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Basis of splines associated with singularly perturbed advection–diffusion problems*

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Abstract. Among fitted-operator methods for solving one-dimensional singular perturbation problems one of the most accurate is the collocation by linear combinations of $\{1, x, \exp(\pm px)\}$, known as tension spline collocation. There exist well established results for determining the 'tension parameter' p, as well as special collocation points, that provide higher order local and global convergence rates. However, if the advection–diffusion–reaction problem is specified in such a way that two boundary internal layers exist, the method is incapable of capturing only one boundary layer, which happens when no reaction term is present. For a pure advection-diffusion problem we therefore modify the basis accordingly, including only one exponential, *i.e.* project the solution to the space locally spanned by $\{1, x, x^2, \exp(px)\}$ where $p > 0$ is the tension parameter. The aim of the paper is to show that in this situation it is still possible to construct a basis of C^1-locally supported functions by a simple knot insertion technique, commonly used in computer aided geometric design. We end by showing that special collocation points can be found, which yield better local and global convergence rates, similar to the tension spline case.

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Key words: singular perturbations, advection–diffusion, Chebyshev theory, exponential tension splines, knot insertion

1. Introduction

Let the following singularly perturbed boundary value problem

$$-\varepsilon\, u''(x) + q\, u'(x) = f(x), \quad u(a) = \alpha,\ u(b) = \beta, \tag{1}$$

$\varepsilon, q > 0$, $\varepsilon \ll q$, and $f \in C^3([a, b])$ be the 'model' equation. Note that the advection term is constant. For this kind of equation the tension spline method [10] may not converge uniformly, since the presence of the reaction term bounded from below is required. Instead of using classical exponential tension splines [20, 2], which are a piecewise linear combination of $\{1, x, e^{px}, e^{-px}\}$, we use a space of splines piecewisely spanned by $\{1, x, x^2, e^{px}\}$. Let us put simply $p := q/\varepsilon > 0$. We call such splines exponential advection–diffusion splines or just AD–splines for short.

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It is not trivial to find a numerically stable algorithm which will produce values and derivatives of the local basis, usually called a B-spline basis. We show first that such a basis can be constructed in the setting of the general theory of Chebyshev splines in Section 2, and then apply the collocation method [8, 17, 5] to find an approximate solution to (1). The second issue is the choice of optimal collocation points. In Section 3 we give an explicit formula for such points.

2. Exponential advection–diffusion B–splines

A choice of the function space is dictated by the differential operator in (1); since we consider the operator fitted method, a natural choice is, at least for a sufficiently smooth right–hand side in (1), the space $\mathrm{L}\{1, x, x^2, e^{px}\} = \mathrm{Ker} D^2(-\varepsilon\, D^2 + q\, D)$. By using product form of the given fourth order differential operator, we can define the measure vector $d\sigma = (d\sigma_2, d\sigma_3, d\sigma_4)^{\mathrm{T}} := (d\tau_2, e^{p\tau_3} d\tau_3, e^{-p\tau_4} d\tau_4)^{\mathrm{T}}$ and a Canonical Complete Chebyshev system (CCC–system) [22, 21, 1]:

$$
\begin{aligned}
u_1(x) &= 1,\\
u_2(x) &= \int_a^x d\tau_2 = x - a,\\
u_3(x) &= \int_a^x d\tau_2 \int_a^{\tau_2} e^{p\tau_3} d\tau_3 = \frac{1}{p^2} e^{px} - \frac{e^{pa}}{p} x + \frac{e^{pa}}{p}\left(a - \frac{1}{p}\right),\\
u_4(x) &= \int_a^x d\tau_2 \int_a^{\tau_2} e^{p\tau_3} d\tau_3 \int_a^{\tau_3} e^{-p\tau_4} d\tau_4\\
&= \frac{e^{-pa}}{p^3} e^{px} - \frac{1}{2p} x^2 + \frac{1}{p}\left(a - \frac{1}{p}\right) x + \frac{1}{p}\left(\frac{a}{p} - \frac{a^2}{2} - \frac{1}{p^2}\right).
\end{aligned}
$$

Now $\mathcal{S}(4, d\sigma) := \mathrm{L}\{1, x, x^2, e^{px}\} = \mathrm{L}\{u_1, u_2, u_3, u_4\}$. The first generalized derivative is the same as the ordinary one

$$L_1 u(x) = D u(x)$$

and it maps the functions from the CCC–system to the first reduced space $\mathrm{L}\{1, x, e^{px}\}$, while the second and the third generalized derivatives

$$L_2 u(x) = e^{-px} D^2 u(x), \qquad L_3 u(x) = e^{px} D\, L_2 u(x)$$

map $\mathcal{S}(4, d\sigma)$ to the second and the third reduced space $\mathrm{L}\{1, e^{-px}\}$, $\mathrm{L}\{1\}$, respectively.

