a morphism of *pro*-(*tow*-$\mathcal{C}$) satisfying $\underline{\boldsymbol{f}}\, \underline{\boldsymbol{p}} = \underline{\boldsymbol{q}} \lfloor \boldsymbol{f} \rfloor$ in *pro*-(*pro*-$\mathcal{C}$). We are to show that $\underline{\boldsymbol{f}}$ does not depend on the chosen special representative (f, f_{μ_j}) of $\boldsymbol{f}$. Let (f', f'_{μ_j}) be an other special representative of $\boldsymbol{f}$. Let $\underline{f}' : \underline{M} \to \underline{\Lambda}$ and $\boldsymbol{f}'_{\underline{\mu}} : \boldsymbol{X}_{\underline{f}'(\underline{\mu})} \to \boldsymbol{Y}_{\underline{\mu}}$, $\underline{\mu} \in \underline{M}$, be defined in the same way by means of (f', f'_{μ_j}). Then, for every $\underline{\mu} \in \underline{M}$, there exists a $\underline{\lambda} \in \underline{\Lambda}$, $\underline{\lambda} \geq \underline{f}(\underline{\mu}), \underline{f}'(\underline{\mu})$, such that

$$\boldsymbol{f}_{\underline{\mu}} \boldsymbol{p}_{\underline{f}(\underline{\mu})\underline{\lambda}} = \boldsymbol{f}'_{\underline{\mu}} \boldsymbol{p}_{\underline{f}'(\underline{\mu})\underline{\lambda}} : \boldsymbol{X}_{\underline{\lambda}} \to \boldsymbol{Y}_{\underline{\mu}}.$$

Indeed, for every $j \in \mathbb{N}$, there exists a $\lambda_j \geq f(\mu_j), f'(\mu_j)$ such that

$$f_{\mu_j} p_{f(\mu_j)\lambda_j} = f'_{\mu_j} p_{f'(\mu_j)\lambda_j}.$$

Since M is directed, one can construct, by induction on $j \in \mathbb{N}$, such an increasing sequence $\underline{\lambda} \geq \underline{f}(\underline{\mu}), \underline{f}'(\underline{\mu})$. Therefore, $(\underline{f}, \boldsymbol{f}_{\underline{\mu}})$ and $(\underline{f}', \boldsymbol{f}'_{\underline{\mu}})$ are equivalent morphisms of *inv*-(*tow*-$\mathcal{A}$), which shows that $\underline{\boldsymbol{f}} = \underline{\boldsymbol{f}}'$.

Consider now the general case of a codomain inverse system $\boldsymbol{Y}$ in $\mathcal{C}$. By [11, Theorem I.1.2] ("Mardešić trick"), $\boldsymbol{Y}$ admits an isomorphic $\boldsymbol{Y}'$ in *pro*-$\mathcal{C}$ indexed by a cofinite M' such that each bonding morphism of $\boldsymbol{Y}'$ is a bonding morphism of $\boldsymbol{Y}$. Let $\boldsymbol{j} : \boldsymbol{Y} \to \boldsymbol{Y}'$ be the isomorphism of *pro*-$\mathcal{C}$ induced by the corresponding identities on the terms. Denote $\boldsymbol{u} \equiv \boldsymbol{j}\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}'$. Then, by the first part of the construction, the morphisms of *pro*-(*tow*-$\mathcal{C}$)

$$\underline{\boldsymbol{j}} : \underline{\boldsymbol{Y}} \to \underline{\boldsymbol{Y}}' \quad \text{and} \quad \underline{\boldsymbol{u}} : \underline{\boldsymbol{X}} \to \underline{\boldsymbol{Y}}'$$

are well defined. It is readily seen, by the construction, that $\underline{\boldsymbol{j}}$ is an isomorphism of *pro*-(*tow*-$\mathcal{C}$). Now we put

$$\underline{\boldsymbol{f}} \equiv \underline{\boldsymbol{j}}^{-1} \underline{\boldsymbol{u}} : \underline{\boldsymbol{X}} \to \underline{\boldsymbol{Y}}.$$

Finally, the verification that $\underline{\boldsymbol{f}}\, \underline{\boldsymbol{p}} = \underline{\boldsymbol{q}} \lfloor \boldsymbol{f} \rfloor$ is straightforward.

An immediate consequence of Theorem 3.7 is as follows:

Corollary 3.8. *Let* $(\mathcal{C},\mathcal{D})$ *be a category pair such that* $\mathcal{D}\subseteq\mathcal{C}$ *is pro-reflective. Then the (abstract) shape categories* $Sh_{(\mathcal{C},\mathcal{D})}$, $Sh_{(pro\text{-}\mathcal{D},\mathcal{D})}$ *and* $Sh_{(pro\text{-}\mathcal{D},tow\text{-}\mathcal{D})}$ *yield the same classification of the* $\mathcal{C}$*-objects (hereby, a "*$\mathcal{C}$*-object in pro-*$\mathcal{D}$*" means an appropriate* $\mathcal{D}$*-expansion, and the shape type of a (pro-*$\mathcal{D}$*)-object coincides with its isomorphism class in pro-*$\mathcal{D}$*).*

Let $\boldsymbol{q}: Y\to\boldsymbol{Y}$ be a morphism of *pro*-$\mathcal{A}$. Suppose that $\boldsymbol{q}$ is a $\mathcal{B}$-expansion with respect to $\mathcal{B}'$, where $\mathcal{B},\mathcal{B}'\subseteq\mathcal{A}$. Let us consider $\boldsymbol{q}$ as the (rudimentary) morphism of *pro*-(*pro*-$\mathcal{A}$), writing correctly, $\lfloor\boldsymbol{q}\rfloor:\lfloor\lfloor Y\rfloor\rfloor\to\lfloor\boldsymbol{Y}\rfloor$. Then, for instance, the assertion "a morphism $\boldsymbol{q}:Y\to\boldsymbol{Y}$ of *pro*-$\mathcal{A}$ is a $\mathcal{B}$-expansion with respect to *pro*-$\mathcal{B}'$ of Y" is logically correct whenever it is meant the rudimentary morphism $\lfloor\boldsymbol{q}\rfloor$ of *pro*-(*pro*-$\mathcal{A}$). In the sequel, we shall not always stress this explicitly, especially if it is quite clear from the context.

Lemma 3.9. *Let* $\mathcal{A}$ *be a category, and let* $\mathcal{B}$, $\mathcal{B}'$ *be a pair of its subcategories,* $\mathcal{B},\mathcal{B}'\subseteq\mathcal{A}$ *. Then, for every morphism* $\boldsymbol{q}:\lfloor Y\rfloor\to\boldsymbol{Y}$ *of pro-*$\mathcal{A}$*, the following assertions are equivalent:*

(i) $\boldsymbol{q}:Y\to\boldsymbol{Y}$ *is a* $\mathcal{B}$*-expansion with respect to* $\mathcal{B}'$ *of* $Y\in Ob\mathcal{A}$*;*

(ii) $\boldsymbol{q}:Y\to\boldsymbol{Y}$ *is a* $\mathcal{B}$*-expansion with respect to tow-*$\mathcal{B}'$ *of* $Y\in Ob\mathcal{A}$*, i.e., precisely, the morphism* $\lfloor\boldsymbol{q}\rfloor:\lfloor\lfloor Y\rfloor\rfloor\to\lfloor\boldsymbol{Y}\rfloor$ *of pro-(pro-*$\mathcal{A}$*) is a (rudimentary) (pro-*$\mathcal{B}$*)-expansion with respect to tow-*$\mathcal{B}'$ *of* $\lfloor Y\rfloor\in Ob(pro\text{-}\mathcal{A})$*;*

(iii) $\boldsymbol{q}:Y\to\boldsymbol{Y}$ *is a* $\mathcal{B}$*-expansion with respect to pro-*$\mathcal{B}'$ *of* $Y\in Ob\mathcal{A}$*, i.e., precisely, the morphism* $\lfloor\boldsymbol{q}\rfloor:\lfloor\lfloor Y\rfloor\rfloor\to\lfloor\boldsymbol{Y}\rfloor$ *of pro-(pro-*$\mathcal{A}$*) is a (rudimentary) (pro-*$\mathcal{B}$*)-expansion with respect to pro-*$\mathcal{B}'$ *of* $\lfloor Y\rfloor\in Ob(pro\text{-}\mathcal{A})$*.*

Proof. By applying (iii) to *tow*-$\mathcal{B}'\subseteq$ *pro*-$\mathcal{B}'$, it immediately follows that (iii) implies (ii). Further, given a $\boldsymbol{Q}=(Q_\lambda,r_{\lambda\lambda'},\Lambda)$ of *pro*-$\mathcal{B}'$, by applying (ii) to the object $\underline{\boldsymbol{Q}}^*=(\lfloor Q_\lambda\rfloor,\lfloor r_{\lambda\lambda'}\rfloor,\Lambda)$ of *pro*-(*tow*-$\mathcal{B}'$), it follows that (ii) implies (i). Namely, each $\lfloor Q_\lambda\rfloor=(Q_\lambda,1_{Q_\lambda},\{1\})$ is a rudimentary object of *tow*-$\mathcal{B}'$. It is left to prove that (i) implies (iii). Let $\underline{\boldsymbol{Q}}=(\boldsymbol{Q}_\lambda,\boldsymbol{r}_{\lambda\lambda'},\Lambda)$ be an arbitrary inverse system in *pro*-$\mathcal{B}'$ (an object of *pro*-(*pro*-$\mathcal{B}'$)), and let $\underline{\boldsymbol{u}}:Y\to\underline{\boldsymbol{Q}}$ be a morphism of *pro*-(*pro*-$\mathcal{A}$), where $Y\equiv\lfloor\lfloor Y\rfloor\rfloor$. Observe that $\underline{\boldsymbol{u}}=(\boldsymbol{u}_\lambda)_{\lambda\in\Lambda}$, where each $\boldsymbol{u}_\lambda:Y\to\boldsymbol{Q}_\lambda$ is a morphism of *pro*-$\mathcal{A}$, $Y\equiv\lfloor Y\rfloor$, such that $\boldsymbol{r}_{\lambda\lambda'}\boldsymbol{u}_{\lambda'}=\boldsymbol{u}_\lambda$, whenever $\lambda\le\lambda'$. Since $\boldsymbol{q}:Y\to\boldsymbol{Y}$ is a $\mathcal{B}$-expansion with respect to $\mathcal{B}'$, we infer that, for every $\lambda\in\Lambda$, there exists a unique morphism $\boldsymbol{v}_\lambda:\boldsymbol{Y}\to\boldsymbol{Q}_\lambda$ of *pro*-$\mathcal{A}$ such that $\boldsymbol{v}_\lambda\boldsymbol{q}=\boldsymbol{u}_\lambda$ in *pro*-$\mathcal{A}$. Further, for every related pair $\lambda\le\lambda'$,

$$\boldsymbol{r}_{\lambda\lambda'}\boldsymbol{v}_{\lambda'}\boldsymbol{q}=\boldsymbol{r}_{\lambda\lambda'}\boldsymbol{u}_{\lambda'}=\boldsymbol{u}_\lambda=\boldsymbol{v}_\lambda\boldsymbol{q}.$$

in *pro*-$\mathcal{A}$. Then the uniqueness of $\boldsymbol{v}_\lambda$ assures that

$$\boldsymbol{r}_{\lambda\lambda'}\boldsymbol{v}_{\lambda'}=\boldsymbol{v}_\lambda$$

in *pro*-$\mathcal{A}$ holds for each $\lambda \in \Lambda$. Thus, the family $(\boldsymbol{v}_\lambda)_{\lambda\in\Lambda}$ determines a morphism

$$\underline{\boldsymbol{v}} = (\boldsymbol{v}_\lambda) : \lfloor \boldsymbol{Y} \rfloor \to \underline{\boldsymbol{Q}}$$

of *pro*-(*pro*-$\mathcal{A}$) such that

$$\underline{\boldsymbol{v}} \lfloor \boldsymbol{q} \rfloor = (\boldsymbol{v}_\lambda) \lfloor \boldsymbol{q} \rfloor = (\boldsymbol{v}_\lambda \boldsymbol{q}) = (\boldsymbol{u}_\lambda) = \underline{\boldsymbol{u}}$$

in *pro*–(*pro*-$\mathcal{A}$). Moreover, such a morphism $\underline{\boldsymbol{v}}$ is unique. Indeed, suppose that

$$\underline{\boldsymbol{v}}^1, \underline{\boldsymbol{v}}^2 : \lfloor \boldsymbol{Y} \rfloor \to \underline{\boldsymbol{Q}}$$

are morphisms of *pro*-(*pro*-$\mathcal{A}$) such that

$$\underline{\boldsymbol{v}}^1 \lfloor \boldsymbol{q} \rfloor = \underline{\boldsymbol{v}}^2 \lfloor \boldsymbol{q} \rfloor : \lfloor \lfloor Y \rfloor \rfloor \to \underline{\boldsymbol{Q}}.$$

This means that, for every $\lambda \in \Lambda$,

$$(\boldsymbol{v}^1_\lambda \boldsymbol{q}) = (\boldsymbol{v}^1_\lambda) \lfloor \boldsymbol{q} \rfloor = \underline{\boldsymbol{v}}^1 \lfloor \boldsymbol{q} \rfloor = \underline{\boldsymbol{v}}^2 \lfloor \boldsymbol{q} \rfloor = (\boldsymbol{v}^2_\lambda) \lfloor \boldsymbol{q} \rfloor = (\boldsymbol{v}^2_\lambda \boldsymbol{q}).$$

Therefore, for every λ,

$$\boldsymbol{v}^1_\lambda \boldsymbol{q} = \boldsymbol{v}^2_\lambda \boldsymbol{q} : Y \to \boldsymbol{Q}_\lambda,$$

in *pro*-$\mathcal{A}$, which by the uniqueness, implies that

$$\boldsymbol{v}^1_\lambda = \boldsymbol{v}^2_\lambda : \boldsymbol{Y} \to \boldsymbol{Q}_\lambda, \lambda \in \Lambda,$$

in *pro*-$\mathcal{A}$. Thus, $\underline{\boldsymbol{v}}^1 = \underline{\boldsymbol{v}}^2$ in *pro*-(*pro*-$\mathcal{A}$), which verifies the universal property of $\lfloor \boldsymbol{q} \rfloor$ with respect to *pro*-$\mathcal{B}'$. □

We are now interested in the relationships between various shape categories induced by a given pair $(\mathcal{C}, \mathcal{D})$, where $\mathcal{D} \subseteq \mathcal{C}$ is a pro-reflective subcategory. In order to do it, we firstly need some additional properties of the previously considered expansion $\underline{\boldsymbol{q}} = (\boldsymbol{q}_{\underline{\mu}}) : \boldsymbol{Y} \to \underline{\boldsymbol{Y}}$.

