Math. Commun., Vol. **15**, No. 1, pp. 107-116 (2010)

Remarks on a common Fixed Point Theorem in compact metric spaces

Mohammad Imdad[1,*] and Javid Ali[1]

[1] *Department of Mathematics, Aligarh Muslim University, Aligarh 202 002, India*

Received July 6, 2007; accepted October 30, 2009

Abstract. A general common fixed point theorem for two pairs of weakly compatible self mappings proved in compact metric spaces employing a slightly modified implicit relation which generalizes almost all existing relevant common fixed point theorems. Some related results are also derived besides furnishing illustrative examples.

AMS subject classifications: 47H10, 54H25

Key words: fixed point, coincidence point, weak compatible mappings and implicit relation

1. Introduction

While considering Lipschitzian mappings, a natural question arises whether it is possible to weaken contraction assumption a little bit in Banach contraction principle and still obtain the existence of a fixed point. In general, the answer to this question is no. In this regard, the following interesting example is available in Khamsi and Kirk [13].

Example 1 (see [13]). *Let $C[0,1]$ denote the complete metric space of real valued continuous functions defined on $[0,1]$ with respect to supremum metric and consider the closed subspace Z of $C[0,1]$ consisting of those functions $f \in C[0,1]$ satisfying $f(1) = 1$. Since Z is a closed subspace of $C[0,1]$, hence Z is also complete. Now, define $T : Z \to Z$ by $Tf(t) = tf(t)\ \forall\ t \in [0,1]$. Then one can easily verify that $d(Tf, Tg) < d(f, g)$ whenever $f \neq g$ but T has no fixed point as $Tf = f \Rightarrow tf = f \Rightarrow f(t) = 0\ \forall\ t \in [0,1)$. On the other hand, $f(1) = 1$ which contradicts the continuity of T and so T cannot have a fixed point in Z. Here one may note that T is a contractive mapping on Z. Let us recall that a mapping T on a metric space (X, d) is said to be contractive if $d(Tx, Ty) < d(x, y)$ for all distinct $x, y \in X$.*

The next natural question arises whether there is a meaningful fixed point theorem for contractive mappings. This time the answer is affirmative but the class of spaces to which it applies is much more restrictive. In this direction, Edelstein [2] proved the first ever fixed point theorem for contractive mappings defined on a compact metric space which runs as follows:

*Corresponding author. *Email addresses:* `mhimdad@yahoo.co.in` (M. Imdad), `javid@math.com` (J. Ali)

Theorem 1 (see [2]). *Let T be a contractive self mapping of a compact metric space (X, d), then T has a unique fixed point x_0. Moreover, for each $x \in X$, $\lim_{n\to\infty} T^n(x) = x_0$.*

In recent years, Theorem 1 has been generalized and improved in several ways. To mention a few, we cite [3, 4, 7, 14, 16, 18-20].

As patterned in Jungck [8], a common fixed point theorem in metric spaces generally involves conditions on contraction, commutativity and continuity of the involved mappings besides suitable containment of the range of one mapping into the range of other. Therefore, in order to prove a new metrical common fixed point theorem one is always required to improve one or more of these conditions.

Sessa [24, 1982] initiated the tradition of improving commutativity condition in common fixed point theorems. Inspired by the definition of weak commutativity of Sessa [24], researchers of this domain introduced several definitions of weak commutativity such as: Compatible mappings, Compatible mappings of type (A), Compatible mappings of type (B), Compatible mappings of type (P), Compatible mappings of type (C), Biased maps, R-weakly commuting mappings and some others whose lucid comparison and illustration can be found in Murthy [17]. In our subsequent work, we use the most natural and minimal of these weak conditions known as 'weak compatibility' (due to Jungck [11]) which runs as follows:

Definition 1 (see [11]). *Let T and S be self mappings of a metric space (X, d). Then the pair (T, S) is said to be weakly compatible if for any $x \in X$ with $Tx = Sx$ implies $TSx = STx$.*

2. Implicit relation

In an attempt to improve and unify a multitude of common fixed point theorems in compact setting (see [5, 10, 25, 26]), Popa [22] defined an implicit relation with several demonstrating examples. Here we emphasize that several other well known contractive conditions (cf. [3, 4, 12, 14, 23]) also fall in the format of this implicit relation and it is also good enough to deduce some natural unknown contractive conditions as well. In order to substantiate earlier claims, we add some more examples to this effect. Before adding our examples, let us recall the implicit relation and with examples (due to Popa [22]) which can be described as follows:

Let $\mathcal{F}$ be the family of real-valued functions $F(t_1, \ldots, t_6) : \Re_+^6 \to \Re$ satisfying the following conditions:

F_1: F is non increasing in variables t_5 and t_6,

F_2: For every $u \geq 0$, $v > 0$

F_{2a}: $F(u, v, v, u, u + v, 0) < 0$, or

F_{2b}: $F(u, v, u, v, 0, u + v) < 0$

we have $u < v$.