Further, we take an arbitrary partition $\Delta = \{x_i\}_{i=0}^{l+1}$ of $[a, b]$ and multiplicity vector $\mathbf{m} = (4, 2, \ldots, 2, 4)$, which together define the extended partition $\mathbf{T}$ whose elements are called knots. $\mathbf{T} := \{t_r\}_{r=1}^{2l+8}$, and all the interior knots are of multiplicity 2:

$$
\begin{aligned}
t_1 \leqslant t_2 \leqslant t_3 \leqslant t_4 &= x_0 = a,\\
t_{2i+3}, t_{2i+4} &= x_i, \qquad i = 1, \ldots, l,\\
b = x_{l+1} = t_{2l+5} &\leqslant t_{2l+6} \leqslant t_{2l+7} \leqslant t_{2l+8}.
\end{aligned}
$$

Chebyshev spline space $\mathcal{S}(4, d\sigma, \mathbf{T})$ is spanned by functions being piecewise in the CCC–space $\mathcal{S}(4, d\sigma)$, where the first generalized derivative is continuous across x_i for $i = 1, \ldots, l$. Thus we have that $\mathcal{S}(4, d\sigma, \mathbf{T}) \subseteq C^1([a, b])$ that will be needed for the collocation method. It is generally agreed that the best choice of the basis are B-splines T_i^4 because of their nice properties like non–negativity, compact supports and the partition of unity [22, 21]. The B-splines associated to the first (second) reduced system are denoted by T_i^3 (T_i^2).

For the collocation matrix we need to calculate the values of derivatives of B-splines at certain points. To reconstruct the solution, we also need the B-splines themselves. One way to do it is the Chebyshev spline knot insertion technique [11, 12, 13, 15, 1, 18, 19], which uses scalar products of positive quantities only. To this end, we use the derivative formula:

$$L_1 T_i^k(x) = \frac{T_i^{k-1}(x)}{C_i^{k-1}} - \frac{T_{i+1}^{k-1}(x)}{C_{i+1}^{k-1}}. \tag{2}$$

Here,

$$\begin{aligned} C_i^1 &= \int_{t_i}^{t_{i+1}} T_i^1(\tau_4) e^{-p\tau_4} d\tau_4, \\ C_i^2 &= \int_{t_i}^{t_{i+2}} T_i^2(\tau_3) e^{p\tau_3} d\tau_3, \\ C_i^3 &= \int_{t_i}^{t_{i+3}} T_i^3(\tau_2) d\tau_2. \end{aligned}$$

If the extended partition contains only interior knots of multiplicity one, the integrals are

$$\begin{aligned} C_i^1 &= \frac{1}{p}\left(e^{-pt_i} - e^{-pt_{i+1}}\right), \\ C_i^2 &= \frac{e^{pt_{i+1}}}{p}\left(\frac{\ell_1(ph_i)}{\ell_0(ph_i)} + \frac{\ell_1(-ph_{i+1})}{-\ell_0(-ph_{i+1})}\right), \\ C_i^3 &= \frac{\frac{\ell_1(ph_i)}{\ell_0(ph_i)}}{\frac{\ell_1(ph_i)}{\ell_0(ph_i)} + \frac{\ell_1(-ph_{i+1})}{-\ell_0(-ph_{i+1})}} \frac{1}{p}\left(\frac{\ell_2(ph_i)}{\ell_1(ph_i)} + \frac{-\ell_2(-ph_{i+1})}{\ell_1(-ph_{i+1})}\right) \\ &+ \frac{h_{i+1}}{2} \frac{\ell_0(-ph_{i+1})\big(2\ell_0(-ph_{i+1}) + ph_{i+1}(e^{-ph_{i+1}} + 1)\big)}{\ell_0(-ph_{i+1})\big(\ell_0(-ph_{i+1}) + ph_{i+1}(e^{-ph_{i+1}} + 1)\big) + p^2 h_{i+1}^2 e^{-ph_{i+1}}} \\ &+ \frac{\frac{\ell_1(-ph_{i+2})}{-\ell_0(-ph_{i+2})}}{\frac{\ell_1(ph_{i+1})}{\ell_0(ph_{i+1})} + \frac{\ell_1(-ph_{i+2})}{-\ell_0(-ph_{i+2})}} \frac{1}{p}\left(\frac{\ell_2(ph_{i+1})}{\ell_1(ph_{i+1})} + \frac{-\ell_2(ph_{i+2})}{\ell_1(-ph_{i+2})}\right), \end{aligned} \tag{3}$$

where

$$\begin{aligned} h_i &:= x_{i+1} - x_i, \\ \ell_i(x) &:= e^x - \sum_{k=0}^{i} \frac{x^k}{k!}, \qquad i = 0, 1, 2, \ldots. \end{aligned}$$