LEMMA 3.10. *Let $(\mathcal{C}, \mathcal{D})$ be a category pair, and let $\boldsymbol{q} = (q_\mu) : Y \to \boldsymbol{Y}$ be a morphism of pro-$\mathcal{C}$. If $\boldsymbol{q} : Y \to \boldsymbol{Y}$ is a $\mathcal{D}$-expansion of Y, then the composite morphism $\underline{\boldsymbol{q}}\boldsymbol{q} = (\boldsymbol{q}_{\underline{\mu}} \boldsymbol{q}) : Y \to \underline{\boldsymbol{Y}}$ of pro-(pro-$\mathcal{C}$) is a (tow-$\mathcal{D}$)-expansion of Y.*

PROOF. The lemma follows by applying the both expansions in the composition (see Lemma 4.4 below). Nevertheless, we will verify conditions (AE1) and (AE2) for $\underline{\boldsymbol{q}}\boldsymbol{q} = (\boldsymbol{q}_{\underline{\mu}} \boldsymbol{q}) : Y \to \underline{\boldsymbol{Y}}$ with respect to *tow*-$\mathcal{D}$ directly. Let $\boldsymbol{P} \in Ob(tow\text{-}\mathcal{D})$ and let $\boldsymbol{u} : Y \to \boldsymbol{P}$ be a morphism of *pro*-$\mathcal{C}$. Since $\boldsymbol{q}$ is a $\mathcal{D}$-expansion of Y, there exists a unique morphism $\boldsymbol{v} : \boldsymbol{Y} \to \boldsymbol{P}$ of *pro*-$\mathcal{C}$ such that $\boldsymbol{v}\boldsymbol{q} = \boldsymbol{u}$. By Theorem 3.5, $\underline{\boldsymbol{q}} = (\boldsymbol{q}_{\underline{\mu}}) : \boldsymbol{Y} \to \underline{\boldsymbol{Y}}$ is a (*tow*-$\mathcal{C}$)-expansion of $\boldsymbol{Y}$. Thus, by property (AE1) of $\underline{\boldsymbol{q}}$, there exist a $\underline{\mu} \in \underline{M}$ and a morphism $\boldsymbol{w}^{\underline{\mu}} : \boldsymbol{Y}_{\underline{\mu}} \to \boldsymbol{P}$ of *pro*-$\mathcal{C}$ (actually, of *tow*-$\mathcal{D}$) such that $\boldsymbol{w}^{\underline{\mu}} \boldsymbol{q}_{\underline{\mu}} = \boldsymbol{v}$. Hence,

$$\boldsymbol{w}^{\underline{\mu}}(\boldsymbol{q}_{\underline{\mu}} \boldsymbol{q}) = (\boldsymbol{w}^{\underline{\mu}} \boldsymbol{q}_{\underline{\mu}})\boldsymbol{q}) = \boldsymbol{v}\boldsymbol{q} = \boldsymbol{u},$$

which verifies condition (AE1) for $\underline{\boldsymbol{q}}\boldsymbol{q}$ with respect to *tow*-$\mathcal{D}$. Further, let $\underline{\mu} \in \underline{M}$ and let $\boldsymbol{w}_1^{\underline{\mu}}, \boldsymbol{w}_2^{\underline{\mu}} : \boldsymbol{Y}_{\underline{\mu}} \to \boldsymbol{P}$ be a pair of morphisms of *pro*-$\mathcal{C}$ such that

$$\boldsymbol{w}_1^{\underline{\mu}}(\boldsymbol{q}_{\underline{\mu}}\boldsymbol{q}) = \boldsymbol{w}_2^{\underline{\mu}}(\boldsymbol{q}_{\underline{\mu}}\boldsymbol{q}).$$

Then,

$$(\boldsymbol{w}_1^{\underline{\mu}}\boldsymbol{q}_{\underline{\mu}})\boldsymbol{q} = (\boldsymbol{w}_2^{\underline{\mu}}\boldsymbol{q}_{\underline{\mu}})\boldsymbol{q},$$

Since $\boldsymbol{q}$ is $\mathcal{D}$-expansion, the uniqueness condition from its universal property implies that $\boldsymbol{w}_1^{\underline{\mu}}\boldsymbol{q}_{\underline{\mu}} = \boldsymbol{w}_2^{\underline{\mu}}\boldsymbol{q}_{\underline{\mu}}$. Finally, property (AE2) of $\underline{\boldsymbol{q}}$ implies that there exists a $\underline{\mu}' \in \underline{M}$, $\underline{\mu}' \geq \underline{\mu}$, such that $\boldsymbol{w}_1^{\underline{\mu}}\boldsymbol{q}_{\underline{\mu}\,\underline{\mu}'} = \boldsymbol{w}_2^{\underline{\mu}}\boldsymbol{q}_{\underline{\mu}\,\underline{\mu}'}$, which verifies condition (AE2) for $\underline{\boldsymbol{q}}\boldsymbol{q}$ with respect to *tow*-$\mathcal{D}$, and completes the proof of the lemma. ∎

THEOREM 3.11. *For every category pair* $(\mathcal{C}, \mathcal{D})$ *and every morphism* $\boldsymbol{q} : Y \to \boldsymbol{Y}$ *of pro-*$\mathcal{C}$*, the following assertions are equivalent:*

(i) $\boldsymbol{q} = (q_\mu) : Y \to \boldsymbol{Y} = (Y_\mu, q_{\mu\mu'}, M)$ *is a* $\mathcal{D}$*-expansion of* Y*;*
(ii) $\underline{\boldsymbol{q}}\boldsymbol{q} = (\boldsymbol{q}_{\underline{\mu}}\boldsymbol{q}) : Y \to \underline{\boldsymbol{Y}} = (\boldsymbol{Y}_{\underline{\mu}}, \boldsymbol{q}_{\underline{\mu}\,\underline{\mu}'}, \underline{M})$ *of pro-(pro-*$\mathcal{C}$*) is a (tow-*$\mathcal{D}$*)-expansion of* Y*.*

PROOF. (i) implies (ii) by Lemma 3.10. Conversely, let the morphism $\underline{\boldsymbol{q}}\boldsymbol{q} : Y \to \underline{\boldsymbol{Y}}$ of *pro*-(*pro*-$\mathcal{C}$) be a (*tow*-$\mathcal{D}$)-expansion of Y. Recall that, by Theorem 3.5, $\underline{\boldsymbol{q}}$ is a (*tow*-$\mathcal{D}$)-expansion of $\boldsymbol{Y}$. Since $\mathcal{D} \subseteq$ *tow*-$\mathcal{D}$, they both are (*tow*-$\mathcal{D}$)-expansions with respect to $\mathcal{D}$ as well. Let $\boldsymbol{P} \in Ob$(*pro*-$\mathcal{D}$) and let $\boldsymbol{u} : Y \to \boldsymbol{P}$ be a morphism of *pro*-$\mathcal{C}$. Then the expansion $\underline{\boldsymbol{q}}\boldsymbol{q}$ provides a unique morphism $\underline{\boldsymbol{w}} : \underline{\boldsymbol{Y}} \to \boldsymbol{P}$ of *pro*-(*pro*-$\mathcal{D}$) such that $\underline{\boldsymbol{w}}\ \underline{\boldsymbol{q}}\boldsymbol{q} = \boldsymbol{u}$. Put $\boldsymbol{v} = \underline{\boldsymbol{w}}\ \underline{\boldsymbol{q}} : \boldsymbol{Y} \to \boldsymbol{P}$, which is a morphism of *pro*-$\mathcal{D}$ (the rudimentary morphism of *pro*-(*pro*-$\mathcal{D}$)). Then,

$$\boldsymbol{v}\boldsymbol{q} = \underline{\boldsymbol{w}}\ \underline{\boldsymbol{q}}\boldsymbol{q} = \boldsymbol{u},$$

in *pro*-(*pro*-$\mathcal{C}$), which means that

$$\boldsymbol{v}\boldsymbol{q} = \boldsymbol{u}.$$

in *pro*-$\mathcal{C}$. Moreover, such a morphism $\boldsymbol{v} : \boldsymbol{Y} \to \boldsymbol{P}$ of *pro*-$\mathcal{D}$ is unique. Indeed, if would exist $\boldsymbol{v}_1, \boldsymbol{v}_2 : \boldsymbol{Y} \to \boldsymbol{P}$ of *pro*-$\mathcal{D}$ such that $\boldsymbol{v}_1\boldsymbol{q} = \boldsymbol{v}_2\boldsymbol{q}$, then, first, the expansion $\underline{\boldsymbol{q}}$ provides a unique pair $\underline{\boldsymbol{w}}_1, \underline{\boldsymbol{w}}_2 : \underline{\boldsymbol{Y}} \to \boldsymbol{P}$ of *pro*-(*pro*-$\mathcal{D}$), such that $\underline{\boldsymbol{w}}_1\underline{\boldsymbol{q}} = \boldsymbol{v}_1$ and $\underline{\boldsymbol{w}}_2\underline{\boldsymbol{q}} = \boldsymbol{v}_2$, and, second,

$$\underline{\boldsymbol{w}}_1\underline{\boldsymbol{q}}\boldsymbol{q} = \boldsymbol{v}_1\boldsymbol{q} = \boldsymbol{v}_2\boldsymbol{q} = \underline{\boldsymbol{w}}_2\underline{\boldsymbol{q}}\boldsymbol{q}.$$

By the uniqueness for $\underline{\boldsymbol{q}}\boldsymbol{q}$, it follows that $\underline{\boldsymbol{w}}_1 = \underline{\boldsymbol{w}}_2$. Then,

$$\boldsymbol{v}_1 = \underline{\boldsymbol{w}}_1\underline{\boldsymbol{q}} = \underline{\boldsymbol{w}}_2\underline{\boldsymbol{q}} = \boldsymbol{v}_2,$$

This shows that $\boldsymbol{q} : Y \to \boldsymbol{Y}$ is a $\mathcal{D}$-expansion of Y, which completes the proof of the theorem. ∎

4. Iterated expansions

The next lemma makes the main step towards forthcoming consideration.

Lemma 4.1. *Let $\mathcal{A}$ be a category and let $\mathcal{B} \subseteq \mathcal{A}$ be a subcategory. Let $\boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ be an inverse system in $\mathcal{A}$ such that, for every $\lambda \in \Lambda$, there exists a $\mathcal{B}$-expansion*

$$\boldsymbol{p}_\lambda = (p_{\mu^\lambda}) : X_\lambda \to \boldsymbol{X}_\lambda = (X_{\mu^\lambda}, p_{\mu^\lambda \mu'^\lambda}, M^\lambda)$$

of X_λ. Then the terms and morphisms of the family of expansions $(\boldsymbol{p}_\lambda)_{\lambda\in\Lambda}$ can be naturally organized in a $\mathcal{B}$-expansion $\boldsymbol{p}' : \boldsymbol{X} \to \boldsymbol{X}'$ of $\boldsymbol{X}$.

Proof. Let $\boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ be an inverse system in $\mathcal{A}$ such that, for every $\lambda \in \Lambda$, there exists a $\mathcal{B}$-expansion

$$\boldsymbol{p}_\lambda = (p_{\mu^\lambda}) : X_\lambda \to \boldsymbol{X}_\lambda = (X_{\mu^\lambda}, p_{\mu^\lambda \mu'^\lambda}, M^\lambda)$$

of X_λ. Then, for every related pair $\lambda \leq \lambda'$ in Λ, there exists a unique morphism $\boldsymbol{p}_{\lambda\lambda'} : \boldsymbol{X}_{\lambda'} \to \boldsymbol{X}_\lambda$ of *pro*-$\mathcal{B}$ making the following diagram (in *pro*-$\mathcal{A}$) commutative

$$\begin{array}{ccc} X_\lambda & \overset{p_{\lambda\lambda'}}{\leftarrow} & X_{\lambda'} \\ \boldsymbol{p}_\lambda \downarrow & & \downarrow \boldsymbol{p}_{\lambda'} \\ \boldsymbol{X}_\lambda & \underset{\boldsymbol{p}_{\lambda\lambda'}}{\leftarrow} & \boldsymbol{X}_{\lambda'} \end{array} .$$

Observe that the collection $(\boldsymbol{X}_\lambda, \boldsymbol{p}_{\lambda\lambda'}, \Lambda)$ is an inverse system in *pro*-$\mathcal{B}$ (an object of *pro*-(*pro*-$\mathcal{B}$)) because the uniqueness assures that

$$\boldsymbol{p}_{\lambda\lambda'}\boldsymbol{p}_{\lambda'\lambda''} = \boldsymbol{p}_{\lambda\lambda''}, \lambda \leq \lambda' \leq \lambda''.$$

Let us first consider the special case of cofinite systems, i.e., let us assume that Λ and all M^λ, $\lambda \in \Lambda$, are cofinite. Thus, every $\boldsymbol{p}_{\lambda\lambda'}$ admits a special representative $(\pi^{\lambda\lambda'}, \pi^{\lambda\lambda'}_{\mu^\lambda})$, $\lambda \leq \lambda'$. Then, for every related pair $\lambda \leq \lambda'$ in Λ and every related pair $\mu^\lambda \leq \mu'^\lambda$ in M^λ,

$$\pi^{\lambda\lambda'}_{\mu^\lambda} p_{\pi^{\lambda\lambda'}(\mu^\lambda)\pi^{\lambda\lambda'}(\mu'^\lambda)} = p_{\mu^\lambda\mu'^\lambda}\pi^{\lambda\lambda'}_{\mu'^\lambda}.$$

Further, the commutativity of the above diagram means that, for every related pair $\lambda \leq \lambda'$ in Λ and every $\mu^\lambda \in M^\lambda$,

$$p_{\mu^\lambda} p_{\lambda\lambda'} = \pi^{\lambda\lambda'}_{\mu^\lambda} p_{\pi^{\lambda\lambda'}(\mu^\lambda)}.$$

Notice that $(\pi^{\lambda'\lambda''}\pi^{\lambda\lambda'}, \pi^{\lambda\lambda'}_{\mu^\lambda}\pi^{\lambda'\lambda''}_{\pi^{\lambda\lambda'}(\mu^\lambda)})$ is also a special representative of $\boldsymbol{p}_{\lambda\lambda''}$ whenever $\lambda \leq \lambda' \leq \lambda''$. Therefore, since Λ is cofinite, we may suppose that

$$\pi^{\lambda\lambda''} = \pi^{\lambda'\lambda''}\pi^{\lambda\lambda'}, \lambda \leq \lambda' \leq \lambda''.$$

Let us denote

$$N \equiv \bigcup_{\lambda\in\Lambda}(\{\lambda\} \times M^\lambda) = \{\nu \equiv (\lambda, \mu^\lambda) \mid \lambda \in \Lambda, \mu^\lambda \in M^\lambda\},$$

and define

$$\nu = (\lambda, \mu^\lambda) \le (\lambda', \mu'^{\lambda'}) = \nu' \Leftrightarrow \begin{cases} (\lambda = \lambda') \wedge (\mu^\lambda \le \mu'^\lambda) \\ \vee \\ (\lambda < \lambda') \wedge (\pi^{\lambda\lambda'}(\mu^\lambda) \le \mu'^{\lambda'}) \end{cases}.$$

It is readily seen that $(N, \le)$ is a directed set. For instance, let us verify the transitivity in the most general case and the directedness. Suppose that $\nu = (\lambda, \mu^\lambda) \le (\lambda', \mu'^{\lambda'}) = \nu'$ and $\nu' = (\lambda', \mu'^{\lambda'}) \le (\lambda'', \mu''^{\lambda''}) = \nu''$, and that $\lambda < \lambda'$ and $\lambda' < \lambda''$. Then, $\lambda < \lambda''$ and

$$\pi^{\lambda\lambda''}(\mu^\lambda) = \pi^{\lambda'\lambda''}(\pi^{\lambda\lambda'}(\mu^\lambda)) \le \pi^{\lambda'\lambda''}(\mu'^{\lambda'})) \le \mu''^{\lambda''}.$$

Thus, $\nu = (\lambda, \mu^\lambda) \le (\lambda'', \mu''^{\lambda''}) = \nu''$. Further, let $\nu = (\lambda, \mu^\lambda), \nu' = (\lambda', \mu'^{\lambda'}) \in N$. Since Λ is directed, there exists a $\lambda'' \ge \lambda, \lambda'$, and since $M^{\lambda''}$ is directed, there exists a $\mu''^{\lambda''} \ge \pi^{\lambda\lambda''}(\mu^\lambda), \pi^{\lambda'\lambda''}(\mu'^{\lambda'})$. Then, $\nu'' = (\lambda'', \mu''^{\lambda''}) \ge \nu, \nu'$. Notice that N is cofinite provided Λ and all M^λ are cofinite and all $\pi^{\lambda\lambda'}$ are *strictly* increasing!.

