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On an application of almost increasing sequences

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Abstract. In the present paper, a general theorem on $\mid \bar{N}, p_n; \delta \mid_k$ summability factors of infinite series has been proved under weaker conditions. Some new results have also been obtained dealing with $\mid \bar{N}, p_n \mid_k$ and $\mid C, 1; \delta \mid_k$ summability factors.

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1. Introduction

Let $\sum a_n$ be a given infinite series with partial sums (s_n). We denote by u_n^α and t_n^α the n-th Cesàro means of order α, with $\alpha > -1$, of the sequence (s_n) and (na_n), respectively, i.e.,

$$u_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=0}^{n} A_{n-v}^{\alpha-1} s_v, \tag{1}$$

$$t_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v, \tag{2}$$

where

$$A_n^\alpha = O(n^\alpha), \ \alpha > -1, \ A_0^\alpha = 1 \text{ and } A_{-n}^\alpha = 0 \text{ for } n > 0. \tag{3}$$

A series $\sum a_n$ is said to be summable $\mid C, \alpha \mid_k$, $k \geq 1$, if (see [8], [11])

$$\sum_{n=1}^{\infty} n^{k-1} \mid u_n^\alpha - u_{n-1}^\alpha \mid^k = \sum_{n=1}^{\infty} \frac{\mid t_n^\alpha \mid^k}{n} < \infty. \tag{4}$$

and it is said to be summable $\mid C, \alpha; \delta \mid_k$, $k \geq 1$ and $\delta \geq 0$, if (see [9])

$$\sum_{n=1}^{\infty} n^{\delta k - 1} \mid t_n^\alpha \mid^k < \infty. \tag{5}$$

Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \text{ as } n \to \infty, \quad (P_{-i} = p_{-i} = 0, i \geq 1). \tag{6}$$

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The sequence-to-sequence transformation

$$\sigma_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \tag{7}$$

defines the sequence (σ_n) of the Riesz mean or simply the $(\bar{N}, p_n)$ mean of the sequence (s_n) generated by the sequence of coefficients (p_n) (see [10]). The series $\sum a_n$ is said to be summable $\mid \bar{N}, p_n \mid_k$, $k \geq 1$, if (see [2], [3])

$$\sum_{n=1}^{\infty} (P_n/p_n)^{k-1} \mid \Delta\sigma_{n-1} \mid^k < \infty, \tag{8}$$

and it is said to be summable $\mid \bar{N}, p_n; \delta \mid_k$, $k \geq 1$ and $\delta \geq 0$, if (see [5])

$$\sum_{n=1}^{\infty} (P_n/p_n)^{\delta k+k-1} \mid \Delta\sigma_{n-1} \mid^k < \infty, \tag{9}$$

where

$$\Delta\sigma_{n-1} = \sigma_n - \sigma_{n-1} = -\frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} a_v, \quad n \geq 1. \tag{10}$$

In the special case $p_n = 1$ for all values of n (resp. $\delta = 0$) $\mid \bar{N}, p_n; \delta \mid_k$ summability is the same as $\mid C, 1; \delta \mid_k$ (resp. $\mid \bar{N}, p_n \mid_k$) summability. Also, if we take $\delta = 0$ and $k = 1$, then we get $\mid \bar{N}, p_n \mid$ summability.

2. Known results

Bor [4] has proved the following theorem for $\mid \bar{N}, p_n \mid_k$ summability factors.

Theorem 1. *Let (X_n) be a positive non-decreasing sequence and let there be sequences (β_n) and (λ_n) such that*

$$\mid \Delta\lambda_n \mid \leq \beta_n, \tag{11}$$

$$\beta_n \to 0 \ \text{as} \ n \to \infty, \tag{12}$$

$$\sum_{n=1}^{\infty} n \mid \Delta\beta_n \mid X_n < \infty, \tag{13}$$

$$\mid \lambda_n \mid X_n = O(1). \tag{14}$$

If

$$\sum_{v=1}^{n} \frac{\mid t_v \mid^k}{v} = O(X_n) \ \text{as} \ n \to \infty, \tag{15}$$

where (t_n) is the n-th (C,1) mean of the sequence (na_n), and (p_n) is a sequence such that

$$P_n = O(np_n), \tag{16}$$

$$P_n \Delta p_n = O(p_n p_{n+1}), \tag{17}$$

then the series $\sum_{n=1}^{\infty} a_n \frac{P_n \lambda_n}{np_n}$ is summable $\mid \bar{N}, p_n \mid_k$, $k \geq 1$.