For extended partitions with multiple knots, we just take the limits by coalescing the knots in (3). We used the integral version of the derivative formula (2) for C_i^2, and for C_i^3 the following Oslo type algorithm [2]: let T_i^3 and T_i^2 be the Chebyshev 3rd and 2nd order B-spline associated with the extended partition $\mathbf{T}$ with all interior knots of multiplicity one, and let us assume that $\widetilde{T}_i^3$, $\widetilde{T}_i^2$ are B-splines associated with the extended partition $\widetilde{\mathbf{T}}$ on the same knot sequence, but with all interior knots of multiplicity two. If $\mathbf{T} = \{t_j\}_{j=1}^{n+4}$ and $\widetilde{\mathbf{T}} = \{\tilde{t}_j\}_{j=1}^{2n}$, and r is an index such that $t_i = \tilde{t}_r < \tilde{t}_{r+1}$, then for $i = 2, \ldots, n$:

$$\begin{aligned} T_i^3 &= \frac{\widetilde{C}_r^2}{C_i^2}\,\widetilde{T}_r^3 + \widetilde{T}_{r+1}^3 + \frac{\widetilde{C}_{r+3}^2}{C_{i+1}^2}\,\widetilde{T}_{r+2}^3, \\ C_j^2 &= \int_{t_j}^{t_{j+2}} T_j^2(\tau_3) e^{p\tau_3}\, d\tau_3, \\ \widetilde{C}_j^2 &= \int_{\tilde{t}_j}^{\tilde{t}_{j+2}} \widetilde{T}_j^2(\tau_3) e^{p\tau_3}\, d\tau_3. \end{aligned}$$

From now on, let $\mathbf{T}$ be the primary extended partition with interior knots of multiplicity two. Next, let $x_i = t_r < t_{r+1}$ and $x \in (x_i, x_{i+1})$, then let us introduce some new extended partitions $\bar{\mathbf{T}} = \{\bar{t}_j\}$, $\widetilde{\mathbf{T}} = \{\tilde{t}_j\}$ and $\widehat{\mathbf{T}} = \{\hat{t}_j\}$ where

$$\bar{t}_j = t_j \text{ for } j = 1, \ldots, r, \quad \bar{t}_{r+1} = x, \quad \bar{t}_j = t_{j-1} \text{ for } j = r+2, \ldots, n+5,$$

$$\begin{aligned} \tilde{t}_j &= t_j \text{ for } j = 1, \ldots, r, \quad \tilde{t}_{r+1} = \tilde{t}_{r+2} = x, \\ \tilde{t}_j &= t_{j-2} \text{ for } j = r+3, \ldots, n+6, \end{aligned}$$

$$\begin{aligned} \hat{t}_j &= t_j \text{ for } j = 1, \ldots, r, \quad \hat{t}_{r+1} = \hat{t}_{r+2} = \hat{t}_{r+3} = x, \\ \hat{t}_j &= t_{j-3} \text{ for } j = r+4, \ldots, n+7, \end{aligned}$$

with $n := 2l+4$. The B-splines associated with these extended partitions are denoted by T_j^{4-l}, $\bar{T}_j^{4-l}$, $\widetilde{T}_j^{4-l}$ and $\widehat{T}_j^{4-l}$ for $l = 0, \ldots, 3$, respectively, as well as the integrals of these B-splines:

$$\begin{aligned} C_j^{4-l} &:= \textstyle\int_{t_j}^{t_{j+4-l}} T_j^{k-l}(\tau_{l+1})\, d\sigma_{l+1}(\tau_{l+1}), \\ \bar{C}_j^{4-l} &:= \textstyle\int_{\bar{t}_j}^{\bar{t}_{j+4-l}} \bar{T}_j^{k-l}(\tau_{l+1})\, d\sigma_{l+1}(\tau_{l+1}), \\ \widetilde{C}_j^{4-l} &:= \textstyle\int_{\tilde{t}_j}^{\tilde{t}_{j+4-l}} \widetilde{T}_j^{k-l}(\tau_{l+1})\, d\sigma_{l+1}(\tau_{l+1}), \\ \widehat{C}_j^{4-l} &:= \textstyle\int_{\hat{t}_j}^{\hat{t}_{j+4-l}} \widehat{T}_j^{k-l}(\tau_{l+1})\, d\sigma_{l+1}(\tau_{l+1}), \end{aligned}$$

for $l = 1, \ldots, 3$. Then, the generalized de Boor algorithm [9, 3, 2, 1] for the third order AD–B-spline s is

$$s(x) = \sum_{j=2}^{n} c_j T_j^3(x) = \sum_{j=2}^{n+2} \tilde{c}_j \widetilde{T}_j^3(x) = \tilde{c}_i,$$

with

$$\tilde{c}_i = \frac{\bar{C}^1_{i+1}\,\widetilde{C}^2_{i+1}}{C^1_i\,C^2_{i-1}}\,c_{i-2} + \left(\frac{\widetilde{C}^2_{i+1}\,\bar{C}^2_{i-1}}{\bar{C}^2_i\,C^2_{i-1}} + \frac{\widetilde{C}^2_i\,\bar{C}^2_{i+1}}{\bar{C}^2_i\,C^2_i}\right)c_{i-1} + \frac{\bar{C}^1_i\,\widetilde{C}^2_i}{C^1_i\,C^2_i}\,c_i.$$

For $x = x_i$ we just take $x \to x_i^+$, whenever it appears in the above algorithm.

