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A trick for investigation of approximate derivations

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Abstract. We utilize the notion of module extension to reduce the problem of stability of derivations to that of ring homomorphisms studied by R. Badora in the context of Banach bimodules over Banach algebras.

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1. Introduction and preliminaries

A classical question in the theory of functional equations is: "When is it true that a mapping which approximately satisfies a functional equation $\mathcal{E}$ must be somehow close to an exact solution of $\mathcal{E}$?". Such a problem was formulated by S. M. Ulam [20] in 1940 and solved in the next year for the Cauchy functional equation by D. H. Hyers [9]. It gave rise to the stability theory for functional equations.

Subsequently, various approaches to the problem have been introduced by several authors. There are cases in which each 'approximate mapping' is actually a 'true mapping'. In such cases, we call the equation $\mathcal{E}$ superstable. For the history and various aspects of this theory we refer the reader to monographs [6, 11, 14].

D.G. Bourgin is the first mathematician dealing with the stability of ring homomorphisms. The topic of approximate ring homomorphisms was studied by a number of mathematicians, see [5, 2, 4, 10, 12, 17, 18, 19] and references therein.

It seems that approximate derivations were first investigated by K.-W. Jun and D.-W. Park [13]. Recently, the stability of derivations has been investigated by some authors, see [1, 3, 13, 15, 16] and references therein.

R. Badora [2, Theorem 1] proved the following result concerning approximate ring homomorphisms.

Theorem 1 (R. Badora)**.** *Let $\mathcal{A}$ be a ring, $\mathcal{B}$ a Banach algebra and δ and ε nonnegative real numbers. Let f be a mapping from $\mathcal{A}$ to $\mathcal{B}$ such that*

$$\|f(a+b)-f(a)-f(b)\| \leq \varepsilon$$

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and

$$\|f(ab) - f(a)f(b)\| \leq \delta$$

for all $a, b \in \mathcal{A}$. *Then there exists a unique ring homomorphism* $\varphi : \mathcal{A} \to \mathcal{B}$ *such that*

$$\|f(a) - \varphi(a)\| \leq \varepsilon, \quad a \in \mathcal{A}.$$

Furthermore,

$$b(f(a) - \varphi(a)) = (f(a) - \varphi(a))b = 0,$$

for all $a \in \mathcal{A}$ *and all* b *in the algebra generated by* $\varphi(\mathcal{A})$.

We should notify that the function φ is defined by $\varphi(x) = \lim_{n\to\infty} \frac{f(nx)}{n}$. In addition, the condition $\|f(ab) - f(a)f(b)\| \leq \delta$ implies that $\lim_{n\to\infty} \frac{f(nxy) - f(nx)f(y)}{n} = 0$, which is applied in establishing the fact that φ is multiplicative. We use Theorem 1 to prove the stability of derivations from a Banach algebra into a Banach bimodule. To do this, we use module extensions as a trick.

Let $\mathcal{X}$ be a Banach bimodule over a Banach algebra $\mathcal{A}$. Recall that $\mathcal{X} \oplus_1 \mathcal{A}$ is a Banach algebra equipped with the following ℓ_1-norm

$$\|(x, a)\| = \|x\| + \|a\|, \quad a \in \mathcal{A}, x \in \mathcal{X},$$

and the product

$$(x_1, a_1)(x_2, a_2) = (x_1 \cdot a_2 + a_1 \cdot x_2, a_1 a_2), \quad a_1, a_2 \in \mathcal{A}, x_1, x_2 \in \mathcal{X}.$$

The algebra $\mathcal{X} \oplus_1 \mathcal{A}$ is called a module extension Banach algebra. We also define (norm decreasing) projection maps $\pi_1 : \mathcal{X} \oplus_1 \mathcal{A} \to \mathcal{X}$ and $\pi_2 : \mathcal{X} \oplus_1 \mathcal{A} \to \mathcal{A}$ by $(x, b) \mapsto x$ and $(x, b) \mapsto b$, respectively. We refer the reader to [7] for more information on Banach modules and to [8] for details on module extensions. Throughout the paper by a ring homomorphism we mean a mapping on an algebra which preserves multiplication and addition.

2. Main result

We start our work with our main result, which can be regarded as an extension of [3, Theorem 2].