Recently, Bor [7] has generalized Theorem 1 for the $\mid \bar{N}, p_n; \delta \mid_k$ summability factors.

Theorem 2. *Let (X_n) be a positive non-decreasing sequence and the sequences (β_n) and (λ_n) are such that conditions (11)-(17) of Theorem A are satisfied with condition (15) replaced by:*

$$\sum_{v=1}^{n} \left(\frac{P_v}{p_v}\right)^{\delta k} \frac{\mid t_v \mid^k}{v} = O(X_n) \text{ as } n \to \infty. \tag{18}$$

If

$$\sum_{n=v+1}^{m+1} (\frac{P_n}{p_n})^{\delta k-1} \frac{1}{P_{n-1}} = O((\frac{P_v}{p_v})^{\delta k} \frac{1}{P_v}) \text{ as } m \to \infty, \tag{19}$$

then the series $\sum_{n=1}^{\infty} a_n \frac{P_n \lambda_n}{n p_n}$ is summable $\mid \bar{N}, p_n; \delta \mid_k$, $k \geq 1$ and $0 \leq \delta < 1/k$.

It should be noted that if we take $\delta = 0$ in Theorem 2, then we get Theorem 1. In this case condition (19) reduces to

$$\sum_{n=v+1}^{m+1} \frac{p_n}{P_n P_{n-1}} = \sum_{n=v+1}^{m+1} \left(\frac{1}{P_{n-1}} - \frac{1}{P_n}\right) = O\left(\frac{1}{P_v}\right) \text{ as } m \to \infty,$$

which always holds.

3. The main result

The aim of this paper is to prove Theorem 2 under weaker conditions. For this we need the concept of an almost increasing sequence. A positive sequence (b_n) is said to be almost increasing if there exist a positive increasing sequence (c_n) and two positive constants A and B such that $Ac_n \leq b_n \leq Bc_n$ (see [1]). Obviously every increasing sequence is almost increasing. However, the converse need not be true as can be seen by taking the example, say $b_n = ne^{(-1)^n}$. Now, we shall prove the following theorem.

Theorem 3. *Let (X_n) be an almost increasing sequence. If conditions (11)-(14) and (16)-(19) are satisfied, then the series $\sum_{n=1}^{\infty} a_n \frac{P_n \lambda_n}{n p_n}$ is summable $\mid \bar{N}, p_n; \delta \mid_k$, $k \geq 1$ and $0 \leq \delta < 1/k$.*

We need the following lemmas for the proof of Theorem 3.

Lemma 1 (see [12])**.** *If (X_n) is an almost increasing sequence, then under conditions (12)-(13) we have that*

$$n X_n \beta_n = O(1), \tag{20}$$

$$\sum_{n=1}^{\infty} \beta_n X_n < \infty. \tag{21}$$

Lemma 2 (see [4])**.** *If conditions (16) and (17) are satisfied, then we have*

$$\Delta\left(\frac{P_n}{n^2p_n}\right) = O\left(\frac{1}{n^2}\right). \tag{22}$$

Lemma 3 (see [4])**.** *If conditions (11)-(14) are satisfied, then we have that*

$$\lambda_n = O(1) \tag{23}$$

$$\Delta\lambda_n = O\left(\frac{1}{n}\right). \tag{24}$$

4. Proof of Theorem 3

Let (T_n) be the sequence of an $(\bar{N}, p_n)$ mean of the series $\sum_{n=1}^{\infty}\frac{a_nP_n\lambda_n}{np_n}$. Then, by definition, we have

$$T_n = \frac{1}{P_n}\sum_{v=1}^{n} p_v \sum_{r=1}^{v}\frac{a_rP_r\lambda_r}{rp_r} = \frac{1}{P_n}\sum_{v=1}^{n}(P_n - P_{v-1})\frac{a_vP_v\lambda_v}{vp_v}. \tag{25}$$