For the fourth order AD–B-spline s and $x \in (x_i, x_{i+1})$, the de Boor algorithm is:

$$s(x) = \sum_{j=1}^{n} c_j T^4_j(x) = \sum_{j=1}^{n+3} \hat{c}_j \widehat{T}^4_j(x) = \hat{c}_i,$$

with

$$\begin{aligned}\hat{c}_i \,=\, & c_{i-3}\frac{\widehat{C}^3_{i+1}\widetilde{C}^2_{i+1}\bar{C}^1_{i+1}}{C^1_i C^2_{i-1} C^3_{i-2}} + c_{i-2}\left(\frac{\widehat{C}^3_{i+1}\widetilde{C}^2_{i+1}\bar{C}^3_{i-2}}{\bar{C}^2_i \bar{C}^3_{i-1} C^3_{i-2}} + \frac{\widehat{C}^3_{i+1}\widetilde{C}^3_{i-1}\bar{C}^2_{i+1}\bar{C}^3_i}{\widetilde{C}^3_i \bar{C}^3_{i-1} C^2_i C^3_{i-1}}\right.\\ & \left. + \frac{\widehat{C}^3_i\widetilde{C}^3_{i+1}\bar{C}^2_{i+1}}{\widetilde{C}^3_i C^2_i C^3_{i-1}}\right) + c_{i-1}\left(\frac{\widehat{C}^3_{i+1}\widetilde{C}^3_{i-1}\bar{C}^2_{i-1}}{\widetilde{C}^3_i C^2_{i-1} C^3_{i-1}} + \frac{\widehat{C}^3_i\widetilde{C}^3_{i+1}\bar{C}^2_{i-1}\bar{C}^3_{i-1}}{\widetilde{C}^3_i \bar{C}^3_i C^2_{i-1} C^3_{i-1}}\right. \\ & \left. + \frac{\widehat{C}^3_i\widetilde{C}^2_i\bar{C}^3_{i+1}}{\bar{C}^2_i \bar{C}^3_i C^3_i}\right) + c_i\frac{\widehat{C}^3_i\widetilde{C}^2_i\bar{C}^1_i}{C^1_i C^2_i C^3_i}.\end{aligned} \tag{4}$$

If $x = x_i$, we do the same as for order three. As mentioned before, for the derivatives of B-splines we use the derivative formula (2).

It is well known that the limiting case of exponential tension splines are cubic splines for $p \to 0$, and linear splines for $p \to \infty$. (Though easily observed, it cannot be easily proved for variable tension parameters, see [4]). There is a similar limiting behavior for AD-splines; for the $p \to 0$ it is the same as for tension splines, while for $p \to \infty$, AD–spline becomes the parabolic splines. That this is indeed the case can be shown by taking limit $p \to 0$ or $p \to \infty$ in (4), and by noting that, among others:

$$\begin{aligned} \lim_{p\to 0} C^3_i &= \tfrac{t_{i+3}-t_i}{3}, & \lim_{p\to\infty} C^3_i &= \tfrac{t_{i+3}-t_{i+1}}{2},\\ \lim_{p\to 0}\frac{\widetilde{C}^2_i\bar{C}^1_i}{C^1_i C^2_i} &= \tfrac{(x-t_i)^2}{(t_{i+1}-t_i)(t_{i+2}-t_i)}, & \lim_{p\to\infty}\frac{\widetilde{C}^2_i\bar{C}^1_i}{C^1_i C^2_i} &= 0,\\ \lim_{p\to 0}\frac{\widetilde{C}^2_{i+1}}{\bar{C}^2_i} &= \tfrac{t_{i+1}-x}{t_{i+1}-t_i}, & \lim_{p\to\infty}\frac{\widetilde{C}^2_{i+1}}{\bar{C}^2_i} &= 1. \end{aligned}$$

Thus AD–splines are ‘in between’ cubic and parabolic ones, at least for one tension parameter, which is the only case we consider here.

3. Collocation

To solve the boundary value problem (1), we seek the approximating solution $s \in \mathcal{S}(4, d\sigma, \mathbf{T})$, $s(x) = \sum_{j=1}^n c_j T^4_j(x)$, satisfying the collocation equations

$$\begin{aligned} -\varepsilon\, s''(\tau_i) + q\, s'(\tau_i) \,&=\, f(\tau_i), \qquad i = 1, \ldots, 2(l+1)\\ s(a) \,&=\, \alpha,\;\; s(b) = \beta, \end{aligned} \tag{5}$$

for $\tau_{2j+m} \in (x_j, x_{j+1})$, $j = 0, \dots l$, $m = 1, 2$. The question is if the choice of the collocation points $\{\tau_i\}_{i=1}^{2(l+1)}$ can decrease the error of the approximation. We propose one possible way, which follows the well known method of finding the optimal collocation points on each interval, leading to the so called superconvergence phenomenon.