Theorem 2. *Let* $\varepsilon, \delta > 0$. *Let* $\mathcal{A}$ *be a Banach algebra and* $\mathcal{X}$ *a Banach A-bimodule. Suppose that a function* $f : \mathcal{A} \to \mathcal{X}$ *satisfies*

$$\|f(a + b) - f(a) - f(b)\| \leq \varepsilon \tag{1}$$

and

$$|f(ab) - f(a)b - af(b)| \leq \delta$$

for all $a, b \in \mathcal{A}$. *Then there exists a unique derivation* $D : A \to \mathcal{X}$ *such that*

$$\|f(a) - D(a)\| \leq \varepsilon, \quad a \in \mathcal{A}.$$

Moreover, we have

$$b(f(a) - D(a)) = (f(a) - D(a))b = 0, \quad a, b \in \mathcal{A}. \tag{2}$$

Proof. Let us define the mapping $\varphi_f : \mathcal{A} \to \mathcal{X} \oplus_1 \mathcal{A}$ by $a \mapsto (f(a), a)$. We have

$$\begin{aligned}\|\varphi_f(a+b) - \varphi_f(a) - \varphi_f(b)\| &= \|(f(a+b), a+b) - (f(a), a) - (f(b), b)\| \\ &= \|(f(a+b) - f(a) - f(b), 0)\| \\ &= \|f(a+b) - f(a) - f(b)\| \\ &\leq \varepsilon\end{aligned}$$

and similarly

$$\|\varphi_f(ab) - \varphi_f(a)\varphi_f(b)\| \leq \delta$$

for all $a, b \in \mathcal{A}$. It follows from Theorem 1 that there exists a unique ring homomorphism $\varphi : \mathcal{A} \to \mathcal{X} \oplus_1 \mathcal{A}$ such that

$$\|\varphi(a) - \varphi_f(a)\| \leq \varepsilon, \quad a \in \mathcal{A}. \tag{3}$$

Moreover,

$$(x, b)(\varphi(a) - \varphi_f(a)) = (\varphi(a) - \varphi_f(a))(x, b) = 0 \tag{4}$$

for all $a \in \mathcal{A}$, and all (x, b) in the algebra generated by $\varphi(\mathcal{A})$.
It follows from (3) that

$$\|(\pi_2 \circ \varphi_f)(na) - (\pi_2 \circ \varphi)(na)\| \leq \|\varphi_f(na) - \varphi(na)\| \leq \varepsilon, \quad n \in \mathbb{N}, a \in \mathcal{A}. \tag{5}$$

By additivity of mappings under consideration

$$(\pi_2 \circ \varphi)(na) = n(\pi_2 \circ \varphi)(a)$$

and

$$(\pi_2 \circ \varphi_f)(na) = \pi_2(f(na), na) = na,$$

whence, by (5),

$$\|a - (\pi_2 \circ \varphi)(a)\| \leq \frac{1}{n}\varepsilon \tag{6}$$

for all $n \in \mathbb{N}, a \in \mathcal{A}$. By letting n tend to ∞ in (6), we obtain

$$(\pi_2 \circ \varphi)(a) = a, \quad a \in \mathcal{A}.$$

Hence

$$\begin{aligned}\big((\pi_1 \circ \varphi)(ab), ab\big) &= \big(\pi_1(\varphi(ab)), \pi_2(\varphi(ab))\big) = \varphi(ab) = \varphi(a)\varphi(b) \\ &= \big(\pi_1(\varphi(a)), \pi_2(\varphi(a))\big)\big(\pi_1(\varphi(b)), \pi_2(\varphi(b))\big) \\ &= \big(\pi_1(\varphi(a)), a\big)\big(\pi_1(\varphi(b)), b\big) \\ &= \big(a\pi_1(\varphi(b)) + \pi_1(\varphi(a))b, ab\big)\end{aligned} \tag{7}$$

for all $a, b \in \mathcal{A}$. Put $D := \pi_1 \circ \varphi$. Then it follows from (7) that D is a derivation from $\mathcal{A}$ into $\mathcal{X}$. It follows from (3) that

$$\|D(a) - f(a)\| = \|\pi_1(\varphi(a)) - \pi_1(\varphi_f(a))\| \leq \|\varphi(a) - \varphi_f(a)\| \leq \varepsilon$$

for all $a \in \mathcal{A}$.