Then, for $n \geq 1$

$$\begin{aligned} T_n - T_{n-1} &= \frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n}\frac{P_{v-1}P_va_v\lambda_v}{vp_v} \\ &= \frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n}\frac{P_{v-1}P_va_vv\lambda_v}{v^2p_v}. \end{aligned}$$

Using Abel's transformation, we get

$$\begin{aligned} T_n - T_{n-1} &= \frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n}\Delta\left(\frac{P_{v-1}P_v\lambda_v}{v^2p_v}\right)\sum_{r=1}^{v} ra_r + \frac{\lambda_n}{n^2}\sum_{v=1}^{n} va_v \\ &= \frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}\frac{P_v}{p_v}(v+1)t_vp_v\frac{\lambda_v}{v^2} + \frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}P_vP_v\Delta\lambda_v(v+1)\frac{t_v}{v^2p_v} \\ &\quad - \frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}P_v\lambda_{v+1}(v+1)t_v\Delta(P_v/v^2p_v) + \lambda_nt_n(n+1)/n^2 \\ &= T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}. \end{aligned}$$

To complete the proof of Theorem 3, by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1} \mid T_{n,r} \mid^k < \infty, \ \text{for } r = 1, 2, 3, 4. \tag{26}$$

Now, applying Hölder's inequality, we have that

$$\begin{aligned}
\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1} \mid T_{n,1}\mid^k &= O(1)\sum_{n=2}^{m+1}(\frac{P_n}{p_n})^{\delta k-1}\frac{1}{P_{n-1}^k}\left\{\sum_{v=1}^{n-1}\frac{P_v}{p_v}p_v \mid t_v \mid\mid \lambda_v \mid \frac{1}{v}\right\}^k \\
&= O(1)\sum_{n=2}^{m+1}(\frac{P_n}{p_n})^{\delta k-1}\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}\left(\frac{P_v}{p_v}\right)^k p_v \mid t_v \mid^k\mid \lambda_v \mid^k \frac{1}{v^k} \\
&\quad \times\left\{\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}p_v\right\}^{k-1} \\
&= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^k p_v \mid t_v \mid^k\mid \lambda_v \mid^k \frac{1}{v^k}\sum_{n=v+1}^{m+1}(\frac{P_n}{p_n})^{\delta k-1}\frac{1}{P_{n-1}} \\
&= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^k \mid \lambda_v \mid^{k-1}\mid \lambda_v \mid p_v \mid t_v \mid^k \frac{1}{v^k}\frac{1}{P_v}(\frac{P_v}{p_v})^{\delta k} \\
&= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{k-1} \mid \lambda_v \mid\mid t_v \mid^k \frac{1}{v^k}(\frac{P_v}{p_v})^{\delta k} \\
&= O(1)\sum_{v=1}^{m}(\frac{P_v}{p_v})^{\delta k}v^{k-1}\frac{1}{v^k} \mid \lambda_v \mid\mid t_v \mid^k \\
&= O(1)\sum_{v=1}^{m}\mid \lambda_v \mid (\frac{P_v}{p_v})^{\delta k}\frac{\mid t_v \mid^k}{v} \\
&= O(1)\sum_{v=1}^{m-1}\Delta \mid \lambda_v \mid \sum_{r=1}^{v}(\frac{P_r}{p_r})^{\delta k}\frac{\mid t_r \mid^k}{r} \\
&\quad + O(1)\mid \lambda_m \mid \sum_{v=1}^{m}(\frac{P_v}{p_v})^{\delta k}\frac{\mid t_v \mid^k}{v} \\
&= O(1)\sum_{v=1}^{m-1}\mid \Delta\lambda_v \mid X_v + O(1)\mid \lambda_m \mid X_m \\
&= O(1)\sum_{v=1}^{m-1}\beta_v X_v + O(1)\mid \lambda_m \mid X_m \\
&= O(1),
\end{aligned}$$