First, we introduce the Green's function [14] for our problem. Without loss of generality, we may assume homogeneous Dirichlet boundary values, $\alpha = \beta = 0$ in (1). Then the Green's function G is:

$$G(x,y) := \begin{cases} \dfrac{(e^{p(x-a)} - 1)(e^{p(b-y)} - 1)}{q\left(e^{p(b-a)} - 1\right)}, & \text{for } x < y, \\ \dfrac{(e^{p(x-b)} - 1)(e^{p(a-y)} - 1)}{q\left(1 - e^{p(a-b)}\right)}, & \text{for } y < x. \end{cases}$$

For fixed x, $G(x,y)$ is piecewisely in $\{1, e^{p\,y}\}$, on each subinterval $[x_j, x_{j+1})$, $j = 0, \dots l$, motivating us to choose collocation points τ_{2j+1}, τ_{2j+2} satisfying

$$(x - \tau_{2j+1})(x - \tau_{2j+2}) \perp \operatorname{span}\{1, e^{p\,x}\}. \tag{6}$$

The sequence $\{\tau_i\}_{i=1}^{2l+2}$ is called generalized Gaussian points [8, 7, 16, 5]. From (6) we get that

$$\tau_{2j+1} = s_j - h\tau_j, \qquad \tau_{2j+2} = s_j + h\tau_j,$$

with

$$\begin{aligned} s_j &:= x_j + \frac{h_j}{3} \frac{f_2(ph_j)}{f_1(ph_j)}, \\ h\tau_j &:= h_j f_3(ph_j), \\ f_1(x) &:= \frac{2(1 - e^{-x}) - x(1 + e^{-x})}{x^3}, \\ f_2(x) &:= \frac{6(e^{-x} - 1 + x) - x^2(2 + e^{-x})}{x^4}, \\ f_3(x) &:= \frac{\sqrt{e^{-x}(1 - \frac{72}{x^4}) + e^{-2x}(\frac{36}{x^4} + \frac{36}{x^3} + \frac{18}{x^2} + \frac{6}{x} + 1) + \frac{36}{x^4} - \frac{36}{x^3} + \frac{18}{x^2} - \frac{6}{x} + 1}}{3\left(\frac{2}{x}(e^{-x} - 1) + e^{-x} + 1\right)}. \end{aligned}$$

Functions f_1, f_2 and f_3 are arranged to get nice behavior in the limits to zero and infinity. On interval $[x_j, x_{j+1}) = [0, 1)$ one easily shows that

$$\begin{aligned} \lim_{p\to 0} \tau_{2j+1} &= \frac{1}{6}(3 - \sqrt{3}), & \lim_{p\to\infty} \tau_{2j+1} &= \frac{1}{3}, \\ \lim_{p\to 0} \tau_{2j+2} &= \frac{1}{6}(3 + \sqrt{3}), & \lim_{p\to\infty} \tau_{2j+2} &= 1 \end{aligned}$$

is in agreement with the known result for collocation by polynomial splines. In our case there is no translation invariance, meaning that Gaussian points must be recomputed on each subinterval.

This choice of collocation points guarantees the following error estimates:

Theorem 1. *Let f from (1) be in $C^3([a,b])$, and let us choose the collocation points $\tau_{2j+m} \in (x_j, x_{j+1})$, $m = 1,2$ for each $j = 0,\dots l$ so that*

$$\int_{x_j}^{x_{j+1}} z(x)(x-\tau_{2j+1})(x-\tau_{2j+2})\,dx = 0, \tag{7}$$

for every $z \in \operatorname{span}\{1, e^{p\,x}\}$. Then there exist constants K_1 and K_2 independent of Δ and ε such that the solution u to (1) and the solution s to the collocation equation (5) satisfy

$$|(u-s)(x)| \leqslant K_1 h^3, \tag{8}$$

for $x \in \{x_0,\dots,x_{l+1}\}$, and globally

$$\|u-s\|_\infty \leqslant \frac{1}{\varepsilon} K_2 h^3,$$

where $h := \max\{h_0,\dots,h_l\}$.