Denote $\boldsymbol{X}' \equiv (X'_\nu, p'_{\nu\nu'}, N)$, where

$$X'_\nu = X_{\mu^\lambda}, \nu = (\lambda, \mu^\lambda) \in N,$$

and

$$p'_{\nu\nu'} = p_{\mu^\lambda \mu'^\lambda} : X'_{\nu'} = X_{\mu'^\lambda} \to X_{\mu^\lambda} = X'_\nu, \nu = (\lambda, \mu^\lambda) \le (\lambda, \mu'^\lambda) = \nu',$$

$$p'_{\nu\nu'} = \pi^{\lambda\lambda'}_{\mu^\lambda} p_{\pi^{\lambda\lambda'}(\mu^\lambda)\mu'^{\lambda'}} : X'_{\nu'} = X_{\mu'^{\lambda'}} \to X_{\mu^\lambda} = X'_\nu,$$
$$\nu = (\lambda, \mu^\lambda) \le (\lambda', \mu'^{\lambda'}) = \nu', \lambda < \lambda'.$$

Then a straightforward examination shows that $p'_{\nu\nu'} p'_{\nu'\nu''} = p'_{\nu\nu''}$ whenever $\nu \le \nu' \le \nu''$. Thus, $\boldsymbol{X}'$ is an inverse system in $\mathcal{B}$.

Let us define a function $p' : N \to \Lambda$ by putting $p'(\nu) = \lambda$, whenever $\nu = (\lambda, \mu^\lambda)$. Further, let $p'_\nu : X_{p'(\nu)} \to X'_\nu$ be defined by

$$p'_\nu = p_{\mu^\lambda} : X_{p'(\nu)} = X_\lambda \to X_{\mu^\lambda} = X'_\nu, \nu = (\lambda, \mu^\lambda) \in N.$$

Then, for every related pair $\nu = (\lambda, \mu^\lambda) \le (\lambda', \mu'^{\lambda'}) = \nu'$ in N,

$$p'_\nu p_{p'(\nu)p'((\nu')} = p_{\mu^\lambda} 1_{X^\lambda} = p_{\mu^\lambda \mu'^\lambda} p_{\mu'^\lambda} = p'_{\nu\nu'} p'_{\nu'}, \qquad \lambda = \lambda',$$

and

$$p'_\nu p_{p'(\nu)p'((\nu')} = p_{\mu^\lambda} p_{\lambda\lambda'} = \pi^{\lambda\lambda'}_{\mu^\lambda} p_{\pi^{\lambda\lambda'}(\mu^\lambda)} = \pi^{\lambda\lambda'}_{\mu^\lambda} p_{\pi^{\lambda\lambda'}(\mu^\lambda)\mu'^{\lambda'}} p_{\mu'^{\lambda'}}$$
$$= p'_{\nu\nu'} p'_{\nu'}, \qquad \lambda < \lambda'.$$

This shows that $(p', p'_\nu) : \boldsymbol{X} \to \boldsymbol{X}'$ is a (special) morphism of *inv*-$\mathcal{A}$. Let us prove that the morphism

$$\boldsymbol{p}' = [(p', p'_\nu)] : \boldsymbol{X} \to \boldsymbol{X}'$$

of *pro*-$\mathcal{A}$ is a $\mathcal{B}$-expansion of $\boldsymbol{X}$. According to Lemma 3.4, it is equivalent to verify conditions (AE1) and (AE2) for $\boldsymbol{p}'$.

First of all, for every $\lambda \in \Lambda$, denote by $\boldsymbol{i}_\lambda : \boldsymbol{X} \to X_\lambda$ the morphism of *pro*-$\mathcal{A}$ determined by the identity 1_{X_λ}. Observe that, for every related pair $\lambda \leq \lambda'$ in Λ,

$$p_{\lambda\lambda'}\boldsymbol{i}_{\lambda'} = \boldsymbol{i}_\lambda : \boldsymbol{X} \to X_\lambda$$

(in *pro*-$\mathcal{C}$). Similarly, for every $\lambda \in \Lambda$, denote by $\boldsymbol{j}_\lambda : \boldsymbol{X}' \to \boldsymbol{X}_\lambda$ the morphism of *pro*-$\mathcal{B}$ determined by all the identities $1_{X_{\mu^\lambda}}$, $\mu^\lambda \in M^\lambda$. It is readily seen that

$$\boldsymbol{j}_\lambda \boldsymbol{p}' = \boldsymbol{p}_\lambda \boldsymbol{i}_\lambda, \ \lambda \in \Lambda,$$

i.e., that the diagram

$$\begin{array}{ccc} \boldsymbol{X}' & \overset{\boldsymbol{p}'}{\leftarrow} & \boldsymbol{X} \\ \boldsymbol{j}_\lambda \downarrow & & \downarrow \boldsymbol{i}_\lambda \\ \boldsymbol{X}_\lambda & \underset{\boldsymbol{p}_\lambda}{\leftarrow} & X_\lambda \end{array}$$

in *pro*-$\mathcal{A}$ commutes. Further, as above, for every $\nu = (\lambda, \mu^\lambda) \in N$, denote by $\boldsymbol{p}'_\nu : \boldsymbol{X} \to X'_\nu$ the morphism of *pro*-$\mathcal{A}$ determined by p_{μ^λ}. Consequently, for every $\lambda \in \Lambda$ and every $\mu^\lambda \in M^\lambda$, the corresponding "subdiagram"

$$\begin{array}{ccc} X'_\nu = X_{\mu^\lambda} & \overset{\boldsymbol{p}'_\nu = (p_{\mu^\lambda})}{\leftarrow} & \boldsymbol{X} \\ j_{\mu^\lambda} = 1_{X_{\mu^\lambda}} \downarrow & & \downarrow \boldsymbol{i}_\lambda = (1_{X_\lambda}) \\ X_{\mu^\lambda} & \underset{p_{\mu^\lambda}}{\leftarrow} & X_\lambda \end{array}$$

in *pro*-$\mathcal{A}$ commutes as well. Clearly, it reduces to the diagram

$$\begin{array}{ccc} X'_\nu = X_{\mu^\lambda} & \overset{p_{\mu^\lambda}}{\leftarrow} & X_\lambda \\ 1 \downarrow & & \downarrow 1 \\ X_{\mu^\lambda} & \underset{p_{\mu^\lambda}}{\leftarrow} & X_\lambda \end{array}$$

in $\mathcal{A}$. Let an arbitrary object P of $\mathcal{B}$ and any morphism $\boldsymbol{u} : \boldsymbol{X} \to P$ of *pro*-$\mathcal{A}$ be given. Then $\boldsymbol{u}$ consists of a unique morphism $u^\lambda : X_\lambda \to P$ of $\mathcal{A}$ such that $u^\lambda \boldsymbol{i}_\lambda = \boldsymbol{u}$. Since $\boldsymbol{p}_\lambda = (p_{\mu^\lambda}) : X_\lambda \to \boldsymbol{X}_\lambda$ is a $\mathcal{B}$-expansion, property (AE1) of $\boldsymbol{p}_\lambda$ implies that there exist a $\mu^\lambda \in M^\lambda$ and a morphism $v^{\mu^\lambda} : X_{\mu^\lambda} \to P$ of $\mathcal{B}$ such that

$$v^{\mu^\lambda} p_{\mu^\lambda} = u^\lambda.$$

Put $\nu = (\lambda, \mu^\lambda) \in N$ and

$$w^\nu = v^{\mu^\lambda} : X'_\nu = X_{\mu^\lambda} \to P.$$

Then (in *pro*-$\mathcal{A}$),

$$w^\nu \boldsymbol{p}'_\nu = v^{\mu^\lambda}(p_{\mu^\lambda}) = v^{\mu^\lambda} 1_{X_{\mu^\lambda}}(p_{\mu^\lambda}) = v^{\mu^\lambda} p_{\mu^\lambda}(1_{X_{\mu^\lambda}}) = u^\lambda \boldsymbol{i}_\lambda = \boldsymbol{u},$$

which verifies condition (AE1) for $\boldsymbol{p}' : \boldsymbol{X} \to \boldsymbol{X}'$. Further, let $\nu = (\lambda, \mu^\lambda) \in N$, and let

$$w_1^\nu, w_2^\nu : X'_\nu = X_{\mu^\lambda} \to P$$

be a pair of $\mathcal{B}$-morphisms such that

$$w_1^\nu \boldsymbol{p}'_\nu = w_2^\nu \boldsymbol{p}'_\nu : \boldsymbol{X} \to P$$

in *pro*-$\mathcal{A}$. This means that $w_1^\nu(p_{\mu^\lambda}) = w_2^\nu(p_{\mu^\lambda})$. Put

$$v_1^{\mu^\lambda} = w_1^\nu,\ v_2^{\mu^\lambda} = w_2^\nu : X_{\mu^\lambda} \to P.$$

Then (in $\mathcal{A}$),

$$v_1^{\mu^\lambda} p_{\mu^\lambda} = v_2^{\mu^\lambda} p_{\mu^\lambda} : X_\lambda \to X_{\mu^\lambda}.$$

By property (AE2) of the expansion $\boldsymbol{p}_\lambda$, there exists a $\mu'^\lambda \in M^\lambda$, $\mu'^\lambda \geq \mu^\lambda$, such that

$$v_1^{\mu^\lambda} p_{\mu^\lambda \mu'^\lambda} = v_2^{\mu^\lambda} p_{\mu^\lambda \mu'^\lambda} : X_{\mu'^\lambda} \to X_{\mu^\lambda}.$$

Put $\nu' = (\lambda, \mu'^\lambda)$. Then $\nu' \in N$ and $\nu' \geq \nu$. Furthermore,

$$w_1^\nu p'_{\nu\nu'} = v_1^{\mu^\lambda} p_{\mu^\lambda \mu'^\lambda} = v_2^{\mu^\lambda} p_{\mu^\lambda \mu'^\lambda} = w_2^\nu p'_{\nu\nu'},$$

which verifies condition (AE2) for $\boldsymbol{p}' : \boldsymbol{X} \to \boldsymbol{X}'$. Therefore, $\boldsymbol{p}'$ is a $\mathcal{B}$-expansion of $\boldsymbol{X}$, which proves the statement of the lemma in the special case.

In the general case of $\boldsymbol{X}$ and $\boldsymbol{X}_\lambda$, i.e., if they are not cofinite, let first $\boldsymbol{i} : \boldsymbol{X} \to \boldsymbol{Y}$ be the natural isomorphism, where $\boldsymbol{Y} = (Y_\mu, q_{\mu\mu'}, M)$ is cofinite (see [11, Theorem I.1.2]). Further, let the isomorphisms

$$\boldsymbol{i}_\lambda : \boldsymbol{X}_\lambda \to \boldsymbol{Y}_\lambda = (Y_{\nu^l}, q_{\nu^l \nu'^\lambda} N^\lambda), \qquad \lambda \in \Lambda,$$

be obtained in the same way. Recall that all the terms and bonds of $\boldsymbol{Y}$ and $\boldsymbol{Y}_\lambda$ are those of $\boldsymbol{X}$ and $\boldsymbol{X}_\lambda$ respectively, $\lambda \in \Lambda$. Moreover, $\boldsymbol{i}$ and $\boldsymbol{i}_\lambda$ consist of the appropriate identities on the terms. Therefore and since every $\boldsymbol{i}_\lambda$ is an isomorphism, for every $\mu \in M$, the composite morphism

$$Y_\mu = X_{\lambda(\mu)} \overset{\boldsymbol{p}_{\lambda(\mu)}}{\to} \boldsymbol{X}_{\lambda(\mu)} \overset{\boldsymbol{i}_{\lambda(\mu)}}{\to} \boldsymbol{Y}_{\lambda(\mu)}$$

is a cofinite $\mathcal{B}$-expansion of Y_μ, which consists of the morphisms of $\boldsymbol{p}_{\lambda(\mu)}$. Consequently, by putting

$$\boldsymbol{q}_\mu \equiv \boldsymbol{i}_{\lambda(\mu)} \boldsymbol{p}_{\lambda(\mu)} : Y_\mu \to \boldsymbol{Y}_\mu \equiv \boldsymbol{Y}_{\lambda(\mu)}, \qquad \mu \in M,$$

we can obtain, by the previous construction (in the special, cofinite, case), a desired $\mathcal{B}$-expansion $\boldsymbol{q}' : \boldsymbol{Y} \to \boldsymbol{Y}'$ of $\boldsymbol{Y}$. Finally, since $\boldsymbol{i} : \boldsymbol{X} \to \boldsymbol{Y}$ is an isomorphism, by putting

$$\boldsymbol{p}' \equiv \boldsymbol{q}' \boldsymbol{i} : \boldsymbol{X} \to \boldsymbol{X}' \equiv \boldsymbol{Y}',$$

we have got a desired $\mathcal{B}$-expansion of $\boldsymbol{X}$. This completes the proof of the lemma. ☐

We are now able to prove our main theoretical fact (see Definition 3.2): "to be a pro-reflective subcategory" and "to be a system pro-reflective subcategory" are equivalent properties.

THEOREM 4.2. *If $\mathcal{D} \subseteq \mathcal{C}$ is a pro-reflective subcategory, then so is $\mathcal{D} \subseteq$ pro-$\mathcal{C}$, i.e., $\mathcal{D} \subseteq \mathcal{C}$ is a system pro-reflective subcategory. Consequently, $\mathcal{D} \subseteq$ tow-$\mathcal{C}$, tow-$\mathcal{D} \subseteq$ tow-$\mathcal{C}$, tow-$\mathcal{D} \subseteq$ pro-$\mathcal{C}$ and pro-$\mathcal{D} \subseteq$ pro-$\mathcal{C}$ are pro-reflective subcategories.*

PROOF. By our convention (Remark 2.8), if $\mathcal{L} \subseteq \mathcal{K}$, then $\mathcal{L} \subseteq$ *tow*-$\mathcal{K}$ and $\mathcal{L} \subseteq$ *pro*-$\mathcal{K}$ mean $\lfloor \mathcal{L} \rfloor \subseteq$ *tow*-$\mathcal{K}$ and $\lfloor \mathcal{L} \rfloor \subseteq$ *pro*-$\mathcal{K}$ respectively. Let $\boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ be an arbitrary inverse system in $\mathcal{C}$. Since $\mathcal{D} \subseteq \mathcal{C}$ is pro-reflective, for every $\lambda \in \Lambda$, there exists a $\mathcal{D}$-expansion $\boldsymbol{p}_\lambda : X_\lambda \to \boldsymbol{X}_\lambda$ of X_λ. Then, by Lemma 4.1, the family $(\boldsymbol{p}_\lambda)_{\lambda\in\Lambda}$ can be organized in a $\mathcal{D}$-expansion $\boldsymbol{p}' : \boldsymbol{X} \to \boldsymbol{X}$ of $\boldsymbol{X}$. Therefore, $\mathcal{D} \subseteq$ *pro*-$\mathcal{C}$ is pro-reflective as well.

Further, since we have proven that $\mathcal{D} \subseteq$ *pro*-$\mathcal{C}$ is pro-reflective and since *tow*-$\mathcal{C} \subseteq$ *pro*-$\mathcal{C}$, we immediately infer that $\mathcal{D} \subseteq$ *tow*-$\mathcal{C}$ is pro-reflective.