F_3: $F(u, u, 0, 0, u, u) \geq 0, \forall\ u > 0$.

A careful examination of Theorem 5 due to Popa [22] reveals the fact that he used the condition '$u < v$ when $v = 0$', which is not permissible in view of F_2. Therefore, one needs to add the following:

F_2': $F(u, 0, 0, u, u, 0) \geq 0$, $\forall\, u > 0$.

Thus implicit relations in this paper are required to obey F_1, F_2, F_2' and F_3 which will be referred to as a modified implicit relation in the sequel.

Example 2. *Define* $F(t_1, t_2, \dots, t_6) : \Re_+^6 \to \Re$ *as*

$$F(t_1, \dots, t_6) = (1 + pt_2)t_1 - p\max\{t_3t_4, t_5t_6\} - \max\left\{t_2, t_3, t_4, \frac{t_5 + t_6}{2}\right\}$$

with $p > 0$.

Example 3. *Define* $F(t_1, t_2, \dots, t_6) : \Re_+^6 \to \Re$ *as*

$$F(t_1, \dots, t_6) = t_1 - \max\left\{t_2, t_3, t_4, \frac{t_5 + t_6}{2}\right\}.$$

Example 4. *Define* $F(t_1, t_2, \dots, t_6) : \Re_+^6 \to \Re$ *as*

$$F(t_1, \dots, t_6) = t_1^2 - a\max\{t_2^2, t_3^2, t_4^2\} - b\max\{t_3t_5, t_4t_6\} - ct_5t_6,$$

where $a + 2b \leq 1$, $a + c \leq 1$ *and* $a, b, c \geq 0$.

Example 5. *Define* $F(t_1, t_2, \dots, t_6) : \Re_+^6 \to \Re$ *as*

$$F(t_1, \dots, t_6) = t_1 - \max\left\{t_2, t_3, t_4, \frac{t_5 + t_6}{2}, b\sqrt{t_5t_6}\right\} \text{ with } b \in (0, 1).$$

Example 6. *Define* $F(t_1, t_2, \dots, t_6) : \Re_+^6 \to \Re$ *as*

$$F(t_1, \dots, t_6) = t_1^3 - at_1^2t_2 - bt_1t_3t_4 - ct_5^2t_6 - dt_5t_6^2,$$

where $a, b, c, d \geq 0$ *and* $a + b + c + d < 1$.

Example 7. *Define* $F(t_1, t_2, \dots, t_6) : \Re_+^6 \to \Re$ *as*

$$F(t_1, \dots, t_6) = t_1^3 - a\frac{t_3^2t_4^2 + t_5^2t_6^2}{1 + t_2 + t_3 + t_4}, \text{ where } a \in (0, 1).$$

Here one may recall that all above examples enjoy the properties F_1, F_2 and F_3 whose verifications are available in Popa [22]. Notice that all earlier examples also enjoy F_2'.

In order to strengthen the view point of Popa [22], we furnish some more examples as follows:

Example 8. *Define* $F(t_1, t_2, \dots, t_6) : \Re_+^6 \to \Re$ *as*

$$F(t_1, \dots, t_6) = t_1 - \phi\left(\max\left\{t_2, t_3, t_4, \frac{t_5 + t_6}{2}\right\}\right),$$

where $\phi : \Re^+ \to \Re^+$ *is an increasing function such that* $\phi(t) \leq t$ *for* $t \geq 0$.