Proof. The proof is a variation of the idea in [6]. Let $x \in [a,b]$, then

$$(u-s)(x) = \sum_{j=0}^{l} E_j(x),$$

where

$$E_j(x) := \int_{x_j}^{x_{j+1}} G(x,y) r(y)\,dy, \qquad j = 0,\dots,l$$

with

$$r(x) := (Lu - Ls)(x), \qquad Lg(x) := -\varepsilon g''(x) + q\,g'(x).$$

If we denote the classical second order divided difference of the function r over the points $\tau_{2j+1}, \tau_{2j+2}, y$ with $\Delta^2(y) := r[\tau_{2j+1}, \tau_{2j+2}, y]$, then

$$r(y) = (y-\tau_{2j+1})(y-\tau_{2j+2}) r[\tau_{2j+1}, \tau_{2j+2}, y],$$

and

$$\Delta^2(y) = \Delta^2(x_j) + (y-x_j)(\Delta^2)'(\theta_1), \qquad \theta_1 \in (x_j, y).$$

Let

$$E_j(x) := \int_{x_j}^{x_{j+1}} G(x,y)(y-\tau_{2j+1})(y-\tau_{2j+2})\big(\Delta^2(x_j) + (y-x_j)(\Delta^2)'(\theta_1)\big)\,dy \tag{9}$$

for $j = 0,\dots,l$. First, let $x \notin (x_j, x_{j+1})$. Because of orthogonality (7)

$$E_j(x) := \int_{x_j}^{x_{j+1}} G(x,y)(y-\tau_{2j+1})(y-\tau_{2j+2})(y-x_j)(\Delta^2)'(\theta_1)\,dy. \tag{10}$$

Since

$$G(x,y) \leqslant \frac{1}{q} \tag{11}$$

and

$$(\Delta^2)'(\theta_1) = r[\tau_{2j+1}, \tau_{2j+2}, \theta_1, \theta_1] = \frac{r'''(\theta_2)}{3!} = \frac{f'''(\theta_2)}{6} \tag{12}$$

equation (10) gives

$$|E_j(x)| \leqslant \frac{M_3}{q} h_j^4,$$

with

$$M_3 := \max_{x\in[a,b]} |f'''(x)| \tag{13}$$

hence (8) follows.

On the other hand, if $x \in (x_j, x_{j+1})$ (9) leads to

$$\begin{aligned} E_j(x) := &\int_{x_j}^{x_{j+1}} \left(G(x,x) + \int_x^y \frac{\partial}{\partial y} G(x,z)\, dz \right) (y-\tau_{2j+1})(y-\tau_{2j+2}) \\ &\times \left(\Delta^2(x_j) + (y-x_j)(\Delta^2)'(\theta_1)\right) dy, \end{aligned}$$

and again, because of orthogonality (7) we have

$$\begin{aligned} E_j(x) := &\int_{x_j}^{x_{j+1}} (y-\tau_{2j+1})(y-\tau_{2j+2}) \Bigg(G(x,x)(y-x_j)(\Delta^2)'(\theta_1) \\ &+ \left(\int_x^y \frac{\partial}{\partial y} G(x,z)\, dz \right) \left(\Delta^2(x_j) + (y-x_j)(\Delta^2)'(\theta_1)\right) \Bigg) dy. \end{aligned}$$

By (11), $\left|\frac{\partial}{\partial y} G(x,y)\right| \leqslant \frac{1}{\varepsilon}$, (12), (13) and

$$\Delta^2(x_j) = r[\tau_{2j+1}, \tau_{2j+2}, x_j] = \frac{r''(\theta_3)}{2!} = \frac{f''(\theta_3)}{2}$$

we finally get

$$|E_j(x)| \leqslant \left(\frac{M_3}{6q} + \frac{M_2}{2\varepsilon} + \frac{M_3 h}{6\varepsilon} \right) h^4 \leqslant \text{const} \frac{1}{\varepsilon} h^4,$$

for $j = 0, \ldots, l$, with $M_2 := \max_{x\in[a,b]} |f''(x)|$. □

4. Example

The exact solution to the problem

$$\varepsilon u''(x) - u'(x) = e^x, \qquad u(0) = u(1) = 0,$$

is

$$u(x) = \frac{e^{\frac{x}{\varepsilon}}(1-e) + e^x \left(e^{\frac{1}{\varepsilon}} - 1\right) + e - e^{\frac{1}{\varepsilon}}}{\left(e^{\frac{1}{\varepsilon}} - 1\right)(\varepsilon - 1)}.$$

We fix ε and compare the approximation by AD–splines and exponential tension splines on the same extended partition. For the tension spline we used the appropriate generalized Gaussian points, as described in [10]. From Figures 1 and 2, with $\varepsilon = 2^{-10}$ and the number of the subintervals in the partition $n = 2^6$, and Tables 1, 2, 3 and 4, where $\varepsilon = 2^{-k}$ for $k = 6, \ldots, 14$, and $n = 2^m$, for $m = 5, \ldots, 11$ it is obvious that the AD–tension spline approximates the solution better.