In order to prove that *tow*-$\mathcal{D} \subseteq$ *tow*-$\mathcal{C}$ is a pro-reflective subcategory, consider the category pair (*tow*-$\mathcal{C}, \mathcal{D}$) and apply Lemma 3.10. Namely, once we have proven that $\mathcal{D} \subseteq$ *tow*-$\mathcal{C}$ is pro-reflective, every inverse sequence $\boldsymbol{X}$ in $\mathcal{C}$ admits a $\mathcal{D}$-expansion $\boldsymbol{p}' : \boldsymbol{X} \to \boldsymbol{X}'$. Then, by Lemma 3.10, the composite morphism $\underline{\boldsymbol{p}'}\boldsymbol{p}' = (\boldsymbol{p}'_{\underline{\nu}}\boldsymbol{p}') : \boldsymbol{X} \to \underline{\boldsymbol{X}'}$ of *pro*-(*pro*-$\mathcal{C}$) (actually, of *pro*-(*tow*-$\mathcal{C}$)) is a (*tow*-$\mathcal{D}$)-expansion of $\boldsymbol{Y} \in Ob($*tow*-$\mathcal{C})$. Therefore, *tow*-$\mathcal{D} \subseteq$ *tow*-$\mathcal{C}$ is a pro-reflective subcategory.

Further, once we have proven that $\mathcal{D} \subseteq$ *pro*-$\mathcal{C}$ is pro-reflective, we may also apply Lemma 3.10 to the category pair (*pro*-$\mathcal{C}, \mathcal{D}$) and prove, in the same way as above, that *tow*-$\mathcal{D} \subseteq$ *pro*-$\mathcal{C}$ is pro-reflective. Namely, every inverse system $\boldsymbol{X}$ in $\mathcal{C}$ admits a morphism $\boldsymbol{p}' : \boldsymbol{X} \to \boldsymbol{X}'$ of *pro*-$\mathcal{C}$, which is a $\mathcal{D}$-expansion of $\boldsymbol{X}$. Then Lemma 3.10 provides the composite morphism $\underline{\boldsymbol{p}'}\boldsymbol{p}' = (\boldsymbol{p}'_{\underline{\nu}}\boldsymbol{p}') : \boldsymbol{X} \to \underline{\boldsymbol{X}'}$ of *pro*-(*pro*-$\mathcal{C}$) to be a (*tow*-$\mathcal{D}$)-expansion of $\boldsymbol{X}$. Thus, *tow*-$\mathcal{D} \subseteq$ *pro*-$\mathcal{C}$ is a pro-reflective subcategory.

Finally, in order to prove that *pro*-$\mathcal{D} \subseteq$ *pro*-$\mathcal{C}$ is pro-reflective, consider the category pair (*pro*-$\mathcal{C}, \mathcal{D}$) again. Then, given an inverse system $\boldsymbol{X}$ in $\mathcal{C}$, there exists a $\mathcal{D}$-expansion $\boldsymbol{p}' : \boldsymbol{X} \to \boldsymbol{X}'$ of $\boldsymbol{X}$. By Lemma 3.9, (i) $\Rightarrow$ (iii), the same $\boldsymbol{p}' : \boldsymbol{X} \to \boldsymbol{X}'$ is a (rudimentary) (*pro*-$\mathcal{D}$)-expansion of $\boldsymbol{X}$ as well. Therefore, *pro*-$\mathcal{D} \subseteq$ *pro*-$\mathcal{C}$ is a pro-reflective subcategory. This completes the proof of the theorem. ∎

REMARK 4.3. By Lemma 4.1, "the terms and morphisms of the family of $\mathcal{B}$-expansions of all terms of an $\mathcal{A}$-system can be naturally organized in a $\mathcal{B}$-expansion of the system". It is also a *rudimentary* (*pro*-$\mathcal{B}$)-expansion of the system (Lemma 3.9). By comparing the main results of [12] and [17] to our Lemma 4.1, one infers that expansions as well as inverse limits and resolutions admit iteration (see also Corollary 4.5 below). On the other hand, given an inverse system $\boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ in $\mathcal{A}$, one can consider the associated inverse system $\underline{\boldsymbol{Y}} = (\boldsymbol{X}_\lambda, \boldsymbol{p}_{\lambda\lambda'}, \Lambda)$ in *pro*-$\mathcal{B}$ (an object of *pro*-(*pro*-$\mathcal{B}$)) obtained by arbitrarily chosen $\mathcal{B}$-expansions $\boldsymbol{p}_\lambda : X_\lambda \to \boldsymbol{X}_\lambda$, $\lambda \in \Lambda$ (as in the proof of Lemma 4.1). Then, for every $\lambda \in \Lambda$, there exists the natural morphism

$\underline{\boldsymbol{q}} = (\boldsymbol{q}_\lambda) : \boldsymbol{X} \to \underline{\boldsymbol{Y}}$ of *pro*-(*pro*-$\mathcal{C}$), where $\boldsymbol{q}_\lambda = \boldsymbol{p}_\lambda \boldsymbol{i}_\lambda : \boldsymbol{X} \to \boldsymbol{X}_\lambda$, and $\boldsymbol{i}_\lambda = (1_{X_\lambda}) : \boldsymbol{X} \to X_\lambda$, $\lambda \in \Lambda$. However, a simple analysis shows that $\underline{\boldsymbol{q}}$ is *not* a (*pro*-$\mathcal{B}$)-expansion of $\boldsymbol{X}$ because condition (AE1) fails. Thus, "the family of $\mathcal{B}$-expansions of all terms of an $\mathcal{A}$-system *cannot* be naturally organized in a (*pro*-$\mathcal{B}$)-expansion of the system".

The next fact will be useful in some considerations in the sequel.

LEMMA 4.4. *Let $\mathcal{A}$ be a category and let $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B} \subseteq \mathcal{A}$ be arbitrary subcategories. Let $\boldsymbol{X}$, $\boldsymbol{Y}$ and $\boldsymbol{Z}$ be inverse systems in $\mathcal{A}$, $\mathcal{B}_1$ and $\mathcal{B}_2$ respectively, and let $\boldsymbol{p} : \boldsymbol{X} \to \boldsymbol{Y}$ and $\boldsymbol{q} : \boldsymbol{Y} \to \boldsymbol{Z}$ be morphisms of pro-$\mathcal{A}$. If two of $\boldsymbol{p}$, $\boldsymbol{q}$, $\boldsymbol{qp}$ are the appropriate ($\mathcal{B}_1$- or $\mathcal{B}_2$-) expansions with respect to $\mathcal{B}$, then so is the third one.*

PROOF. Let $\boldsymbol{p} : \boldsymbol{X} \to \boldsymbol{Y}$ be a $\mathcal{B}_1$-expansion with respect to $\mathcal{B}$ of $\boldsymbol{X}$, and let $\boldsymbol{q} : \boldsymbol{Y} \to \boldsymbol{Z}$ be a $\mathcal{B}_2$-expansion with respect to $\mathcal{B}$ of $\boldsymbol{Y}$. Let $\boldsymbol{P} \in Ob($*pro*-$\mathcal{B})$ and let $\boldsymbol{u} : \boldsymbol{X} \to \boldsymbol{P}$ be a morphism of *pro*-$\mathcal{A}$. Then $\boldsymbol{p}$ yields a unique $\boldsymbol{v} : \boldsymbol{Y} \to \boldsymbol{P}$ of *pro*-$\mathcal{A}$ such that $\boldsymbol{vp} = \boldsymbol{u}$, and $\boldsymbol{q}$ yields a unique $\boldsymbol{w} : \boldsymbol{Z} \to \boldsymbol{P}$ of *pro*-$\mathcal{A}$ such that $\boldsymbol{wq} = \boldsymbol{v}$. Consequently, $\boldsymbol{w}$ is unique for $\boldsymbol{u}$ such that $\boldsymbol{wqp} = \boldsymbol{u}$. Thus, $\boldsymbol{qp} : \boldsymbol{X} \to \boldsymbol{Z}$ is a $\mathcal{B}_2$-expansion with respect to $\mathcal{B}$ of $\boldsymbol{X}$.

Let $\boldsymbol{p} : \boldsymbol{X} \to \boldsymbol{Y}$ be a $\mathcal{B}_1$-expansion with respect to $\mathcal{B}$ of $\boldsymbol{X}$, and let $\boldsymbol{qp} : \boldsymbol{X} \to \boldsymbol{Z}$ be a $\mathcal{B}_2$-expansion with respect to $\mathcal{B}$ of $\boldsymbol{X}$. Let $\boldsymbol{P} \in Ob($*pro*-$\mathcal{B})$ and let $\boldsymbol{v} : \boldsymbol{Y} \to \boldsymbol{P}$ be a morphism of *pro*-$\mathcal{A}$. Put $\boldsymbol{u} = \boldsymbol{vp} : \boldsymbol{X} \to \boldsymbol{P}$. Since $\boldsymbol{qp}$ is an expansion with respect to $\mathcal{B}$, there exists a unique $\boldsymbol{w} : \boldsymbol{Z} \to \boldsymbol{P}$ of *pro*-$\mathcal{A}$ such that $\boldsymbol{wqp} = \boldsymbol{u} = \boldsymbol{vp}$. Since $\boldsymbol{p}$ is an expansion with respect to $\mathcal{B}$, it implies that $\boldsymbol{wq} = \boldsymbol{v}$. Further, if $\boldsymbol{w}_1, \boldsymbol{w}_2 : \boldsymbol{Z} \to \boldsymbol{P}$ satisfy $\boldsymbol{w}_1\boldsymbol{q} = \boldsymbol{w}_2\boldsymbol{q}$, then $\boldsymbol{w}_1\boldsymbol{qp} = \boldsymbol{w}_2\boldsymbol{qp}$. Since $\boldsymbol{qp}$ is an expansion with respect to $\mathcal{B}$, it follows that $\boldsymbol{w}_1 = \boldsymbol{w}_2$. Therefore, $\boldsymbol{q} : \boldsymbol{Y} \to \boldsymbol{Z}$ is a $\mathcal{B}_2$-expansion with respect to $\mathcal{B}$ of $\boldsymbol{Y}$.

Finally, let $\boldsymbol{q} : \boldsymbol{Y} \to \boldsymbol{Z}$ be a $\mathcal{B}_2$-expansion with respect to $\mathcal{B}$ of $\boldsymbol{Y}$, where $\boldsymbol{Y} \in Ob($*pro*-$\mathcal{B}_1)$, and let $\boldsymbol{qp} : \boldsymbol{X} \to \boldsymbol{Z}$ be a $\mathcal{B}_2$-expansion with respect to $\mathcal{B}$ of $\boldsymbol{X}$. Let $\boldsymbol{P} \in Ob($*pro*-$\mathcal{B})$ and let $\boldsymbol{u} : \boldsymbol{X} \to \boldsymbol{P}$ be a morphism of *pro*-$\mathcal{A}$. Then $\boldsymbol{qp}$ yields a unique morphism $\boldsymbol{w} : \boldsymbol{Z} \to \boldsymbol{P}$ of *pro*-$\mathcal{A}$ such that $\boldsymbol{wqp} = \boldsymbol{u}$. Put $\boldsymbol{v} \equiv \boldsymbol{wq} : \boldsymbol{Y} \to \boldsymbol{P}$. Then, $\boldsymbol{vp} = \boldsymbol{wqp} = \boldsymbol{u}$. Suppose that there exist two morphisms $\boldsymbol{v}_1, \boldsymbol{v}_2 : \boldsymbol{Y} \to \boldsymbol{P}$ of *pro*-$\mathcal{A}$ such that $\boldsymbol{v}_1\boldsymbol{p} = \boldsymbol{v}_2\boldsymbol{p}$. Then $\boldsymbol{q}$ yields a unique pair $\boldsymbol{w}_1, \boldsymbol{w}_2 : \boldsymbol{Z} \to \boldsymbol{P}$ in *pro*-$\mathcal{A}$ such that $\boldsymbol{w}_1\boldsymbol{q} = \boldsymbol{v}_1$ and $\boldsymbol{w}_2\boldsymbol{q} = \boldsymbol{v}_2$. It follows that

$$\boldsymbol{w}_1\boldsymbol{qp} = \boldsymbol{v}_1\boldsymbol{p} = \boldsymbol{v}_2\boldsymbol{p} = \boldsymbol{w}_2\boldsymbol{qp}.$$

Since $\boldsymbol{qp}$ is an appropriate expansion, it follows that $\boldsymbol{w}_1 = \boldsymbol{w}_2$, and consequently, that $\boldsymbol{v}_1 = \boldsymbol{v}_2$. Thus, $\boldsymbol{p} : \boldsymbol{X} \to \boldsymbol{Y}$ is a $\mathcal{B}_1$-expansion with respect to $\mathcal{B}$ of $\boldsymbol{X}$. ∎

COROLLARY 4.5. (i) *Let $\mathcal{B}, \mathcal{B}' \subseteq \mathcal{A}$, let $\boldsymbol{p} : X \to \boldsymbol{X}$ be a $\mathcal{B}$-expansion with respect to $\mathcal{B}'$ of an $X \in Ob\mathcal{A}$, and let, for every $\lambda \in \Lambda$, there exists a $\mathcal{B}'$-expansion $\boldsymbol{p}_\lambda : X_\lambda \to \boldsymbol{X}_\lambda$ of X_λ. Then the terms and morphisms* *of the family* $(\boldsymbol{p}_\lambda)_{\lambda\in\Lambda}$ *can be naturally organized in a morphism* $\boldsymbol{p}' : \boldsymbol{X} \to \boldsymbol{X}'$ *of pro-*$\mathcal{A}$ *such that* $\boldsymbol{p}'\boldsymbol{p} : X \to \boldsymbol{X}'$ *is a* $\mathcal{B}'$*-expansion of* X.

(ii) *If the image* $\underline{E}(\boldsymbol{f}) : \underline{\boldsymbol{X}} \to \underline{\boldsymbol{Y}}$ *is an isomorphism of pro-(tow-*$\mathcal{C}$*) (Theorem 3.7), then* $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ *is an isomorphism of pro-*$\mathcal{C}$.