Example 9. *Define* $F(t_1,t_2,\dots,t_6):\Re_+^6\to\Re$ *as*

$$F(t_1,\dots,t_6)=t_1-\max\left\{t_2,\frac{t_3+t_4}{2},\frac{t_5+t_6}{2}\right\}.$$

Example 10. *Define* $F(t_1,t_2,\dots,t_6):\Re_+^6\to\Re$ *as*

$$F(t_1,\dots,t_6)=t_1-\max\left\{t_2,t_3,t_4,\frac{t_5}{2},\frac{t_6}{2}\right\}.$$

Example 11. *Define* $F(t_1,t_2,\dots,t_6):\Re_+^6\to\Re$ *as*

$$F(t_1,\dots,t_6)=\begin{cases} t_1-\max\left\{t_3,t_4,t_2\left(\dfrac{t_5+t_6}{t_3+t_4}\right)\right\}, & \text{if } t_3+t_4\neq 0\\ t_1-t_2, & \text{if } t_3+t_4=0.\end{cases}$$

The verifications of above examples are easy, hence details are omitted.

Recently, Aliouche [1] introduced yet another implicit relation by slightly modifying the implicit relation given by Popa [22]. In order to describe the implicit relation due to Aliouche [1], let Φ be a family of all continuous mappings $F(t_1,\dots,t_6):\Re_+^6\to\Re$ satisfying

(C_1): For all $u\geq 0, v>0$ and $w\geq 0$ with

(C_a): $F(u,v,v,u,w,0)<0$, or

(C_b): $F(u,v,u,v,0,w)<0$, we have $u<v$.

(C_2): For all $u>0$, $F(u,0,0,u,u,0)\geq 0$.

(C_3): For all $u>0$, $F(u,u,0,0,u,u)\geq 0$.

Both of these implicit relations described by $\mathcal{F}$ and Φ are in fact independent. In order to substantiate this view point notice that Example 3 satisfies all the requirements of the implicit relation $\mathcal{F}$ but not of the implicit relation Φ, whereas Example 2 (contained in [1]) satisfies all the conditions of the implicit relation Φ but not of implicit relation $\mathcal{F}$. In fact, most of the well known strict contractions are governed by the class $\mathcal{F}$ but only rarely by the implicit relation Φ (due to Aliouche [1]). Moreover, Aliouche [1] assumed Φ to be a class of continuous mappings, whereas we never need any such continuity requirement.

In this paper, we prove a general common fixed point theorem for two pairs of self mappings employing a slightly modified implicit relation as described earlier and also lessen the commutativity requirement to the points of coincidence of the pairs. In process sharpened versions of results due to Fisher [3-5], Jungck [10], Kasahara and Rhoades [12], Khan and Imdad [14], Popa [22] and others are deduced as special cases.

3. Results

Recently, Popa [22] proved an interesting fixed point theorem in compact metric spaces with the minimal continuity requirement as most of the earlier common fixed point theorems require the continuity of all involved mappings whereas on the other hand this is the first ever result in compact setting employing implicit relations. Now we state and prove our main result as follows:

Theorem 2. *Let S, T, I and J be self mappings of a compact metric space (X, d) with $S(X) \subset J(X)$ and $T(X) \subset I(X)$ such that*

$$F(d(Sx,Ty), d(Ix,Jy), d(Ix,Sx), d(Jy,Ty), d(Ix,Ty), d(Jy,Sx)) < 0, \tag{1}$$

for all distinct $x, y \in X$ for which one of $d(Ix,Jy), d(Ix,Sx), d(Jy,Ty)$ is positive, where $F \in \mathcal{F}$ and the mappings S and I are continuous, then

(a) the pairs (S, I) and (T, J) have a point of coincidence, and

(b) the mappings S, T, I and J have a unique common fixed point provided both pairs (S, I) and (T, J) are weakly compatible.

Proof. Let $m = \inf\{d(Sx, Ix) : x \in X\}$. Since X is compact, therefore there is a convergent sequence $\{x_n\}$ with $\lim_{n\to\infty} x_n = x_0$ in X such that $\lim_{n\to\infty} d(Ix_n, Sx_n) = m$. By the continuity of mappings S and I along with $\lim_{n\to\infty} x_n = x_0$, we get $d(Ix_0, Sx_0) = m$.