as $m\to\infty$, by (11), (14), (16), (18), (19), (22) and (24).
Now using the fact that $(P_v/v) = O(p_v)$ by (16), we have that

$$\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1} \mid T_{n,2}\mid^k = O(1)\sum_{n=2}^{m+1}(\frac{P_n}{p_n})^{\delta k-1}\frac{1}{P_{n-1}^k}\left\{\sum_{v=1}^{n-1}\frac{P_v}{p_v}\mid \Delta\lambda_v \mid p_v \mid t_v \mid\right\}^k$$

$$
\begin{aligned}
&= O(1) \sum_{n=2}^{m+1} (\frac{P_n}{p_n})^{\delta k-1} \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} \left(\frac{P_v}{p_v}\right)^k \mid \Delta\lambda_v \mid^k \mid t_v \mid^k p_v \\
&\quad \times \left\{ \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_v \right\}^{k-1} \\
&= O(1) \sum_{v=1}^{m} \left(\frac{P_v}{p_v}\right)^k \mid \Delta\lambda_v \mid^k \mid t_v \mid^k p_v \sum_{n=v+1}^{m+1} (\frac{P_n}{p_n})^{\delta k-1} \frac{1}{P_{n-1}} \\
&= O(1) \sum_{v=1}^{m} \left(\frac{P_v}{p_v}\right)^{k-1} \mid \Delta\lambda_v \mid^k \mid t_v \mid^k (\frac{P_v}{p_v})^{\delta k} \\
&= O(1) \sum_{v=1}^{m} (\frac{P_v}{p_v})^{\delta k} \left(\frac{P_v}{p_v}\right)^{k-1} \mid \Delta\lambda_v \mid^{k-1} \mid \Delta\lambda_v \mid\mid t_v \mid^k \\
&= O(1) \sum_{v=1}^{m} (\frac{P_v}{p_v})^{\delta k} v^{k-1} \frac{1}{v^{k-1}} \mid \Delta\lambda_v \mid\mid t_v \mid^k \\
&= O(1) \sum_{v=1}^{m} \beta_v (\frac{P_v}{p_v})^{\delta k} \mid t_v \mid^k = O(1) \sum_{v=1}^{m} v\beta_v (\frac{P_v}{p_v})^{\delta k} \frac{\mid t_v \mid^k}{v} \\
&= O(1) \sum_{v=1}^{m-1} \Delta(v\beta_v) \sum_{r=1}^{v} (\frac{P_r}{p_r})^{\delta k} \frac{\mid t_r \mid^k}{r} + O(1) m\beta_m \sum_{v=1}^{m} (\frac{P_v}{p_v})^{\delta k} \frac{\mid t_v \mid^k}{v} \\
&= O(1) \sum_{v=1}^{m-1} v \mid \Delta\beta_v \mid X_v + O(1) \sum_{v=1}^{m-1} \beta_v X_v + O(1) m\beta_m X_m \\
&= O(1) \text{ as } m \to \infty,
\end{aligned}
$$

by (11), (13), (16), (18), (19), (21), (22) and (25).

Now, since $\Delta(\frac{P_v}{p_v v^2}) = O(\frac{1}{v^2})$ by Lemma 2, we have that

$$
\begin{aligned}
\sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k+k-1} \mid T_{n,3} \mid^k &= O(1) \sum_{n=2}^{m+1} (\frac{P_n}{p_n})^{\delta k-1} \frac{1}{P_{n-1}^k} \left\{ \sum_{v=1}^{n-1} P_v \mid \lambda_{v+1} \mid\mid t_v \mid \frac{1}{v} \frac{v+1}{v} Q \right\}^k \\
&= O(1) \sum_{n=2}^{m+1} (\frac{P_n}{p_n})^{\delta k-1} \frac{1}{P_{n-1}^k} \left\{ \sum_{v=1}^{n-1} \frac{P_v}{p_v} p_v \mid \lambda_{v+1} \mid \frac{1}{v} \mid t_v \mid \right\}^k \\
&= O(1) \sum_{n=2}^{m+1} (\frac{P_n}{p_n})^{\delta k-1} \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} \left(\frac{P_v}{p_v}\right)^k p_v \frac{1}{v^k} \mid \lambda_{v+1} \mid^k \mid t_v \mid^k \\
&\quad \times \left\{ \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_v \right\}^{k-1} \\
&= O(1) \sum_{v=1}^{m} \left(\frac{P_v}{p_v}\right)^k p_v \frac{1}{v^k} \mid \lambda_{v+1} \mid^{k-1} \mid \lambda_{v+1} \mid\mid t_v \mid^k
\end{aligned}
$$