To prove the uniqueness of D, assume that D^* is another derivation from $\mathcal{A}$ into $\mathcal{X}$ satisfying

$$\|D^*(a) - f(a)\| \leq \varepsilon, \quad a \in \mathcal{A}.$$

Then

$$\|D(a) - D^*(a)\| = \frac{1}{n}\|D(na) - D^*(na)\| \leq \frac{1}{n}\|D^*(a) - f(a)\| + \frac{1}{n}\|D(a) - f(a)\| \leq \frac{2}{n}\varepsilon$$

for all $a \in \mathcal{A}, n \in \mathbb{N}$. By letting $n \to \infty$ in the last inequality above, we conclude that $D(a) = D^*(a)$ for all $a \in \mathcal{A}$.
Moreover,

$$\begin{aligned}
(f(a) - D(a))b &= \pi_1\big((f(a) - D(a))b, 0\big) \\
&= \pi_1\Big(\big(f(a) - D(a), 0\big)\big(D(b), b\big)\Big) \\
&= \pi_1\Big(\big(\pi_1(\varphi(a) - \varphi_f(a)), 0\big)\big(\pi_1(\varphi(b)), b\big)\Big) \\
&= \pi_1\Big(\big(\pi_1(\varphi(a) - \varphi_f(a)), 0\big)\varphi(b)\Big) \\
&= \pi_1\Big(\big(\big(\pi_1(\varphi(a)), a\big) - \big(\pi_1(\varphi_f(a)), a\big)\big)\varphi(b)\Big) \\
&= \pi_1\Big(\big(\varphi(a) - \varphi_f(a)\big)\varphi(b)\Big) \\
&= \pi_1(0, 0) \qquad\qquad \text{by (4)} \\
&= 0
\end{aligned}$$

for all $a, b \in \mathcal{A}$. Similarly, we have $b(f(a) - D(a)) = 0$ for all $a, b \in \mathcal{A}$. □

Remark 1. *To achieve an algebra homomorphism, i.e. a homogeneous ring homomorphism, one can replace inequality* (1) *by*

$$\|f(\lambda a + b) - \lambda f(a) - f(b)\| \leq \varepsilon$$

where $\lambda \in \{z \in \mathbb{C} : |z| = 1\}$ *and* $a, b \in \mathcal{A}$. *Then a standard argument shows that* D *turns into a linear derivation (see [9, 15]).*

As a first consequence we obtain the main result of [3].

Corollary 1 (see [3], Theorem 2)**.** *Let* $\mathcal{A}_1$ *be a closed subalgebra of a Banach algebra* $\mathcal{A}$. *Assume that* $f : \mathcal{A}_1 \to \mathcal{A}$ *satisfies*

$$\|f(x + y) - f(x) - f(y)\| \leq \delta, \quad x, y \in \mathcal{A}_1$$

and

$$\|f(xy) - xf(x) - f(x)y)\| \leq \varepsilon, \quad x, y \in \mathcal{A}$$

for some constants $\delta, \varepsilon \geq 0$. *Then there exists a unique derivation* $d : \mathcal{A}_1 \to \mathcal{A}$ *such that*

$$\|f(x) - d(x)\| \leq \delta, \quad x \in \mathcal{A}_1.$$

Furthermore,

$$x(f(y) - h(y)) = (f(y) - h(y))x = 0$$

for all $x, y \in \mathcal{A}_1$.

Proof. Consider $\mathcal{A}$ as an $\mathcal{A}_1$-bimodule and use Theorem 2. □

We now present two superstability results concerning derivations as follows.

Corollary 2. *Let* $\varepsilon, \delta > 0$. *Let* $\mathcal{A}$ *be a Banach algebra and* $\mathcal{X}$ *a Banach* $\mathcal{A}$*-bimodule without order, i.e.* $\mathcal{A}x = 0$ *or* $x\mathcal{A} = 0$ *implies that* $x = 0$, *where* $x \in \mathcal{X}$. *Suppose that a function* $f : \mathcal{A} \to \mathcal{X}$ *satisfies*

$$\|f(a + b) - f(a) - f(b)\| \leq \varepsilon$$

and

$$|f(ab) - f(a)b - af(b)| \leq \delta$$

for all $a, b \in \mathcal{A}$. *Then* f *is a derivation.*

Proof. Due to $\mathcal{X}$ is a Banach $\mathcal{A}$-bimodule without order, (2) in Theorem 2 implies that $f = d$ is a derivation. □