Since $S(X) \subset J(X)$, there exists a point y_0 in X such that $Jy_0 = Sx_0$ which in turn yields $d(Ix_0, Jy_0) = m$. Suppose that $m > 0$. Then using (1), we have

$$F(d(Sx_0,Ty_0), d(Ix_0,Jy_0), d(Ix_0,Sx_0), d(Jy_0,Ty_0), d(Ix_0,Ty_0), d(Jy_0,Sx_0)) < 0$$

or

$$F(d(Jy_0,Ty_0), m, m, d(Jy_0,Ty_0), m + d(Jy_0,Ty_0), 0) < 0,$$

yielding thereby (due to F_{2a})

$$d(Jy_0, Ty_0) < m. \tag{2}$$

Since $T(X) \subset I(X)$, then there exists a point z in X such that $Iz = Ty_0$ and thus $d(Iz, Jy_0) < m$. Since $d(Iz, Sz) \geq m > 0$, then by (1), we have

$$F(d(Sz,Ty_0), d(Iz,Jy_0), d(Iz,Sz), d(Jy_0,Ty_0), d(Iz,Ty_0), d(Jy_0,Sz)) < 0,$$
$$F(d(Iz,Sz), d(Jy_0,Ty_0), d(Iz,Sz), d(Jy_0,Ty_0), 0, d(Jy_0,Ty_0) + d(Iz,Sz)) < 0,$$

implying thereby (in view of F_{2b})

$$d(Iz, Sz) < d(Jy_0, Ty_0). \tag{3}$$

Now, making use of (2) and (3), one obtains

$$m \leq d(Iz, Sz) < d(Jy_0, Ty_0) < m,$$

which is a contradiction. Therefore, $m = 0$, which implies $Ix_0 = Sx_0 = Jy_0$.
If $d(Jy_0, Ty_0) > 0$, then by (1), we have

$$F(d(Sx_0, Ty_0), d(Ix_0, Jy_0), d(Ix_0, Sx_0), d(Jy_0, Ty_0), d(Ix_0, Ty_0), d(Jy_0, Sx_0)) < 0$$

or

$$F(d(Jy_0, Ty_0), 0, 0, d(Jy_0, Ty_0), d(Jy_0, Ty_0), 0) < 0,$$

yielding thereby $d(Jy_0, Ty_0) = 0$ (due to F_2'), which gives $Jy_0 = Ty_0$. Therefore, $Ix_0 = Sx_0 = Jy_0 = Ty_0$, which shows that the pairs (S, I) and (T, J) have a point of coincidence.
Since the pair (S, I) is weakly compatible, $S^2x_0 = SIx_0 = ISx_0 = I^2x_0$. If $I^2x_0 \neq Ix_0$, then $ISx_0 \neq Jy_0$ and by (1), one can have

$$\begin{aligned} &F(d(S^2x_0, Ty_0), d(ISx_0, Jy_0), d(ISx_0, S^2x_0), d(Jy_0, Ty_0), \\ &\quad d(ISx_0, Ty_0), d(Jy_0, S^2x_0)) < 0 \end{aligned}$$

or

$$F(d(S^2x_0, Ix_0), d(I^2x_0, Ix_0), 0, 0, d(I^2x_0, Ix_0), d(I^2x_0, Ix_0)) < 0$$

a contradiction to F_3. Therefore, $Ix_0 = I^2x_0$. Hence $SIx_0 = Ix_0 = I^2x_0$, which shows that Ix_0 is a common fixed point of the pair (S, I). Similarly, we can show $TJy_0 = Jy_0 = J^2y_0$. Since $Ix_0 = Jy_0$, therefore Ix_0 is a common fixed point of the mappings S, T, I and J. The unicity of the common fixed point is a direct consequence of the condition (F_3). This completes the proof. □

Remark 1. *Theorem 2 is a sharpened form of a result due to Popa [22, Theorem 5] for four mappings as 'compatibility' of the pair (S, I) is weakened to 'weak compatibility'. Notice that up to coincidence points, we never need any condition of weak commutativity.*

Corollary 1. *The conclusions of Theorem 2 remain true if we replace condition* (1) *by any one of the following:*

(a_1)

$$\begin{aligned} [1 + p.d(Ix, Jy)]d(Sx, Ty) < {} & p \max\{d(Ix, Sx).d(Jy, Ty), d(Ix, Ty).d(Jy, Sx)\} \\ & + \max\left\{d(Ix, Jy), d(Ix, Sx), d(Jy, Ty), \right. \\ & \left. \frac{d(Ix, Ty) + d(Jy, Sx)}{2}\right\}, \end{aligned}$$

where $p > 0$.