k \ m	5	6	7	8	9	10	11
6	1.97E-07	1.33E-08	8.44E-10	5.32E-11	1.89E-12	1.63E-11	7.11E-11
7	3.19E-07	2.57E-08	1.73E-09	1.10E-10	6.55E-12	4.70E-12	2.17E-11
8	3.26E-07	4.06E-08	3.30E-09	2.23E-10	1.35E-11	2.60E-13	6.60E-12
9	1.77E-07	4.14E-08	5.14E-09	4.19E-10	2.86E-11	5.33E-13	2.38E-12
10	1.40E-08	2.23E-08	5.21E-09	6.50E-10	5.29E-11	3.87E-12	2.83E-12
11	9.72E-08	1.61E-09	2.81E-09	6.54E-10	8.16E-11	6.71E-12	1.69E-12
12	1.61E-07	1.25E-08	1.92E-10	3.52E-10	8.20E-11	1.02E-11	8.75E-13
13	1.95E-07	2.07E-08	1.59E-09	2.34E-11	4.41E-11	1.05E-11	9.81E-13
14	2.13E-07	2.51E-08	2.62E-09	2.01E-10	2.86E-12	5.65E-12	1.73E-12

Table 1. *The maximal error at the knots for the AD–spline*

k \ m	5	6	7	8	9	10	11
6	3.96E-04	2.69E-05	1.71E-06	1.08E-07	6.75E-09	4.07E-10	1.64E-11
7	2.48E-03	2.08E-04	1.41E-05	9.02E-07	5.67E-08	3.56E-09	1.92E-10
8	8.59E-03	1.27E-03	1.07E-04	7.28E-06	4.65E-07	2.92E-08	1.84E-09
9	1.55E-02	4.40E-03	6.45E-04	5.46E-05	3.70E-06	2.36E-07	1.48E-08
10	2.01E-02	7.93E-03	2.23E-03	3.27E-04	2.75E-05	1.87E-06	1.19E-07
11	2.28E-02	1.03E-02	4.01E-03	1.12E-03	1.64E-04	1.38E-05	9.39E-07
12	2.42E-02	1.17E-02	5.21E-03	2.02E-03	5.61E-04	8.24E-05	6.94E-06
13	2.49E-02	1.24E-02	5.90E-03	2.62E-03	1.01E-03	2.81E-04	4.13E-05
14	2.53E-02	1.28E-02	6.26E-03	2.96E-03	1.31E-03	5.06E-04	1.41E-04

Table 2. *The maximal error at the knots for the tension spline*

k \ m	5	6	7	8	9	10	11
6	2.72E-07	2.08E-08	1.46E-09	9.81E-11	6.46E-12	1.63E-11	7.11E-11
7	4.60E-07	3.40E-08	2.60E-09	1.83E-10	1.23E-11	5.53E-12	2.17E-11
8	8.51E-07	5.83E-08	4.25E-09	3.25E-10	2.30E-11	1.83E-12	6.74E-12
9	1.21E-06	1.08E-07	7.41E-09	5.31E-10	4.06E-11	3.14E-12	2.58E-12
10	1.48E-06	1.52E-07	1.35E-08	9.34E-10	6.64E-11	4.99E-12	3.23E-12
11	1.63E-06	1.87E-07	1.91E-08	1.70E-09	1.17E-10	8.27E-12	1.69E-12
12	1.70E-06	2.05E-07	2.35E-08	2.40E-09	2.12E-10	1.47E-11	1.01E-12
13	1.73E-06	2.14E-07	2.57E-08	2.94E-09	3.00E-10	2.68E-11	1.65E-12
14	1.75E-06	2.18E-07	2.68E-08	3.21E-09	3.68E-10	3.76E-11	3.77E-12

Table 3. *The maximal global error for the AD–spline*

5. Conclusion

The emphasis of the paper is placed more on the construction of the local basis suitable for collocation for advection–diffusion problems, rather than on the collocation method itself. Indeed, for a general problem there is much to be done, for instance

k \ m	5	6	7	8	9	10	11
6	4.17E-04	2.85E-05	1.82E-06	1.14E-07	7.15E-09	4.32E-10	1.74E-11
7	2.54E-03	2.15E-04	1.46E-05	9.28E-07	5.83E-08	3.66E-09	1.99E-10
8	8.84E-03	1.30E-03	1.09E-04	7.38E-06	4.71E-07	2.96E-08	1.86E-09
9	1.62E-02	4.44E-03	6.53E-04	5.49E-05	3.73E-06	2.38E-07	1.49E-08
10	2.11E-02	8.10E-03	2.23E-03	3.28E-04	2.76E-05	1.87E-06	1.20E-07
11	2.39E-02	1.06E-02	4.05E-03	1.12E-03	1.65E-04	1.39E-05	9.40E-07
12	2.54E-02	1.19E-02	5.27E-03	2.03E-03	5.61E-04	8.25E-05	6.95E-06
13	2.62E-02	1.27E-02	5.97E-03	2.64E-03	1.01E-03	2.81E-04	4.13E-05
14	2.66E-02	1.31E-02	6.34E-03	2.98E-03	1.32E-03	5.07E-04	1.41E-04