Corollary 3. *Let* $\varepsilon, \delta > 0$. *Let* $\mathcal{A}$ *be a Banach algebra with a bounded approximate identity. Suppose that a function* $f : \mathcal{A} \to \mathcal{A}$ *satisfies*

$$\|f(a + b) - f(a) - f(b)\| \leq \varepsilon$$

and

$$|f(ab) - f(a)b - af(b)| \leq \delta$$

for all $a, b \in \mathcal{A}$. *Then* f *is a derivation.*

Proof. This follows from Corollary 2 since every Banach algebra with an approximate unit, as a Banach bimodule over itself, is without order. □

Similarly, we can use [2, Theorem 2] to prove the Hyers–Ulam stability of derivations as follows.

Theorem 3. *Let* $\varepsilon, \delta > 0$, *and let* p, q *be real numbers such that* $p, q < 1$, *or* $p, q > 1$. *Let* $\mathcal{A}$ *be a Banach algebra and* $\mathcal{X}$ *a Banach* A*-bimodule. Suppose that a function* $f : \mathcal{A} \to \mathcal{X}$ *satisfies*

$$\|f(a + b) - f(a) - f(b)\| \leq \varepsilon(\|a\|^p + \|b\|^p)$$

and

$$|f(ab) - f(a)b - af(b)| \leq \delta(\|a\|^q \|b\|^q)$$

for all $a, b \in \mathcal{A}$. *Then there exists a unique derivation* $D : A \to \mathcal{X}$ *and a constant* k *such that*

$$\|f(a) - D(a)\| \leq k\varepsilon\|x\|^p$$

for all $a \in \mathcal{A}$.

The following counterexample, which is a modification of Luminet's example (see [12]), shows that this result fails for $p = 1$.

Example 1. *Let*

$$X = \begin{bmatrix} 0 & 0 & 0 \\ \mathbb{R} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and define a function $\varphi : \mathbb{R} \to \mathbb{R}$ *by*

$$\varphi(x) = \begin{cases} 0, & |x| \le 1 \\ x\ln(|x|), & |x| > 1 \end{cases}$$

Let $f : \mathbb{R} \to X$ *be defined by*

$$f\left(\begin{bmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 & 0 \\ \varphi(x) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

for all $x \in \mathbb{R}$. *Then*

$$\|f(a+b) - f(a) - f(b)\| \le \varepsilon(\|a\| + \|b\|)$$

and

$$|f(ab) - f(a)b - af(b)| \le \delta(\|a\|^2\|b\|^2)$$

for some $\delta > 0, \varepsilon > 0$ *and all* $a, b \in \mathcal{A}$*; see [2]. Therefore* f *satisfies the conditions of Theorem 3 with* $p = 1$, $q = 2$. *There is however no derivation* $D : \mathcal{A} \to \mathcal{A}$ *and no constant* $k > 0$ *such that*

$$\|f(a) - D(a)\| \le k\varepsilon\|a\|, \quad a \in \mathcal{A}.$$

In contrary, assume that there exist a derivation $D : \mathcal{A} \to \mathcal{A}$ *and a constant* $k > 0$ *such that*

$$\|f(a) - D(a)\| \le k\varepsilon\|a\|, \quad a \in \mathcal{A}.$$

Representing D *as a* 3×3 *matrix* $[D_{ij}]$*, we infer that*

$$D_{21} : \mathbb{R} \to \mathbb{R}$$

is an additive mapping such that

$$|\varphi(x) - D_{21}(x)| \le k\varepsilon|x|$$

for all $x \in \mathbb{R}$. *From the continuity of* φ*, it follows that* D_{21} *is bounded on some neighborhood of zero. Then there exists a fixed* c *such that* $D_{21}(x) = cx$*, for all* $x \in \mathbb{R}$. *Hence*

$$|x\ln(x) - cx| \le k\varepsilon x$$

for $x > 1$*, whence*

$$|\ln(x) - c| \le k\varepsilon$$

for $x > 1$*, which yields a contradiction.*

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