(a_2) $d(Sx, Ty) < \max\left\{d(Ix, Jy), d(Ix, Sx), d(Jy, Ty), \frac{d(Ix,Ty)+d(Jy,Sx)}{2}\right\}$

(a_3)

$$\begin{aligned} d^2(Sx, Ty) < {} & c \max\{d^2(Ix, Sx), d^2(Jy, Ty), d^2(Ix, Jy)\} \\ & + \frac{1-c}{2} \max\{d(Ix,Sx), d(Jy,Ty), d(Ix, Jy), d(Ix, Ty), d(Jy, Sx)\} \\ & + (1-c)d(Ix, Ty).d(Sx, Ty)\} \end{aligned}$$

(a_4)

$$d(Sx,Ty) < \max\Big\{d(Ix,Jy), d(Ix,Sx), d(Jy,Ty), \frac{d(Ix,Ty)+d(Jy,Sx)}{2}, b\sqrt{d(Ix,Ty)d(Jy,Sx)}\Big\}$$

for all $x, y \in X$ *with* $b \in (0,1)$.

(a_5)

$$d^3(Sx,Ty) < ad^2(Sx,Ty)d(Ix,Jy) + bd(Sx,Ty)d(Ix,Sx)d(Jy,Ty) + cd^2(Ix,Jy)d(Jy,Sx) + d.d(Ix,Ty)d^2(Jy,Sx)$$

where $a, b, c, d \geq 0$ *and* $a+b+c+d < 1$.

(a_6) $d^3(Sx,Ty) < a\frac{d^2(Ix,Sx)d^2(Jy,Ty)+d^2(Ix,Ty)d^2(Jy,Sx)}{1+d(Ix,Jy)+d(Ix,Sx)+d(Jy,Ty)}$, *where* $a \in (0,1)$.

(a_7) $d(Sx,Ty) < \phi\left(\max\left\{d(Ix,Jy), d(Ix,Sx), d(Jy,Ty), \frac{d(Ix,Ty)+d(Jy,Sx)}{2}\right\}\right)$

where $\phi : \Re^+ \to \Re^+$ *is an increasing function such that* $\phi(t) \leq t$ *for* $t \geq 0$.

(a_8) $d(Sx,Ty) < \max\left\{d(Ix,Jy), \frac{d(Ix,Sx)+d(Jy,Ty)}{2}, \frac{d(Ix,Ty)+d(Jy,Sx)}{2}\right\}$

(a_9) $d(Sx,Ty) < \max\left\{d(Ix,Jy), d(Ix,Sx), d(Jy,Ty), \frac{d(Ix,Ty)}{2}, \frac{d(Jy,Sx)}{2}\right\}$

(a_{10}) $$d(Sx,Ty) < \begin{cases} \max\left\{d(Ix,Sx), d(Jy,Ty), d(Ix,Jy)\dfrac{d(Ix,Ty)+d(Jy,Sx)}{d(Ix,Sx)+d(Jy,Ty)}\right\}, \\ \text{if } d(Ix,Sx)+d(Jy,Ty) \neq 0, \\ \\ d(Ix,Jy), \ \text{if } d(Ix,Sx)+d(Jy,Ty) = 0. \end{cases}$$

Proof. The proof of the above corollary follows from Theorem 2 and Examples 2-11. □

Remark 2. *Corollary 1 corresponding to condition* (a_1) *extends a theorem due to Fisher et al. [5] to two pairs of weakly compatible mappings whereas the same (i.e. Corollary 1) corresponding to condition* (a_4) *further refines a theorem due to Telci et al. [26] where 'commutativity' and 'continuity' requirements are weakened. Similarly, Corollary 1 corresponding to condition* (a_2) *enriches a well known fixed point theorem due to Jungck [10] (also Khan and Imdad [14] and Fisher [3]) in 'commutativity' and 'continuity' considerations. Corollary 1 corresponding to condition* (a_3) *extends a theorem of Tas et al. [25] to two pairs of weak compatible mappings and Corollary 1 corresponding to condition* (a_8) *is an improved form of a theorem of Kasahara and Rhoades [12] for four self mappings.*

Remark 3. *Besides the above deductions and improvements, one can employ implicit functions to derive new fixed point theorems. For example, Corollary 1 corresponding to the conditions* $(a_5, a_6, a_7, a_9$ *and* $a_{10})$ *is possibly new to the existing literature.*

In the sequel, $F(T)$ denotes a set of fixed points of the mapping $T : X \to X$.