Table 4. *The maximal global error for the tension spline*

the method has not been proved to be robust in the sense it is nowadays generally agreed upon, *i.e.* the estimate is not parameter uniform (except at the knots). Also, non-constant advection is much more interesting, and one must find suitable piecewise constant approximations to the advection term to apply the method. Numerical tests performed well for various choices, but the proof may be much more involved than the one given in Theorem 1. Finally, the choice of tension parameters in a general situation is an open question. By analogy to the tension spline collocation, it should follow from either asymptotic expansions of the solution or the properties of the collocation matrix. The situation is thus very much alike to the early days of tension splines, when only one tension parameter was used, the so called uniform tension case. However, the construction of the appropriate 'variable tension' AD–

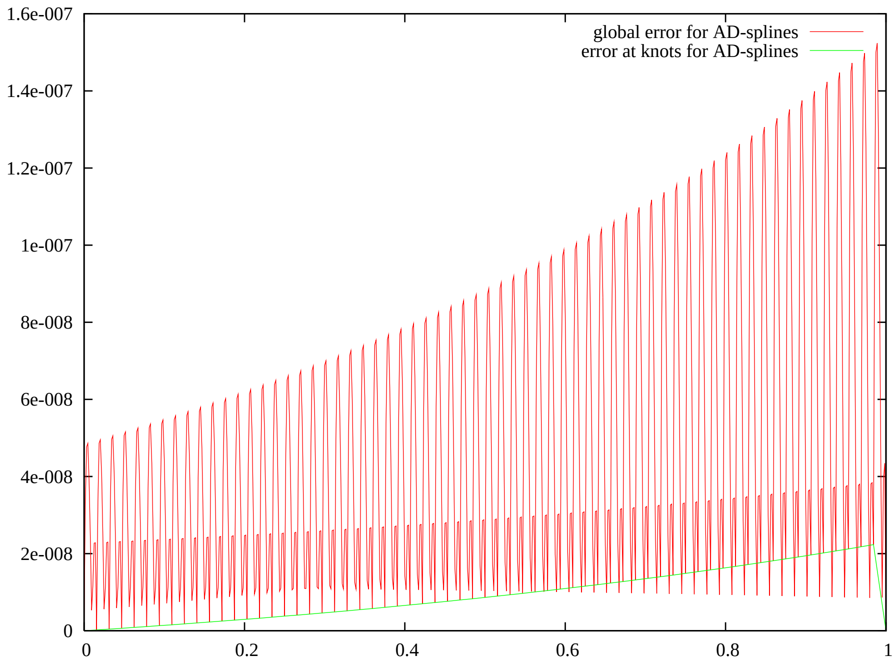

Figure 1. *Global error (red) and the error at the knots (green) for the collocation by AD–spline*

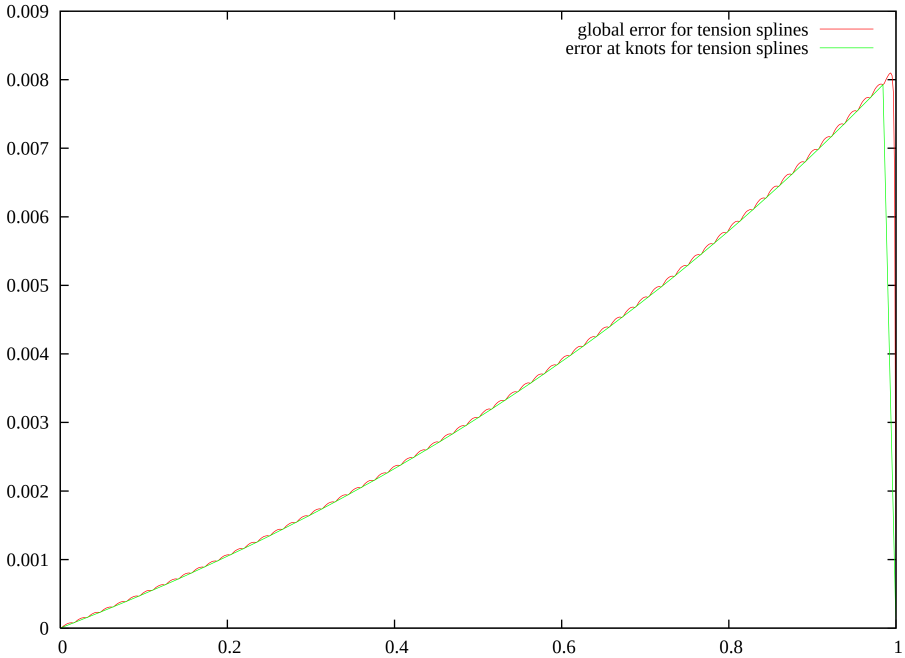

Figure 2. *Global error (red) and the error at the knots (green) for the collocation by tension spline*

spline spaces should present no difficulties, following the same approach as for the uniform ones.

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