Proof. According to Lemma 4.1, the naturally constructed morphism $\boldsymbol{p}' : \boldsymbol{X} \to \boldsymbol{X}'$ is a $\mathcal{B}'$-expansion of $\boldsymbol{X}$. Then, by Lemma 4.4, $\boldsymbol{p}'\boldsymbol{p} : X \to \boldsymbol{X}'$ is a $\mathcal{B}'$-expansion of X, and statement (i) follows. Consider the appropriate commutative diagram in *pro*-(*pro*-$\mathcal{C}$) (by using the rudimentary embedding *pro*-$\mathcal{C} \to$ *pro*-(*pro*-$\mathcal{C}$), i.e., $\lfloor pro\text{-}\mathcal{C} \rfloor \subseteq pro\text{-}(pro\text{-}\mathcal{C})$) concerning the fully faithful functor $\underline{E} : pro\text{-}\mathcal{C} \to pro\text{-}(tow\text{-}\mathcal{C})$ of Theorem 3.7:

$$\begin{array}{ccc} \underline{E}(\boldsymbol{X}) & \overset{\underline{\boldsymbol{p}}}{\leftarrow} & \lfloor \boldsymbol{X} \rfloor \\ \underline{E}(\boldsymbol{f}) \downarrow & & \downarrow \lfloor \boldsymbol{f} \rfloor \\ \underline{E}(\boldsymbol{Y}) & \underset{\underline{\boldsymbol{q}}}{\leftarrow} & \lfloor \boldsymbol{Y} \rfloor \end{array}$$

Assume that $\underline{E}(\boldsymbol{f}) \equiv \underline{\boldsymbol{f}}$ is an isomorphism. Then, obviously, it is a (*tow*-$\mathcal{C}$)-expansion of $\underline{\boldsymbol{X}} \equiv \underline{E}(\boldsymbol{X}) \in Ob(pro\text{-}(tow\text{-}\mathcal{C})) \subseteq Ob(pro\text{-}(pro\text{-}\mathcal{C}))$. By Theorem 3.5, $\underline{\boldsymbol{p}}$ is a (*tow*-$\mathcal{C}$)-expansion of $\lfloor \boldsymbol{X} \rfloor \in Ob(pro\text{-}(pro\text{-}\mathcal{C}))$. Put now $\mathcal{A} \equiv pro\text{-}\mathcal{C}$ and $\mathcal{B}_1 = \mathcal{B}_2 = \mathcal{B} \equiv tow\text{-}\mathcal{C}$, and apply Lemma 4.4. Then the composite $\underline{\boldsymbol{f}}\,\underline{\boldsymbol{p}}$ is a (*tow*-$\mathcal{C}$)-expansion of $\lfloor \boldsymbol{X} \rfloor$. Since $\underline{\boldsymbol{f}}\,\underline{\boldsymbol{p}} = \underline{\boldsymbol{q}}\lfloor \boldsymbol{f} \rfloor$ and $\underline{\boldsymbol{q}}$ is a (*tow*-$\mathcal{C}$)-expansion of $\lfloor \boldsymbol{Y} \rfloor$, Lemma 4.4 (put $\mathcal{A} = \mathcal{B}_1 \equiv pro\text{-}\mathcal{C}$ and $\mathcal{B}_2 = \mathcal{B} \equiv tow\text{-}\mathcal{C}$) implies that $\lfloor \boldsymbol{f} \rfloor : \lfloor \boldsymbol{X} \rfloor \to \lfloor \boldsymbol{Y} \rfloor$ is a (rudimentary) (*pro*-$\mathcal{C}$)-expansion with respect to *tow*-$\mathcal{C}$ of (the rudimentary object) $\lfloor \boldsymbol{X} \rfloor \in Ob(pro\text{-}(pro\text{-}\mathcal{C}))$. Then, especially, $\lfloor \boldsymbol{f} \rfloor : \lfloor \boldsymbol{X} \rfloor \to \lfloor \boldsymbol{Y} \rfloor$ is a (*pro*-$\mathcal{C}$)-expansion with respect to $\lfloor \mathcal{C} \rfloor$ of $\lfloor \boldsymbol{X} \rfloor$. This finally implies that the morphism $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ of *pro*-$\mathcal{C}$ is a $\mathcal{C}$-expansion of $\boldsymbol{X}$. However, $\boldsymbol{1_X} : \boldsymbol{X} \to \boldsymbol{X}$ is a $\mathcal{C}$-expansion of $\boldsymbol{X}$ as well. Consequently, $\boldsymbol{f}$ is an isomorphism of *pro*-$\mathcal{C}$, which proves statement (ii) and completes the proof of the corollary. ∎

5. Applications

Let us now apply the obtained general results to certain category pairs $(\mathcal{C}, \mathcal{D})$, whereas $\mathcal{D} \subseteq \mathcal{C}$ is a pro-reflective subcategory, and, especially, to some familiar such category pairs.

5.1. *Application I.* By Lemma 4.1 (and Remark 2.8), given a category pair $(\mathcal{C}, \mathcal{D})$ such that $\mathcal{D} \subseteq \mathcal{C}$ is pro-reflective, there are, beside the shape category $Sh_{(\mathcal{C},\mathcal{D})}$, the following shape categories: $Sh_{(tow\text{-}\mathcal{C},\mathcal{D})}$, $Sh_{(tow\text{-}\mathcal{C},tow\text{-}\mathcal{D})}$, $Sh_{(pro\text{-}\mathcal{C},\mathcal{D})}$, $Sh_{(pro\text{-}\mathcal{C},tow\text{-}\mathcal{D})}$ and $Sh_{(pro\text{-}\mathcal{C},pro\text{-}\mathcal{D})}$.

Theorem 5.1. *Let* $(\mathcal{C}, \mathcal{D})$ *be a category pair such that* $\mathcal{D} \subseteq \mathcal{C}$ *is pro-reflective. Let* $\mathcal{A}, \mathcal{A}' \in \{\mathcal{C}, tow\text{-}\mathcal{C}, pro\text{-}\mathcal{C}\}$ *and let* $\mathcal{B}, \mathcal{B}' \in \{\mathcal{D}, tow\text{-}\mathcal{D}, pro\text{-}\mathcal{D}\}$. *Then, the following statements hold:*

(i) *If* $\mathcal{B} \subseteq \mathcal{A}$, $\mathcal{B}' \subseteq \mathcal{A}'$ *and* $\mathcal{A} \subseteq \mathcal{A}'$, $\mathcal{B} \subseteq \mathcal{B}'$, *then* $Sh_{(\mathcal{A},\mathcal{B})} \to Sh_{(\mathcal{A}',\mathcal{B}')}$ *is a full (functorial) embedding.*
Moreover,

(ii) *The shape categories* $Sh_{(pro\text{-}\mathcal{C},\mathcal{D})}$, $Sh_{(pro\text{-}\mathcal{C},tow\text{-}\mathcal{D})}$ *and* $Sh_{(pro\text{-}\mathcal{C},pro\text{-}\mathcal{D})}$ *are mutually isomorphic;*

(iii) *The shape categories* $Sh_{(tow\text{-}\mathcal{C},\mathcal{D})}$ *and* $Sh_{(tow\text{-}\mathcal{C},tow\text{-}\mathcal{D})}$ *are isomorphic.*

Proof. Statement (i) is a consequence of the definition of an abstract shape category. Statements (ii) and (iii) are the special cases. Let us define

$$F : Sh_{(pro\text{-}\mathcal{C},\mathcal{D})} \to Sh_{(pro\text{-}\mathcal{C},tow\text{-}\mathcal{D})}$$

by putting $\boldsymbol{X} \mapsto F(\boldsymbol{X}) = \boldsymbol{X}$ and $\Phi \mapsto F(\Phi) = \Phi'$, where Φ' is represented by $\underline{\boldsymbol{f}} = E(\boldsymbol{f}) : \underline{\boldsymbol{X}'} \to \underline{\boldsymbol{Y}'}$ in *pro*-(*tow*-$\mathcal{D}$) (see Theorem 3.7), whenever Φ is represented by $\boldsymbol{f} : \boldsymbol{X}' \to \boldsymbol{Y}'$ in *pro*-$\mathcal{D}$. Then, by Theorem 3.7 and Lemma 3.10 (see also Corollary 4.5(ii)), F is an isomorphism of those categories. Further, according to the proof of Theorem 4.2, the same rule defines the functors

$$F' : Sh_{(pro\text{-}\mathcal{C},\mathcal{D})} \to Sh_{(pro\text{-}\mathcal{C},pro\text{-}\mathcal{D})},$$

and

$$F'' : Sh_{(tow\text{-}\mathcal{C},\mathcal{D})} \to Sh_{(tow\text{-}\mathcal{C},tow\text{-}\mathcal{D})},$$

which are isomorphisms of the corresponding categories. Thus, (ii) and (iii) hold. □

Given a category $\mathcal{A}$, let us denote $pro^1\mathcal{A} \equiv pro\text{-}\mathcal{A}$ and define, by induction on $n \in \mathbb{N}$,

$$pro^{n+1}\mathcal{A} = pro^1(pro^n\mathcal{A}).$$

Then,

$$(\forall m, n \in \mathbb{N}) pro^{m+n}\mathcal{A} = pro^m(pro^n\mathcal{A}).$$

By Theorem 4.2, the next fact holds.

Corollary 5.2. *If* $\mathcal{D} \subseteq \mathcal{C}$ *is a pro-reflective subcategory, then so is* $\mathcal{D} \subseteq pro^n\mathcal{C}$, *for every* $n \in \mathbb{N}$. *Consequently, the appropriate generalizations of Theorems 4.2 and 5.1 hold.*

Let us observe that Lemmata 4.4 and 5.15 imply the following more general fact.

Theorem 5.3. *Let* $(\mathcal{B},\mathcal{C},\mathcal{D})$ *be a category triple. If* $\mathcal{D} \subseteq \mathcal{C}$ *and* $\mathcal{C} \subseteq \mathcal{B}$ *are pro-reflective subcategories, then so is* $\mathcal{D} \subseteq \mathcal{B}$.

Proof. Let X be a $\mathcal{B}$-object. Since $\mathcal{C} \subseteq \mathcal{B}$ is pro-reflective, there exists a $\mathcal{C}$-expansion $\boldsymbol{p} : X \to \boldsymbol{X}$ of X. Since $\mathcal{D} \subseteq \mathcal{C}$ is pro-reflective, so is, by Theorem 4.2, $\mathcal{D} \subseteq pro\text{-}\mathcal{C}$. Thus, there exists a $\mathcal{D}$-expansion $\boldsymbol{p}' : \boldsymbol{X} \to \boldsymbol{X}'$ of $\boldsymbol{X}$. Then, by Lemma 4.4 (since $\mathcal{D} \subseteq \mathcal{C}$), the composite morphism $\boldsymbol{p}'\boldsymbol{p} : X \to \boldsymbol{X}'$ of *pro*-$\mathcal{B}$ is a $\mathcal{D}$-expansion of X. Therefore, $\mathcal{D} \subseteq \mathcal{B}$ is a pro-reflective subcategory. □

Corollary 5.4. *The relation "to be a pro-reflective subcategory" is a partial order on the conglomerate of all categories.*

5.2. *Application II.* Let us now apply the obtained general results to the standard case, i.e., to some suitable full subcategories of $HTop$ (the homotopy category of topological spaces and homotopy classes of mappings).

Theorem 5.5. *Let $\mathcal{A} \subseteq Top$ be a full subcategory, and let $\mathcal{D} \subseteq \mathcal{C} \equiv H\mathcal{A}$ ($\subseteq HTop$) be a pro-reflective subcategory. Let $\boldsymbol{X}$ be an inverse system in $\mathcal{A}$ that admits a morphism $\boldsymbol{p} : X \to \boldsymbol{X}$ in pro-$\mathcal{A}$ such that the corresponding morphism $H\boldsymbol{p} : X \to H\boldsymbol{X}$ of pro-$\mathcal{C}$ is a $\mathcal{C}$-expansion with respect to $\mathcal{D}$. Then, $Sh_{(pro\text{-}\mathcal{C},\mathcal{D})}(H\boldsymbol{X}) = Sh_{(pro\text{-}\mathcal{C},\mathcal{D})}(\lfloor X \rfloor) = Sh_{(\mathcal{C},\mathcal{D})}(X)$.*

Proof. First, by Lemma 4.1, $\mathcal{D} \subseteq pro\text{-}\mathcal{C}$ is pro-reflective. Thus, by Theorem 5.1(iii), $Sh_{(\mathcal{C},\mathcal{D})} \subseteq Sh_{(pro\text{-}\mathcal{C},\mathcal{D})}$ is a full subcategory. Further, let $\boldsymbol{X} \in Ob(pro\text{-}\mathcal{A})$ admit a morphism $\boldsymbol{p} : X \to \boldsymbol{X}$ such that $H\boldsymbol{p} : X \to H\boldsymbol{X}$ is a $\mathcal{C}$-expansion with respect to $\mathcal{D}$. By Lemma 4.1, there exists a $\mathcal{D}$-expansion $\boldsymbol{p}' : H\boldsymbol{X} \to \boldsymbol{X}'$ of $H\boldsymbol{X}$. Consequently, since $H\boldsymbol{p} : X \to H\boldsymbol{X}$ is a $\mathcal{C}$-expansion with respect to $\mathcal{D}$ of X, Lemma 4.4 implies that the composite morphism $\boldsymbol{p}'(H\boldsymbol{p}) : X \to \boldsymbol{X}'$ is a $\mathcal{D}$-expansion of X. Hence,

$$Sh_{(pro\text{-}\mathcal{C},\mathcal{D})}((\lfloor X \rfloor, H\boldsymbol{X}) \approx pro - \mathcal{D}(\boldsymbol{X}', \boldsymbol{X}'),$$

and the conclusion follows. ◻

Corollary 5.6. *Let $\boldsymbol{X}$ be an inverse system of compact Hausdorff spaces. Then, $Sh(H\boldsymbol{X}) = Sh(\lim \boldsymbol{X})$, and this realizes in pro-HcPol.*

Proof. Put $\mathcal{A} = cT_2 \subseteq Top$ - the full subcategory of compact Hausdorff spaces. Then, $\mathcal{D} \equiv HcPol \subseteq HTop$ - the homotopy (sub)category of compact polyhedra is a pro-reflective subcategory of $\mathcal{C} \equiv HcT_2$ (see [11, I.5]). Further, it is a well known fact that every inverse system $\boldsymbol{X}$ in cT_2 admits a limit $\boldsymbol{p} : X \to \boldsymbol{X}$ in $pro\text{-}cT_2$. By [11, Theorem I.5.9], $H\boldsymbol{p} : X \to H\boldsymbol{X}$ is HcT_2-expansion of $X = \lim \boldsymbol{X}$. The conclusion now follows by Theorem 5.5. ◻

Corollary 5.7. *Let $\boldsymbol{X}$ be an inverse sequence of compact metrizable spaces. Then, $Sh(H\boldsymbol{X}) = Sh(\lim \boldsymbol{X})$, and this realizes in tow-HcPol.*

Proof. Observe that the category of compact metrizable spaces (full subcategory of cT_2) is closed with respect to the limits of inverse sequences. ◻

5.3. *Application III.* Consider now the full subcategory $cpl\text{-}T_3 \subseteq Top$ determined by all completely regular spaces, and its full subcategories cT_2 of compact Hausdorff spaces and $\mathbb{R}$-*cpt* of realcompact spaces (see Example 3.3). By [11, Example I.2.1], cT_2 and $\mathbb{R}$-*cpt* are pro-reflective subcategories of $cpl\text{-}T_3$ (via the Stone-Čech compactification, $j_X : X \to \beta X$, and the Hewitt realcompactification, $k_X : X \to \nu X$, respectively). Then, consequently, cT_2 is a pro-reflective subcategory of $\mathbb{R}$-*cpt*. Let us denote the corresponding (abstract) shape categories by $Sh_{(cpl\text{-}T_3, cT_2)} \equiv Sh_\beta$ and $Sh_{(cpl\text{-}T_3,\mathbb{R}\text{-}cpt)} \equiv Sh_\nu$. Their realization categories are (the subpro-categories!) $\lfloor cT_2 \rfloor \subseteq pro\text{-}cT_2$ and $\lfloor \mathbb{R}\text{-}cpt \rfloor \subseteq pro\text{-}(\mathbb{R}\text{-}cpt)$ respectively. Therefore, clearly, $Sh_\beta(X) = Sh_\beta(Y)$ and $Sh_\nu(X) = Sh_\nu(Y)$ mean $\beta X \approx \beta Y$ and $\nu X \approx \nu Y$ respectively. Further, it is readily seen that

$$\beta : cpl\text{-}T_3 \to cT_2$$

and

$$\nu : cpl\text{-}T_3 \to \mathbb{R}\text{-}cpt$$

are functors. Hereby, we identify $\beta|(cT_2) = 1_{cT_2}$ and $\nu|(\mathbb{R}\text{-}cpt) = 1_{\mathbb{R}\text{-}cpt}$. The following results are well known:

- $\beta\nu X \approx \nu X$;
- there are dense embeddings $X \to \nu X \to \beta X$ and νX embeds as the minimal realcompact space between X and βX;
- $\nu X \approx \beta X$ if and only if X is pseudocompact.