Theorem 3. *Let S, T, I and J be self mappings of a compact metric space (X, d) which satisfy implicit relation (1), then*

$$(F(I) \cap F(J)) \cap F(S) = (F(I) \cap F(J)) \cap F(T).$$

Proof. The proof is similar to the proof of Theorem 3 from [21]. □

Theorems 2 and 3 imply the following one.

Theorem 4. *Let I, J and $\{T_i\}_{i\in N\cup\{0\}}$ be self mappings of a compact metric space (X, d) such that*

(a) $T_0(X) \subset J(X)$ and $T_i(X) \subset I(X), i \in N$,

(b) $F(d(T_0x, T_iy), d(Ix, Jy), d(Ix, T_0x), d(Jy, T_iy), d(Ix, T_iy), d(Jy, T_0x)) < 0$, holds for all distinct $x, y \in X$ for which one of $d(Ix, Jy), d(Ix, T_0x), d(Jy, T_iy)$ is positive, where $F \in \mathcal{F}$ and the mappings T_0 and I are continuous.

Then mappings I, J and $\{T_i\}$ have a unique common fixed point provided the pairs (T_0, I) and (T_i, J) are weakly compatible.

Remark 4. *Theorem 4 generalizes the main result of Aliouche [1] for a sequence of self mappings.*

Remark 5. *One may obtain a corollary similar to Corollary 1 for a sequence of self mappings. The details are avoided due to repetition.*

Now we furnish an example to demonstrate the validity of hypotheses and a degree of generality of Theorem 4 over earlier results.

Example 12. *Consider $X = [2, 20]$ with a usual metric. Define self mappings I, J and $\{T_i\}$ on X as*

$$\begin{aligned} J2 &= 2, \\ Jx &= 12, \ \text{if } 2 < x \le 5, \\ Jx &= x - 3, \ \text{if } x > 5 \\ Ix &= x, \ \text{for all } x \in X, \\ T_1x &= 2, \ \text{for all } x \in X, \end{aligned}$$

$$T_2x = \begin{cases} 2, & \text{if } x = 2, \text{ or } x > 5 \\ 6, & \text{if } 2 < x \le 5, \end{cases} \quad , \quad T_{i(>2)}x = \begin{cases} 2, & \text{if } x \le 2 + \frac{1}{i}, \text{ or } x > 5 \\ 6, & \text{if } 2 + \frac{1}{i} < x \le 5. \end{cases}$$

Clearly, $T_1(X) = \{2\} \subset [2, 17] = J(X)$ and $T_i(X)(i > 1) = \{2, 6\} \subset X = I(X)$. Also, mappings T_1 and I are continuous. The pairs (T_1, I) and (T_i, J) commute at 2 which is their common coincidence point. Also, all needed pairwise commutativities at coincidence point 2 are immediate. Define $F : \Re^6_+ \to \Re$ as follows

$$F(t_1, \dots, t_6) = t_1 - \max\left\{t_2, t_3, t_4, \frac{t_5 + t_6}{2}\right\}.$$

By a routine calculation one can verify that the above contraction condition is satisfied for all $x, y \in X$. Thus all the conditions of Theorem 4 are satisfied and 2 is a unique common fixed point of I, J and $\{T_i\}$.

We now furnish an example which establishes the genuineness of our extension. To accomplish this purpose, we construct four self mappings S, T, I and J which satisfy the unknown contractive condition (a_{10}) but do not satisfy the known condition (a_2) in Corollary 1.

Example 13. *Consider* $X = \{1, 2, 3, 4\}$ *endowed with the following metric d:* $d(x, y) = 0 \Leftrightarrow x = y$, $d(x, y) = d(y, x)$, $d(1, 2) = 6, d(1, 3) = 4, d(1, 4) = 5, d(2, 3) = 5, d(2, 4) = 8$ *and* $d(3, 4) = 7$. *Clearly,* (X, d) *is a compact metric space. Define self mappings* S, T, I *and* J *on* X *as*

$$\begin{aligned}
&Sx = 4, \text{ for all } x \in X,\\
&Ix = x \text{ for all } x \in X,\\
&T1 = T2 = T4 = 4, T3 = 3 \text{ and}\\
&J1 = J2 = 3, J3 = 2, J4 = 4.
\end{aligned}$$

Notice that

$$S(X) = \{4\} \subset \{2, 3, 4\} = J(X)$$

and

$$T(X) = \{3, 4\} \subset \{1, 2, 3, 4\} = I(X).$$

By a routine calculation, one can verify that all the conditions of Corollary 1 in respect of (a_{10}) *are satisfied and* 4 *is a unique common fixed point of mappings* S, T, I *and* J. *On the other hand, condition* (a_2) *of Corollary 1 is not satisfied when* $x = 4$ *and* $y = 3$.