$$
\begin{aligned}
&\times \sum_{n=v+1}^{m+1} (\frac{P_n}{p_n})^{\delta k-1} \frac{1}{P_{n-1}} \\
&= O(1) \sum_{v=1}^{m} \left(\frac{P_v}{p_v}\right)^{k-1} \frac{1}{v^k} \mid \lambda_{v+1} \mid\mid t_v \mid^k (\frac{P_v}{p_v})^{\delta k} \\
&= O(1) \sum_{v=1}^{m} (\frac{P_v}{p_v})^{\delta k} v^{k-1} \frac{1}{v^k} \mid \lambda_{v+1} \mid\mid t_v \mid^k \\
&= O(1) \sum_{v=1}^{m} (\frac{P_v}{p_v})^{\delta k} \mid \lambda_{v+1} \mid \frac{\mid t_v \mid^k}{v} \\
&= O(1) \sum_{v=1}^{m-1} \Delta \mid \lambda_{v+1} \mid \sum_{r=1}^{v} (\frac{P_r}{p_r})^{\delta k} \frac{\mid t_r \mid^k}{r} + O(1) \mid \lambda_{m+1} \mid \sum_{v=1}^{m} (\frac{P_v}{p_v})^{\delta k} \frac{\mid t_v \mid^k}{v} \\
&= O(1) \sum_{v=1}^{m-1} \mid \Delta\lambda_{v+1} \mid X_v + O(1) \mid \lambda_{m+1} \mid X_m \\
&= O(1) \sum_{v=1}^{m-1} \mid \Delta\lambda_{v+1} \mid X_{v+1} + O(1) \mid \lambda_{m+1} \mid X_{m+1} \\
&= O(1) \sum_{v=2}^{m} \mid \Delta\lambda_{v} \mid X_{v} + O(1) \mid \lambda_{m+1} \mid X_{m+1} \\
&= O(1) \sum_{v=1}^{m} \beta_v X_v + O(1) \mid \lambda_{m+1} \mid X_{m+1} = O(1)
\end{aligned}
$$

as $m \to \infty$, by (11), (14), (16), (18), (19), (22) and (24). Finally, as in $T_{n,3}$, we have that

$$
\begin{aligned}
\sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{\delta k+k-1} \mid T_{n,4} \mid^k &= O(1) \sum_{n=1}^{m} (\frac{P_n}{p_n})^{\delta k} \left(\frac{P_n}{p_n}\right)^{k-1} \left(\frac{n+1}{n}\right)^k \frac{1}{n^k} \mid \lambda_n \mid^k \mid t_n \mid^k \\
&= O(1) \sum_{n=1}^{m} (\frac{P_n}{p_n})^{\delta k} n^{k-1} \frac{1}{n^k} \mid \lambda_n \mid^{k-1} \mid \lambda_n \mid\mid t_n \mid^k \\
&= O(1) \sum_{n=1}^{m} \mid \lambda_n \mid (\frac{P_n}{p_n})^{\delta k} \frac{\mid t_n \mid^k}{n} = O(1) \text{ as } m \to \infty.
\end{aligned}
$$

Therefore, we get that

$$
\sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{\delta k+k-1} \mid T_{n,r} \mid^k = O(1) \text{ as } m \to \infty, \text{ for } r = 1,2,3,4.
$$

This completes the proof of the Theorem.

If we take $\delta = 0$, then we get a result of Bor [6] for $\mid \bar{N}, p_n \mid_k$ summability factors. Also, if we take $p_n = 1$ for all values of n, then we get a new result dealing with $\mid C, 1; \delta \mid_k$ summability factors.

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