Observe that by following [5, Ch. 8, 8.4.–8.8.], they are consequences of a slightly different approach to the Hewitt realcompactification (for instance, in [5], it is $\nu X \subseteq \beta X$ by definition, while in our approach it is not the case). Let us show how they follow by means of expansions.

THEOREM 5.8. $(\nu|(cT_2))\beta = \beta \cong (\beta|(\mathbb{R}\text{-}cpt)\nu$. *Thus, for every completely regular space X,*

$$Sh_\beta(X) = Sh_\beta(\nu X).$$

PROOF. The first equality is trivial. For the second one, since $cT_2 \subseteq \mathbb{R}$-*cpt*, Lemma 4.4 implies that, for every completely regular space X, the composite mapping

$$X \overset{k_X}{\to} \nu X \overset{j_{\nu X}}{\to} \beta\nu X$$

is a (rudimentary) cT_2-expansion of X. Since $j_X : X \to \beta X$ is also such an expansion, Remark I.1.2 of [11] implies that $\lfloor \beta\nu X \rfloor \cong \lfloor \beta X \rfloor$ in $pro\text{-}cT_2$. Further, for every pair $X, Y \in Ob(cpl\text{-}T_3)$, there exists a natural bijection

$$pro\text{-}cT_2(\lfloor \beta\nu X \rfloor, \lfloor \beta\nu Y \rfloor) \approx pro\text{-}cT_2(\lfloor \beta X \rfloor, \lfloor \beta Y \rfloor).$$

The conclusion follows. Consequently, $Sh_\beta(X) = Sh_\beta(\nu X)$. ∎

COROLLARY 5.9. *Let X be a completely regular space. Then,*

(i) *$\beta\nu X \approx \beta X$ (especially, $X \not\approx \nu X$ and $\beta X \approx \beta\nu X$ hold whenever X is not realcompact);*
(ii) *νX embeds densely into βX as a minimal realcompact space containing X (moreover, the natural mapping $f : \nu X \to \beta X$, $fk_X = j_X$, is a dense embedding);*

(iii) *$\nu X \approx \beta X$ if and only if X is pseudocompact (moreover, the natural mapping f is a homeomorphism).*

PROOF. For (i), notice that $\beta X \approx \beta\nu X$ is equivalent to $Sh_\beta(X) = Sh_\beta(\nu X)$ and apply Theorem 5.8. For (ii), first, since βX is compact, and thus realcompact, there exists a unique mapping $f : \nu X \to \beta X$ such that $fk_X = j_X$. By Theorem 5.8, the following diagram commutes

$$\begin{array}{ccccc} \beta\nu X & \overset{j_{\nu X}}{\leftarrow} & \nu X & \overset{k_X}{\leftarrow} & X \\ \beta f \downarrow & & \downarrow f & \swarrow j_X & \\ \beta X & \underset{1}{\leftarrow} & \beta X & & \end{array} .$$

By Lemma 4.4, all the mappings in the diagram are appropriate expansions, and βf is a homeomorphism. Since $(\beta f)j_{\nu X} = f$, it follows that f is a dense embedding. Further, if $X \subseteq \widetilde{X} \subseteq \nu X$ are densely embedded in βX, whereas $\nu\widetilde{X} = \widetilde{X}$, then one easily derives that $\widetilde{X}$ has the universal property of νX. Therefore, $\widetilde{X} = \nu X$.

In order to prove (iii), let $\nu X \approx \beta X$. Then there exists a homeomorphism $h : \nu X \to \beta X$. Observe that in the next commutative diagram (with a unique f, a unique g and a unique g')

$$\begin{array}{ccc} \nu X & \overset{k_X}{\leftarrow} & X \\ h\downarrow & f \searrow & \downarrow j_X \\ \beta X & \overset{g}{\underset{g'}{\rightleftarrows}} & \beta X \end{array}$$

all the mappings are appropriate expansions. It is readily seen that g and $g' = g^{-1}$ are homeomorphisns. Thus, $f : \nu X \to \beta X$ is a homeomorphism as well. Let $u : X \to \mathbb{R}$ be an arbitrary mapping. Then there exists a unique mapping $w : \nu X \to \mathbb{R}$ such that $wk_X = u$. Since νX is compact, w is bounded, and thus, u is bounded. Consequently, X is a pseudocompact space. Conversely, let X be pseudocompact. Consider an arbitrary mapping $w : \nu X \to \mathbb{R}$. Put $u = wk_X : X \to \mathbb{R}$. Since X is pseudocompact, $u[X] \subseteq \mathbb{R}$ is a bounded subset, and thus, it is contained in a compact subspace $K \subseteq \mathbb{R}$. Since $j_{\nu X}k_X : X \to \beta\nu X \approx \beta X$ is a cT_2-expansion of X (Lemma 4.4), there exists a unique mapping g such that the following diagram commutes:

$$\begin{array}{ccccc} \beta X \approx \beta\nu X & \overset{j_{\nu X}}{\leftarrow} & \nu X & \overset{k_X}{\leftarrow} & X \\ & g \searrow & & \swarrow u = wk_X & \\ & & K & & \end{array} .$$

Then, $gj_{\nu X}k_X = u = wk_X$. Since k_X is an expansion, it follows that $gj_{\nu X} = w$. Thus, $w : \nu X \to K \subseteq \mathbb{R}$ is bounded, which shows that νX is pseudocompact. Since it is realcompact, it must be compact ([5, Ch. 5, Problem 5H.1, p. 79]). The conclusion follows. ∎

REMARK 5.10. As we know, βX is the "largest" among all the compactifications $\widehat{X}$ of X such that each $\widehat{X}$ is the natural quotient space of βX (see [2, Theorem 8.2.(3), p. 243]). On the other hand, the proof of Corollary 5.9 shows that, although νX is the "largest" among all the realcompactifications $\widetilde{X}$ of X, the space $\widetilde{X}$ is not, generally, the natural quotient space of νX. Indeed, observe that $j_X : X \to \beta X$ is a realcompactification of X. Suppose that βX is homeomorphic to $\nu X/(\sim_f)$, where $\sim_f$ *is* induced by the unique dense embedding (cT_2-expansion) $f : \nu X \to \beta X$, $fk_X = j_X$. Since f is injective, the equivalence relation $\sim_f$ is trivial, Thus, $\beta X \approx \nu X$ holds, which is impossible unless X is pseudocompact.

Observe that every inverse system $\boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ in *cpl*-T_3 admits a unique inverse system $\beta\boldsymbol{X} = (\beta X_\lambda, \beta p_{\lambda\lambda'}, \Lambda)$ in cT_2 and the induced morphism $\boldsymbol{X} \to \beta\boldsymbol{X}$ of *pro*-(*cpl*-T_3) (a unique inverse system $\nu\boldsymbol{X} = (\nu X_\lambda, \nu p_{\lambda\lambda'}, \Lambda)$ in $\mathbb{R}$-*cpt* and the induced morphism $\boldsymbol{X} \to \nu\boldsymbol{X}$ of *pro*-($\mathbb{R}$-*cpt*)) - see the proof of Lemma 4.1. Since in these categories inverse limits exist, the following questions (about the continuity of β and ν) naturally occur:

(1) *Is* $\beta(\lim \boldsymbol{X})$ *homeomorphic to* $\lim(\beta\boldsymbol{X})$*?*
(2) *Is* $\nu(\lim \boldsymbol{X})$ *homeomorphic to* $\lim(\nu\boldsymbol{X})$*?*
(3) *Are* $\beta\nu(\lim \boldsymbol{X})$*,* $\beta(\lim \nu\boldsymbol{X})$*,* $\beta(\lim \boldsymbol{X})$*,* $\lim(\beta\nu\boldsymbol{X})$ *and* $\lim(\beta\boldsymbol{X})$ *mutually homeomorphic?*

Recall that $\beta(\prod_{\lambda\in\Lambda} X_\lambda) \approx \prod_{\lambda\in\Lambda} \beta X_\lambda$ if and only if $\prod_{\lambda\in\Lambda} X_\lambda$ is pseudocompact (provided, for each $\lambda_0 \in \Lambda$, $\prod_{\lambda\neq\lambda_0} X_\lambda$ is not finite) see [4, 5]; [3, d-17, "The Čech-Stone compactification", pp. 210-212]. Further, if $X \times Y$ is pseudocompact, then $\nu(X \times Y) = \nu X \times \nu Y$, but not conversely ([5, 18], [3, d-09, "Realcompactness", pp. 185-188]). Therefore, in general, one should expect the answers to all the above questions in negative. Notice also that the answer to question (1) is negative whenever $\lim \boldsymbol{X} = \varnothing$. A simple example is given below (the fact that the terms X_i are realcompact and not pseudocompact is not used hereby!).

EXAMPLE 5.11. Let $\boldsymbol{X} = (X_i, p_{ii'}, \mathbb{N})$ be the decreasing inverse sequence of countable discrete spaces

$$X_1 = \mathbb{N}$$

and

$$X_{i+1} = X_i \setminus \{i\} = \mathbb{N} \setminus [1, i]_{\mathbb{N}},\ i \in \mathbb{N},$$

with the inclusion bonding mappings (see Example 2.4). Then, obviously, $\lim \boldsymbol{X}$ is the empty space. Therefore, $\beta(\lim \boldsymbol{X}) = \varnothing$. On the other hand, for every $i \in \mathbb{N}$, $\beta X_i \approx \beta\mathbb{N}$ is a nonempty compact Hausdorff space. Therefore, $\lim(\beta\boldsymbol{X}) \neq \varnothing$ ([2, Appendix Two, 2.4 (3)]). Notice that *none* of $\beta p_{ii'} : \beta X_{i'} \to \beta X_i$, $i < i'$, is a *homeomorphism*. Indeed, if a $\beta p_{ii'}$ were a homeomorphism for some pair $i < i'$, then it would be an expansion with respect to cT_2. Then, by Lemma 4.4, the inclusion $p_{ii'}$ would be an expansion with respect to cT_2. However, since

$$X_i \setminus X_{i'} \supseteq \{i\} \neq \varnothing$$

and since X_i is a discrete space, the desired uniqueness cannot be achieved.

The next almost trivial example detects the behavior of β and ν with respect to (category) limits of finite diagrams. It shows that the answers to all stated questions are affirmative whenever $\boldsymbol{X}$ (i.e., Λ) is finite or infinite having a maximal term.

Example 5.12. Let $\boldsymbol{X}$ be a finite inverse system given by the following commutative diagram

$$\begin{array}{ccc} X_2 & \stackrel{p_{24}}{\leftarrow} & X_4 \\ \downarrow & & \downarrow p_{34} \\ X_1 = \{*\} & \leftarrow & X_3 \end{array}$$

in *cpl*-T_3. Notice that $\lim \boldsymbol{X} = X_4$. Further, the diagram

$$\begin{array}{ccc} \beta X_2 & \stackrel{\beta(p_{24})}{\leftarrow} & \beta X_4 \\ \downarrow & & \downarrow \beta(p_{34}) \\ \beta X_1 = \{*\} & \leftarrow & \beta X_3 \end{array}$$

commutes and represents $\beta\boldsymbol{X}$. Thus, $\lim(\beta\boldsymbol{X}) = \beta X_4$. Consequently,

$$\beta(\lim \boldsymbol{X}) = \beta X_4 = \lim(\beta\boldsymbol{X}).$$

Especially, if $X_4 = X_2 \times X_3$ and the bonding mappings are the corresponding projections, then

$$\beta(\lim \boldsymbol{X}) = \beta(X_2 \times X_3) = \lim(\beta\boldsymbol{X}).$$

The same holds for the Hewitt realcompactification, i.e., $\nu(\lim \boldsymbol{X}) = \lim(\nu\boldsymbol{X})$. On the other hand, let Δ be a finite diagram

$$\begin{array}{ccc} X_2 & & \\ \downarrow & & \\ \{*\} & \leftarrow & X_3 \end{array}$$

in *cpl*-T_3, and let $\beta\Delta$ be the corresponding diagram

$$\begin{array}{ccc} \beta X_2 & & \\ \downarrow & & \\ \{*\} & \leftarrow & \beta X_3 \end{array}$$

in $cT_2 \subseteq$ *cpl*-T_3. Put

$$X_2 = X_3 = X$$

to be a completely regular pseudocompact space such that $X \times X$ is not pseudocompact. For instance, the space X constructed (by the J. Novak's result concerning $\beta\mathbb{N}$) in [2, XI. 8.2, p. 245]. Then, clearly, $\lim \Delta = X \times X$ and $\lim(\beta\Delta) = \beta X \times \beta X$. Since $X \times X$ is not pseudocompact, we infer that

$$\beta(\lim \Delta) = \beta(X \times X) \not\approx \beta X \times \beta X = \lim(\beta\Delta).$$

Concerning question (3), if $\beta(X \times X) \not\approx \beta X \times \beta X$, then (even X is realcompact)

$$\beta\nu(X \times X) = \beta(X \times X) \not\approx \beta X \times \beta X \approx \beta\nu X \times \beta\nu X.$$

This shows that questions (1), (2) and (3) should be restricted to an infinite inverse system $\boldsymbol{X}$ (having no maximal element) such that $\lim \boldsymbol{X} \neq \varnothing$. If $\boldsymbol{X}$ belongs to *pro*-($\mathbb{R}$-*cpt*), then $\nu\boldsymbol{X} = \boldsymbol{X}$ and $\lim \boldsymbol{X}$ belongs to $\mathbb{R}$-*cpt*. Thus, in this special case, the answer to (2) is trivially affirmative, the first three spaces in (3) are mutually homeomorphic, the last two are also mutually homeomorphic, while the question (1) remains open. Recall that realcompactness and pseudocompactness imply compactness, and thus, in that case all the answers are trivially affirmative.

Let us now consider the general case, trying to find sufficient conditions for positive answers in terms of expansions. The next results are the consequences of the more general ones exhibited by Lemma 4.1.

COROLLARY 5.13. *Let $\boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ be an inverse system in cpl-T_3. Then the induced morphism*

(i) $\boldsymbol{j} = [(1_\Lambda, j_\lambda = j_{X_\lambda})] : \boldsymbol{X} \to \beta\boldsymbol{X} = (\beta X_\lambda, \beta p_{\lambda\lambda'}, \Lambda)$ *of pro-(cpl-T_3) is a (cT_2)-expansion of $\boldsymbol{X}$;*

(ii) $\boldsymbol{k} = [(1_\Lambda, k_\lambda = k_{X_\lambda})] : \boldsymbol{X} \to \nu\boldsymbol{X} = (\nu X_\lambda, \nu p_{\lambda\lambda'}, \Lambda)$ *of pro-(cpl-T_3) is an ($\mathbb{R}$-cpt)-expansion of $\boldsymbol{X}$.*

PROOF. Apply Lemma 4.1 and its proof to the category pairs (*cpl*-T_3, *c*T_2) and (*cpl*-T_3, $\mathbb{R}$-*cpt*) respectively. ∎

Corollary 5.13 implies that β and ν admit extensions to the functors of the corresponding pro-categories, i.e., to

$$\beta : \textit{pro-}(\textit{cpl-}T_3) \to \textit{pro-c}T_2$$

and

$$\nu : \textit{pro-}(\textit{cpl-}T_3) \to \textit{pro-}(\mathbb{R}\textit{-cpt})$$

(the same notation will not cause ambiguity) such that $\beta|(\textit{pro-c}T_2) = 1_{\textit{pro-c}T_2}$ and $\nu|(\textit{pro-}(\mathbb{R}\textit{-cpt})) = 1_{\textit{pro-}(\mathbb{R}\textit{-cpt})}$, which are, in addition, expansions on the objects. This implies that, for every stable system $\boldsymbol{X}$, the answer to each of the above questions is affirmative. Further, the next facts hold.