Acknowledgment

Authors are grateful to Prof. V. Popa for academic help and also for providing the reprints of his papers. Authors are also thankful to both learned referees for their useful suggestions and comments.

References

[1] A. Aliouche, *A common fixed point theorem for weakly compatible mappings in compact metric spaces*, Sarajevo J. Math. **15**(2007), 123–130.

[2] M. Edelstein, *On fixed and periodic points under contractive mappings*, J. London Math. Soc. **37**(1962), 74–79.

[3] B. Fisher, *A fixed point mapping*, Bull. Calcutta Math. Soc. **68**(1976), 265–266.

[4] B. Fisher, *Mappings with a common fixed point*, Math. Sem. Notes **7**(1979), 81–84.

[5] B. Fisher, M. Telci, K. Tas, *A generalization of a fixed point theorem of Nesic*, Doğa-Tr. Math. **17**(1993), 247–250.

[6] M. Imdad, S. Kumar, M. S. Khan, *Remarks on some fixed point theorems satisfying implicit relations*, Rad. Mat. **11**(2002), 135–143.

[7] G. Jungck, *An iff fixed point criterion*, Math. Magazine **49**(1976), 32–34.

[8] G. Jungck, *Commuting mappings and fixed points*, Amer. Math. Monthly **83**(1976), 261–263.

[9] G. Jungck, *Compatible mappings and common fixed points*, Internat. J. Math. Math. Sci. **9**(1986), 771–779.

[10] G. Jungck, *Common fixed points for commuting and compatible maps on compacta*, Proc. Amer. Math. Soc. **103**(1988), 977–983.

[11] G. Jungck, *Common fixed points for noncontinuous nonself maps on nonmetric spaces*, Far East J. Math. Sci. **4**(1996), 199–215.

[12] S. Kasahara, B. E. Rhoades, *Common fixed point theorems in compact metric spaces*, Math. Japonica **23**(1978), 227–229.

[13] M. A. Khamsi, W. A. Kirk, *An introduction to metric spaces and fixed point theory*, John Willy & Sons, New York, 2001.

[14] M. S. Khan, M. Imdad, *Some common fixed point theorems*, Glasnik Mat. **18**(1983), 321–326.

[15] M. A. Kutbi, A. Latif, *Fixed points of multivalued maps in modular function spaces*, Fixed Point Theory Appl. **2009**(2009), Article ID 786357.

[16] B. A. Meade, S. P. Singh, *An extension of fixed point theorem*, Math. Sem. Notes **5**(1977), 443–446.

[17] P. P. Murthy, *Important tools and possible applications of metric fixed point theory*, Nonlinear Anal. (TMA) **47**(2001), 3479–3490.

[18] S. Park, *A generalization of a theorem of Janos and Edelstein*, Proc. Amer. Math. Soc. **66**(1977), 344–346.

[19] S. Park, *Fixed points of contractive maps on compact metric spaces*, Bull. Korean Math. Soc. **14**(1977), 33–37.

[20] S. Park, *Fixed points of f-contractive maps*, Rocky Mountain J. Math. **8**(1978), 743–750.

[21] V. Popa, *Some fixed point theorems for compatible mappings satisfying an implicit relation*, Demons. Math. **32**(1999), 157–163.

[22] V. Popa, *A general fixed point theorem for weakly compatible mappings in compact metric spaces*, Turkish J. Math. **25**(2001), 465–474.

[23] B. E. Rhoades, *A comparison of various definitions of contractive mappings*, Trans. Amer. Math. Soc. **226**(1977), 257–290.

[24] S. Sessa, *On a weak commutativity condition of mappings in fixed point considerations*, Publ. Inst. Math. **32**(1982), 149–153.

[25] K. Tas, M. Telci, B. Fisher, *Common fixed point theorems for compatible mappings*, Internat. J. Math. Math. Sci. **19**(1996), 451–456.

[26] M. Telci, K. Tas, B. Fisher, *A generalization of the fixed point theorem of Bhola and Sharma*, Math. Balkanica (NS) **9**(1995), 113–116.