THEOREM 5.14. *For every inverse system $\boldsymbol{X}$ of completely regular spaces, $\beta\nu\boldsymbol{X} \cong \beta\boldsymbol{X}$ in pro-cT_2. Therefore, $\beta\nu(\lim \boldsymbol{X}) \approx \beta(\lim \boldsymbol{X})$ and $\lim(\beta\nu\boldsymbol{X}) \approx \lim(\beta\boldsymbol{X})$. Further, if $\lim \boldsymbol{X}$ does not belong to $\mathbb{R}$-cpt, then $\nu\boldsymbol{X}$ and $\boldsymbol{X}$ are not isomorphic in pro-(cpl-T_3), while $\beta\nu\boldsymbol{X} \cong \beta\boldsymbol{X}$ in pro-cT_2.*

Proof. By Theorem 5.8 and Corollary 5.13, $Sh_\beta(\nu\boldsymbol{X}) = Sh_\beta(\boldsymbol{X})$ holds, and the conclusions follow. ∎

Lemma 5.15. *Let* $\boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ *be an inverse system in cpl-*T_3 *and let* $X = \lim \boldsymbol{X}$*. If* $X \neq \varnothing$*. Then,*

(i) $(\widehat{X} = \lim \beta\boldsymbol{X}, j)$ *is a compactification of* X*, where* $j : X \to \widehat{X}$ *is the limit mapping of* $\boldsymbol{j} : \boldsymbol{X} \to \beta\boldsymbol{X}$*;*

(ii) $(\widetilde{X} = \lim \nu\boldsymbol{X}, k)$ *is a realcompactification of* X*, where* $k : X \to \widetilde{X}$ *is the limit mapping of* $\boldsymbol{k} : \boldsymbol{X} \to \nu\boldsymbol{X}$*.*

Proof. Observe that $X = \lim \boldsymbol{X}$ is a completely regular space. It is a well known fact that $\widehat{X} = \lim(\beta\boldsymbol{X})$ belongs to cT_2 ($\widetilde{X} = \lim(\gamma\boldsymbol{X})$ belongs to $\mathbb{R}$-*cpt*). Therefore, to prove (i), it suffices to show that $j = \lim \boldsymbol{j}$ is a dense embedding. Since $\lim \boldsymbol{X} = X$ is not the empty space, there exists an inverse system $\boldsymbol{X}'$ in *cpl*-T_3 with surjective bonding mappings such that $\boldsymbol{X} \cong \boldsymbol{X}'$ in *pro*-(*cpl*-T_3). Then $\beta\boldsymbol{X} \cong \beta\boldsymbol{X}'$ in *pro*-cT_2, and hence, $\lim \beta\boldsymbol{X} \approx \lim \beta\boldsymbol{X}'$. Thus, we may assume that $\boldsymbol{X}$ and $\beta\boldsymbol{X}$ have surjective bonding mappings. Then, since every $j_\lambda : X_\lambda \to \beta X_\lambda$, $\lambda \in \Lambda$, is a dense embedding, it follows straightforwardly that $j = \lim \boldsymbol{j} : X \to \widehat{X}$ is a dense embedding as well. To prove (ii), first notice that $\lim \boldsymbol{X} = X \neq \varnothing$ implies that also $\lim(\nu\boldsymbol{X}) = \widetilde{X} \in Ob(\mathbb{R}\text{-}cpt)$ is not empty. Then, similarly to the proof of (i), since every $k_\lambda : X_\lambda \to \gamma X_\lambda$, $\lambda \in \Lambda$, is a dense embedding, it follows that so is $k = \lim \boldsymbol{k}$. ∎

Lemma 5.16. *Let* $\boldsymbol{p} : X = \lim \boldsymbol{X} \to \boldsymbol{X}$*,* $\boldsymbol{r} = (r_\lambda) : \widehat{X} = \lim \beta\boldsymbol{X} \to \beta\boldsymbol{X}$ *and* $\boldsymbol{s} = (s_\lambda) : \widetilde{X} = \lim \nu\boldsymbol{X} \to \nu\boldsymbol{X}$ *be inverse limits in cpl-*T_3*. Then,*

(i) *the quotient mapping* $\widehat{q} : \beta X \to \widehat{X}$*,* $\widehat{q} j_X = j$*, and the limit mapping* $\widehat{p} : \beta X \to \widehat{X}$*,* $r_\lambda \widehat{p} = \beta p_\lambda$*,* $\lambda \in \Lambda$*, coincide;*

(ii) *the quotient mapping* $\widetilde{q} : \nu X \to \widetilde{X}$*,* $\widetilde{q} k_X = k$*, and the limit mapping* $\widetilde{p} : \nu X \to \widetilde{X}$*,* $s_\lambda \widetilde{p} = \nu p_\lambda$*,* $\lambda \in \Lambda$*, coincide.*

Thus, $(\lim \beta\boldsymbol{X}, \lim \beta\boldsymbol{p})$ *is a compactification of* $\lim \boldsymbol{X}$*, and* $(\lim \nu\boldsymbol{X}, \lim \nu\boldsymbol{p})$ *is a realcompactification of* $\lim \boldsymbol{X}$*.*

Proof. Observe that, for every $\lambda \in \Lambda$,

$$r_\lambda \widehat{q} j_X = r_\lambda j = p_\lambda j_\lambda = (\beta p_\lambda) j_X = r_\lambda \widehat{p} j_X.$$

Since j_X is an expansion (with respect to cT_2), it follows that,

$$(\forall \lambda \in \Lambda) r_\lambda \widehat{q} = r_\lambda \widehat{p}.$$

Since $\boldsymbol{r} : \widehat{X} \to \beta\boldsymbol{X}$ is a limit, $\widehat{q} = \widehat{p}$ must hold, and (i) is proved. The proof of (ii) is quite similar. The conclusion follows by Lemma 5.15. ∎

Let us abandon for a while the Hewitt realcompactification, and focus our attention to the Stone-Čech compactification. Notice that, in these considerations, to each purely categorical fact concerning the pair (*cpl*-T_3, cT_2) it corresponds the analogous one concerning the pair (*cpl*-$T_3, \mathbb{R}$-*cpt*).

LEMMA 5.17. *Every mapping $f : X \to Z$ of cpl-T_3, which is a (cpl-T_3)-expansion with respect to cT_2 of X, is injective, while every mapping $g : \beta X \to Z$, which is a (cpl-T_3)-expansion with respect to cT_2 of βX, is an embedding.*

PROOF. Let a mapping $f : X \to Z$ be a (rudimentary) (*cpl*-T_3)-expansion with respect to cT_2 of X. Then, for the embedding $j_X : X \to \beta X$, there exists a unique mapping $u : Z \to \beta X$ such that $uf = j_X$. This implies that f is an injection. Similarly, if $g : \beta X \to Z$ is a mapping that is a (rudimentary) (*cpl*-T_3)-expansion with respect to cT_2 of βX, then, for the identity mapping $1_{\beta X}$, there exists a mapping $v : Z \to \beta X$ such that $vg = 1_{\beta X}$. This implies that g is a continuous injection. Since βX is a compact Hausdorff space, g is an embedding. □

THEOREM 5.18. *Let $\boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ be an inverse system in cpl-T_3, and let $X = \lim \boldsymbol{X}$ and $\widehat{X} = \lim(\beta \boldsymbol{X})$. Then, with the notation from above, the following assertions are mutually equivalent:*

(i) *The quotient mapping $\widehat{q} : \beta X \to \widehat{X}$ is a homeomorphism;*
(ii) *$\widehat{q} : \beta X \to \widehat{X}$ of cT_2 is a (cT_2)-expansion of βX;*
(iii) *The limit mapping $j : X \to \widehat{X}$ of cpl-T_3 is a (cT_2)-expansion of X.*

Further, they imply the following mutually equivalent assertions:

(iv) *There exists a homeomorphism $h : \beta X \to \widehat{X}$, i.e., $\beta(\lim \boldsymbol{X}) \approx \lim(\beta \boldsymbol{X})$;*
(v) *There exists a mapping $g : \beta X \to \widehat{X}$ of cT_2 that is a (cT_2)-expansion of βX;*
(vi) *There exists a mapping $f : X \to \widehat{X}$ of cpl-T_3 that is a (cT_2)-expansion of X.*

PROOF. (i) trivially implies (ii). Further, (ii) implies (iii) by Lemma 4.4. Namely, the composition of (cT_2)-expansions $\widehat{q} j_X = j$ is a (cT_2)-expansion. The converse, (iii) ⇒ (ii), holds also by Lemma 4.4. Namely, $\widehat{q}$ is an expansion since j_X and $j = \widehat{q} j_X$ are the appropriate expansions. To prove that (ii) implies (i), apply Lemma 5.17 to $Z = \widehat{X}$. It follows that $\widehat{q}$ is an embedding. Since $\widehat{q}$ is a surjective quotient mapping (Remark 5.10), it must be a homeomorphism.

Further, observe that (i), (ii) and (iii) imply (iv), (v) and (vi) respectively, and that (iv) trivially implies (v). Let us suppose that there exists a mapping $g : \beta X \to \widehat{X}$ which is a (cT_2)-expansion of βX. By Lemma 4.4, the composition $g j_X \equiv f : X \to \widehat{X}$ is a (cT_2)-expansion of X, which shows that (v) implies (vi). Let there exist a mapping $f : X \to \widehat{X}$ that is a (cT_2)-expansion of X. Since j_X is an expansion of the same kind, Remark I.2.2 of [11] implies that $\widehat{X} \approx \beta X$. Thus, (vi) implies (iv), which completes the proof of the theorem. ∎

Problem 5.1. *Are all the assertions of Theorem 5.18 mutually equivalent?*

Theorem 5.19. *Let* $\boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ *be an inverse system in cpl-*T_3*, and let* $\boldsymbol{p} = (p_\lambda) : X \to \boldsymbol{X}$ *and* $\boldsymbol{r} = (r_\lambda) : \widehat{X} \to \beta\boldsymbol{X}$ *be inverse limits. Then, with the notation from above, the following five statements are equivalent:*

(i) *the limit morphism* $\boldsymbol{p} : X \to \boldsymbol{X}$ *of pro-(cpl-*T_3*) is a (cpl-*T_3*)-expansion with respect to* cT_2 *of* X*;*

(ii) *the morphism* $\boldsymbol{jp} : X \to \beta\boldsymbol{X}$ *of pro-(cpl-*T_3*) is a (*cT_2*)-expansion of* X*;*

(iii) *the morphism* $(\beta\boldsymbol{p})j_X : X \to \beta\boldsymbol{X}$ *of pro-(cpl-*T_3*) is a (*cT_2*)-expansion of* X*;*

(iv) *the morphism* $\beta\boldsymbol{p} : \beta X \to \beta\boldsymbol{X}$ *of pro-(*cT_2*) is a (*cT_2*)-expansion of* βX*;*

(v) $\beta\boldsymbol{p} : \beta X \to \beta\boldsymbol{X}$ *is an isomorphism of pro-*cT_2*.*

Further, they imply the following three mutually equivalent statements:

(vi) *there exists a morphism* $\boldsymbol{p}' : X \to \boldsymbol{X}$ *of pro-(cpl-*T_3*) that is a (cpl-*T_3*)-expansion with respect to* cT_2 *of* X*;*

(vii) *the induced morphism* $\beta\boldsymbol{p}' : \beta X \to \beta\boldsymbol{X}$ *of pro-*cT_2 *is a (*cT_2*)-expansion of* X*;*

(viii) $\beta\boldsymbol{p}' : \beta X \to \beta\boldsymbol{X}$ *is an isomorphism of pro-*cT_2*.*

Further, these imply the following three mutually equivalent statements:

(ix) *there exists a morphism* $\boldsymbol{f} : X \to \beta\boldsymbol{X}$ *of pro-(cpl-*T_3*) that is a (*cT_2*)-expansion of* X*;*

(x) *there exists a morphism* $\boldsymbol{g} : \beta X \to \beta\boldsymbol{X}$ *of pro-(*cT_2*) that is a (*cT_2*)-expansion of* βX*;*

(xi) $\beta\boldsymbol{X} \cong \lfloor \beta X \rfloor$ *in pro-*cT_2*.*

Finally, these imply that

(xii) $\beta(\lim \boldsymbol{X}) \approx \lim(\beta\boldsymbol{X})$.

Proof. Consider the next commutative diagram

$$
\begin{array}{ccc}
\boldsymbol{X} & \overset{\boldsymbol{p}}{\leftarrow} & X \\
\downarrow \boldsymbol{j} & & \downarrow j_X \\
\beta\boldsymbol{X} & \overset{\beta\boldsymbol{p}}{\leftarrow} & \beta X \\
\boldsymbol{r} \nwarrow & & \swarrow \widehat{p} = \widehat{q} \\
& \widehat{X} &
\end{array}
$$

in *pro*-(*cpl*-T_3) with a unique morphism $\beta\boldsymbol{p} : \beta X \to \beta\boldsymbol{X}$ of *pro*-cT_2 and the unique limit mapping $\widehat{p} : \beta X \to \widehat{X}$. Then, (i) $\Leftrightarrow$ (ii) holds by Corollary 5.13 and Lemma 4.4. Since $(\beta\boldsymbol{p})j_X = \boldsymbol{jp}$, the equivalence (ii) $\Leftrightarrow$ (iii) is trivial, while the equivalence (iii) $\Leftrightarrow$ (iv) follows by Lemma 4.4. Further, if $\beta\boldsymbol{p}$ is a (cT_2)-expansion of βX, then by Corollary 5.13 and Lemma 4.4, $\boldsymbol{jp} = (\beta\boldsymbol{p})j_X$ is a (cT_2)-expansion of X. Since $\beta\boldsymbol{p}$ is unique (making the square commutative), Remark I.2.2 of [11] and Lemma 4.4 assure that it is an isomorphism of *pro*-cT_2. Thus, (iv) implies (v). Conversely, (v) implies (iv) by Remark I.2.3 of [11].

Observe that (i) trivially implies (vi). Let there exist a morphism $\boldsymbol{p}' : X \to \boldsymbol{X}$ that is a (*cpl*-T_3)-expansion with respect to cT_2 of X. Then there exists a commutative diagram similar to that from the above, with a unique $\beta\boldsymbol{p}' : \beta X \to \beta\boldsymbol{X}$ and the unique limit mapping $\beta p' : \beta X \to \widehat{X}$. By Corollary 5.13 and Lemma 4.4, all the morphisms in the square are appropriate expansions. Thus, (vi) is equivalent to (vii). Notice that (vii) $\Leftrightarrow$ (viii) works in the same way as (iv) $\Leftrightarrow$ (v).

Further, (vi) implies (ix) by Lemma 4.4 ($\boldsymbol{f} = (\beta\boldsymbol{p}')j_X$). Let $\boldsymbol{f} : X \to \beta\boldsymbol{X}$ be a (cT_2)-expansion of X. Then there exists the following commutative diagram

$$\begin{array}{ccc} \beta\boldsymbol{X} & \overset{\boldsymbol{f}}{\leftarrow} & X \\ \downarrow \mathbf{1} & & \downarrow j_X \\ \beta\boldsymbol{X} & \overset{\beta\boldsymbol{f}}{\leftarrow} & \beta X \end{array}$$

in *pro*-(*cpl*-T_3) with a unique $\beta\boldsymbol{f} : \beta X \to \beta\boldsymbol{X}$ of *pro*-cT_2. By Lemma 4.4, $\boldsymbol{g} \equiv \beta\boldsymbol{f}$ is a (cT_2)-expansion of βX, and thus, (ix) implies (x). Let there exist a morphism $\boldsymbol{g} : \beta X \to \beta\boldsymbol{X}$ of *pro*-(cT_2) that is a (cT_2)-expansion of βX. Since $1_{\beta X}$ is also a (cT_2)-expansion of βX, Remark I.2.2 of [11] implies that $\beta\boldsymbol{X} \cong \lfloor\beta X\rfloor$ in *pro*-cT_2. Hence, (x) implies (xi). Conversely, if $\beta\boldsymbol{X} \cong \lfloor\beta X\rfloor$ in *pro*-cT_2, then Remark I.2.3 of [11] implies that there exists a morphism $\boldsymbol{f} : X \to \beta\boldsymbol{X}$ of *pro*-(*cpl*-T_3) which is a (cT_2)-expansion of X. Thus, (xi) implies (ix). Finally, if $\beta\boldsymbol{X} \cong \lfloor\beta X\rfloor$ in *pro*-cT_2, then

$$\beta X = \lim(\lfloor\beta X\rfloor) \approx \lim(\beta\boldsymbol{X}).$$

Therefore,

$$\beta(\lim \boldsymbol{X}) = \beta X \approx \lim(\beta\boldsymbol{X}).$$

which shows that (xi) implies (xii), and completes the proof of the theorem. □

Let us now consider an inverse system $\boldsymbol{Y}$ in *cpl*-T_3 that is an *expansion system*, i.e., let there exist a morphism $\boldsymbol{q}^* : Y^* \to \boldsymbol{Y}$ of *pro*-(*cpl*-T_3) that is a (*cpl*-T_3)-expansion with respect to cT_2 of Y^* (for instance, every stable $\boldsymbol{Y}$ is an expansion system). Then there exists the following commutative diagram

$$\begin{array}{ccc} Y^* & \overset{\boldsymbol{q}^*}{\rightarrow} & \boldsymbol{Y} \\ j_{Y^*} \downarrow & & \downarrow \boldsymbol{j} \\ \beta Y^* & \overset{\beta \boldsymbol{q}^*}{\rightarrow} & \beta \boldsymbol{Y} \end{array}$$

in *pro*-(*cpl*-T_3) consisting of appropriate expansions (Corollary 5.13 and Lemma 4.4). Thus, the morphisms

$$j_{Y^*} : Y^* \to (\beta Y^*)$$

and

$$(\beta \boldsymbol{q}^*) j_{Y^*} = \boldsymbol{j}\boldsymbol{q}^* : Y^* \to \beta \boldsymbol{Y}$$

of *pro*-(*cpl*-T_3) are cT_2-expansions of Y^*. By Remark I.2.2 of [11], the (unique) morphism $\beta \boldsymbol{q}^* : \beta Y^* \to \beta \boldsymbol{Y}$, such that $(\beta \boldsymbol{q}^*) j_{Y^*} = \boldsymbol{j}\boldsymbol{q}^*$, is the corresponding natural isomorphism of *pro*-cT_2. Consequently, by Remark I.2.3 of [11] and Lemma 4.4, the morphism

$$(\beta \boldsymbol{q}^*)^{-1} \boldsymbol{j} : \boldsymbol{Y} \to \lfloor \beta Y^* \rfloor$$

of *pro*-(*cpl*-T_3) is a (rudimentary) cT_2-expansion of the system $\boldsymbol{Y}$.

Further, notice that if $\boldsymbol{Y}$ is the expanding system of another $Y^{*\prime}$, i.e., if in addition to $\boldsymbol{q}^* : Y^* \to \boldsymbol{Y}$ there exists a $\boldsymbol{q}^{*\prime} : Y^{*\prime} \to \boldsymbol{Y}$ of *pro*-(*cpl*-T_3) which is (*cpl*-T_3)-expansions with respect to cT_2 of $Y^{*\prime}$, then $\beta Y^* \approx \beta Y^{*\prime}$. Indeed, $\beta \boldsymbol{q}^*$ and $\beta \boldsymbol{q}^{*\prime}$ are isomorphisms of *pro*-cT_2, and thus,

$$\lfloor \beta Y^* \rfloor \cong \beta \boldsymbol{Y} \cong \lfloor \beta Y^{*\prime} \rfloor$$

in *pro*-cT_2, which implies (see the commutative diagram in *pro*-(*cpl*-T_3) below) that

$$\beta Y^* \approx \lim(\beta \boldsymbol{Y}) \approx \beta Y^{*\prime}.$$

Therefore, by denoting $ex(\boldsymbol{Y})$ to be any of those spaces Y^*, $Y^{*\prime}$, ... (if some exists), the Stone-Čech compactification $\beta(ex(\boldsymbol{Y}))$ is well defined (up to a homeomorphism). Observe that if an $ex(\boldsymbol{Y})$ exists, then also an $ex(\beta \boldsymbol{Y})$ exists and, moreover, there exists a compact one, such as $\beta(ex(\boldsymbol{Y}))$. Especially, if there exists a compact $ex(\boldsymbol{Y})$, then it is homeomorphic to $ex(\beta \boldsymbol{Y})$.

$$\begin{array}{ccccc} ex(\boldsymbol{Y}) & \overset{\boldsymbol{q}^*}{\rightarrow} & \boldsymbol{Y} & \overset{\boldsymbol{q}}{\leftarrow} & Y = \lim \boldsymbol{Y} \\ j_* \downarrow & & \downarrow \boldsymbol{j} & & \downarrow j_Y \\ \beta(ex(\boldsymbol{Y})) & \overset{\beta \boldsymbol{q}^*}{\rightarrow} & \beta \boldsymbol{Y} & \overset{\beta \boldsymbol{p}}{\leftarrow} & \beta Y \\ & \searrow \widehat{q}^* & \uparrow \boldsymbol{s} & \widehat{q} \swarrow & \\ & & \widehat{Y} = \lim(\beta \boldsymbol{Y}) & & \end{array} .$$

Hence, this consideration yields the following lemma.

LEMMA 5.20. *Let $\boldsymbol{X}$ be an inverse system in cpl-T_3.*

(i) *If there exists a* $\boldsymbol{p}^* : ex(\boldsymbol{X}) \to \boldsymbol{X}$ *which is a (cpl-*T_3*)-expansion with respect to c*T_2*, then* $\beta\boldsymbol{X} \cong \lfloor\beta(ex(\boldsymbol{X}))\rfloor$ *in pro-c*T_2 *and the morphisms*

$$(\beta\boldsymbol{p}^*)^{-1}\boldsymbol{j} : \boldsymbol{X} \to \lfloor\beta(ex(\boldsymbol{X}))\rfloor$$

and

$$\boldsymbol{hj} : \boldsymbol{X} \to \lfloor\lim(\beta\boldsymbol{X})\rfloor$$

of pro-cpl-T_3*, where* $\boldsymbol{h} : \beta\boldsymbol{X} \to \lfloor\beta(ex(\boldsymbol{X}))\rfloor$ *is any isomorphism, are the rudimentary c*T_2*-expansions of the system* $\boldsymbol{X}$*. Further, the limit morphism*

$$\underline{\boldsymbol{r}} : \lim(\beta\boldsymbol{X}) \to \beta\boldsymbol{X}$$

is a cT_2*-expansion as well. Consequently,*

$$\lim(\beta\boldsymbol{X}) \approx \beta(ex(\beta\boldsymbol{X})) \approx \beta(ex(\boldsymbol{X})).$$

(ii) *More general, if* $\beta\boldsymbol{X}$ *is an expansion system, i.e., if there exists a c*T_2*-expansion* $\boldsymbol{q}^* : ex(\beta\boldsymbol{X}) \to \beta\boldsymbol{X}$*, then the limit morphism* $\boldsymbol{r}$ *is a c*T_2*-expansion, and thus,*

$$\lim(\beta\boldsymbol{X}) \approx \beta(ex(\beta\boldsymbol{X})).$$

Proof. Consider the above diagram,with $\boldsymbol{X}$, $\boldsymbol{p}^*$, ... instead of $\boldsymbol{Y}$, $\boldsymbol{q}^*$, ... ($\boldsymbol{p} : X \to \boldsymbol{X}$ and $\boldsymbol{r} : \widehat{X} \to \beta\boldsymbol{X}$ are the inverse limits). Then, $\beta\boldsymbol{p}^* : \lfloor\beta(ex(\boldsymbol{X}))\rfloor \to \beta\boldsymbol{X}$ is the natural isomorphism by the above consideration. We have also noticed that $(\beta\boldsymbol{p}^*)^{-1}\boldsymbol{j} : \boldsymbol{X} \to \lfloor\beta(ex(\boldsymbol{X}))\rfloor$ is cT_2-expansion. Let $\boldsymbol{h} : \beta\boldsymbol{X} \to \lfloor\beta(ex(\boldsymbol{X}))\rfloor$ be an arbitrary isomorphism of *pro-c*T_2. Then, it is also a rudimentary cT_2-expansion of $\beta\boldsymbol{X}$. By Corollary 5.13 and Lemma 4.4, $\boldsymbol{hj} : \boldsymbol{X} \to \lfloor\lim(\beta\boldsymbol{X})\rfloor$ is a (rudimentary) cT_2-expansions of $\boldsymbol{X}$. Further, since $\beta\boldsymbol{p}^*$ is an isomorphism, the limit mapping

$$\lim(\beta\boldsymbol{p}^*) \equiv \widehat{p^*} : \beta(ex(\boldsymbol{X})) \to \lim(\beta\boldsymbol{X})$$

is a homeomorphism. Since, $\boldsymbol{r}(\widehat{p^*})) = \beta\boldsymbol{p}^*$, Lemma 4.4 implies that the limit morphism

$$\underline{\boldsymbol{r}} : \lim(\beta\boldsymbol{X}) \to \beta\boldsymbol{X}$$

is a cT_2-expansion. Finally, it follows by the above consideration that there exists an $ex(\beta\boldsymbol{X})$ and that, for every $ex(\boldsymbol{X})$ and every $ex(\beta\boldsymbol{X})$,

$$\beta(ex(\beta\boldsymbol{X})) \approx \beta(\beta(ex((\boldsymbol{X}))) = \beta(ex(\boldsymbol{X})) \approx \lim(\beta\boldsymbol{X}).$$

Assertion (ii) follows immediately by (i) (consider $\boldsymbol{Y} = \beta\boldsymbol{X}$). □

Theorem 5.21. *For every inverse system* $\boldsymbol{X}$ *in cpl-*T_3*, the following assertions are mutually equivalent:*

(i) $\beta\boldsymbol{X}$ *is stable (in pro-Top);*
(ii) $\beta\boldsymbol{X}$ *is stable in pro-c*T_2*;*
(iii) *there exists a compact* $ex(\beta\boldsymbol{X})$*;*
(iv) *there exists an* $ex(\beta\boldsymbol{X})$*;*
(v) *the limit morphism* $\underline{\boldsymbol{r}} : \widehat{X} \to \beta\boldsymbol{X}$ *is a c*T_2*-expansion of* $\widehat{X} = \lim(\beta\boldsymbol{X})$*;*

(vi) *the limit morphism* $\underline{\boldsymbol{r}} : \widehat{X} \to \beta\boldsymbol{X}$ *is an isomorphism of pro-*cT_2.

PROOF. Let $\beta\boldsymbol{X}$ be a stable inverse system, i.e., let there exists a space Y such that $\beta\boldsymbol{X} \cong \lfloor Y \rfloor$ in *pro-Top*. Then, $\lim(\beta\boldsymbol{X}) \approx \lim \lfloor Y \rfloor = Y$, and thus, Y must be compact and Hausdorff. Hence, (i) implies (ii). Further, (ii) implies (iii) since every isomorphism of (a compact Hausdorff) Y to $\beta\boldsymbol{X}$ is a cT_2-expansion of Y. (iii) trivially implies (iv). Further, (iv) implies (v) by Lemma 5.20(ii). Let the limit morphism

$$\underline{\boldsymbol{r}} : \widehat{X} \to \beta\boldsymbol{X},\ \ \widehat{X} = \lim(\beta\boldsymbol{X}),$$

be a cT_2-expansion of $\widehat{X}$. Since $\widehat{X}$ is compact, the identity mapping

$$1_{\widehat{X}} : \widehat{X} \to \widehat{X}$$

is also a cT_2-expansion of $\widehat{X}$. Therefore, $\beta\boldsymbol{X} \cong \left\lfloor \widehat{X} \right\rfloor = \lfloor \lim(\beta\boldsymbol{X}) \rfloor$ in *pro-*cT_2, and moreover, the uniqueness implies that $\underline{\boldsymbol{r}} : \lim(\beta\boldsymbol{X}) = \widehat{X} \to \beta\boldsymbol{X}$ is an isomorphism. Thus, (v) implies (vi). Finally, (vi) trivially implies (i). □

COROLLARY 5.22. *For every inverse system* $\boldsymbol{X}$ *in cpl-*T_3*, the following assertions are equivalent:*

(i) $\lim \boldsymbol{X}$ *is an* $ex(\beta\boldsymbol{X})$*;*

(ii) $\beta\boldsymbol{X}$ *is stable and* $\beta(\lim \boldsymbol{X}) \approx \lim(\beta\boldsymbol{X})$.

PROOF. Let $\boldsymbol{X}$ be an inverse system in *cpl-*T_3. Suppose that $\beta\boldsymbol{X}$ is stable and that $\beta(\lim \boldsymbol{X}) \approx \lim(\beta\boldsymbol{X})$. By Theorem 5.21, the limit morphism $\underline{\boldsymbol{r}} : \lim(\beta\boldsymbol{X}) \to \beta\boldsymbol{X}$ is a cT_2-expansion. Let

$$h : \beta(\lim \boldsymbol{X}) \to \lim(\beta\boldsymbol{X})$$

be a homeomorphism. By Lemma 4.4, the composition

$$\lim \boldsymbol{X} \stackrel{j_X}{\to} \beta(\lim \boldsymbol{X}) \stackrel{h}{\to} \lim(\beta\boldsymbol{X}) \stackrel{\boldsymbol{r}}{\to} \beta\boldsymbol{X}$$

is a cT_2-expansion. Thus, $\lim \boldsymbol{X}$ is an $ex(\beta\boldsymbol{X})$. The converse holds by Theorem 5.21 and Lemma 5.20(ii). □

Observe that Corollary 5.22 might provide another example of an inverse system $\boldsymbol{X}$ in *pro-*(*cpl-*T_3) such that $\beta(\lim \boldsymbol{X}) \not\approx \lim(\beta\boldsymbol{X})$. Namely, such is, (if it exists) every unstable $\boldsymbol{X}$ having $\beta\boldsymbol{X}$ stable and $\lim \boldsymbol{X}$ that is not any of $ex(\beta\boldsymbol{X})$.

REMARK 5.23. The full analogues of Theorems 5.18, 5.19 and 5.21 hold for the category pair (*cpl-*$T_3, \mathbb{R}$*-cpt*), i.e., for the Hewitt realcompactification of completely regular spaces, as well as for the corresponding inverse systems.

REMARK 5.24. An analysis similar to that given in Application III can be carried out for the category pair $(\mathcal{H}, T_2)$, where $\mathcal{H}$ is the full subcategory of *Top* determined by all topological spaces admitting *Hausdorff reflection* ([7,8], [3, j-04, "Digital Topology", p. 430]). Namely, every Hausdorff reflection

$h_X : X \to X_H$ is a rudimentary T_2-expansion of X. Consequently, there exists a functor $\chi : \mathcal{H} \to T_2$, $X \mapsto \chi(X) = X_H$, which admits an extension to $\chi : pro\text{-}\mathcal{H} \to pro\text{-}T_2$. Therefore, we can apply the previous theory.

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