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The focus and the median of a non-tangential quadrilateral in the isotropic plane

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Abstract. In this paper non-tangential quadrilaterals in the isotropic plane are studied. A quadrilateral is called standard if a parabola with the equation $x = y^2$ is inscribed in it. The properties of the standard quadrilateral related to the focus and the median of the quadrilateral are presented. Furthermore, the orthic of the quadrilateral is introduced.

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A projective plane with the absolute (in the sense of Cayley-Klein) consisting of a line ω and a point Ω on ω is called an isotropic plane. Usually such an affine coordinate system is chosen that in homogeneous coordinates $\omega : z = 0$, i.e., ω is the line at infinity, and $\Omega = (0, 1, 0)$. The line ω is said to be the absolute line and the point Ω the absolute point. A line incident with the absolute point is called an isotropic line, and a conic touching ω at the absolute point Ω is an isotropic circle (sometimes "isotropic" will be omitted). All the notions related to the geometry of the isotropic plane can be found in [7] and [8].

Let $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ be any four lines in the isotropic plane where any two of them are not parallel and any three of them do not pass through the same point. The figure consisting of these four lines and their six points of intersection is called a complete quadrilateral and will be denoted by $\mathcal{ABCD}$. The lines $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ are the sides of the quadrilateral, and $\mathcal{A} \cap \mathcal{B}, \mathcal{A} \cap \mathcal{C}, \mathcal{A} \cap \mathcal{D}, \mathcal{B} \cap \mathcal{C}, \mathcal{B} \cap \mathcal{D}, \mathcal{C} \cap \mathcal{D}$ are its vertices. Pairs of vertices $\mathcal{A} \cap \mathcal{B}, \mathcal{C} \cap \mathcal{D}$; $\mathcal{A} \cap \mathcal{C}, \mathcal{B} \cap \mathcal{D}$ and $\mathcal{A} \cap \mathcal{D}, \mathcal{B} \cap \mathcal{C}$ are called opposite. There is exactly one conic touching the lines $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ and the absolute line ω. If this conic touches ω at the point Ω, then it is an isotropic circle inscribed in $\mathcal{ABCD}$. Otherwise, if the contact point Φ is different from Ω, then the conic is a parabola inscribed in $\mathcal{ABCD}$. In this case the quadrilateral $\mathcal{ABCD}$ is non-tangential. Let us denote this parabola by $\mathcal{P}$. Except for the tangent ω, the parabola $\mathcal{P}$ has another tangent $\mathcal{T}$, from the point Ω, being an isotropic line. This line is called the directrix of the parabola $\mathcal{P}$ and its point T of contact with the parabola is called the focus of the parabola $\mathcal{P}$. The joint line of the focus T and the point at infinity Φ is called the

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axis of the parabola and all lines parallel to the axis are diameters of the parabola $\mathcal{P}$. The definitions of the center, the axis, the foci and the directrices of the conic in the isotropic plane can be found in [7] and [2].

Without loss of generality, the affine coordinate system can be chosen in such a way that the point T coincides with the origin of the coordinate system, and the lines $T\Phi$ and $\mathcal{T}$ are the x-axis and the y-axis, respectively. In the euclidean model of the isotropic plane this coordinate system will be drawn as a rectangular coordinate system, although a right angle does not have any geometrical meaning in the isotropic plane. Let us prove the following lemma:

Lemma 1. *The parabola with the focus at the origin touching the ordinate axis, with the x-axis as its axis, has the equation of the form*

$$ny^2 + ox = 0, \quad n, o \in \mathbb{R}, \quad n, o \neq 0. \tag{1}$$

Proof. Any conic in the isotropic plane has the equation of the form

$$lx^2 + mxy + ny^2 + ox + py + q = 0. \tag{2}$$

Its points of intersection with the y-axis have ordinates that are solutions of the equation $ny^2 + py + q = 0$. Since the y-axis touches the conic (2) at the origin, the previous equation has double solution $y = 0$, which is obtained by the condition $p = q = 0$. Hence, the conic expressed in homogeneous coordinates has the equation of the form

$$lx^2 + mxy + ny^2 + oxz = 0. \tag{3}$$

Without loss of generality, we can choose the x-axis as the axis of our parabola. Since the point at infinity of the x-axis has the coordinates $\Phi = (1, 0, 0)$, the point Φ has to be the point of contact of the line at infinity ω with the equation $z = 0$ and the conic (3), i.e. the equation $lx^2 + mxy + ny^2 = 0$ has a double solution $y = 0$, which is achieved iff $l = m = 0$. Therefore the conic (3) has the final form (1), otherwise, if $n = 0$ or $o = 0$ we get a conic degenerated into two lines. □

By substitution $x \to -\dfrac{n}{o}x$, equation (1) turns into the form

$$\mathcal{P} \ldots x = y^2. \tag{4}$$

Let A, B, C, D be the points of contact of the lines $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ with the inscribed parabola $\mathcal{P}$ of the quadrilateral $\mathcal{ABCD}$ with equation (4). Let the coordinates of these points be:

$$A = (a^2, a), B = (b^2, b), C = (c^2, c), D = (d^2, d), \tag{5}$$

where a, b, c, d are different real numbers. We prove two further lemmas.

Lemma 2. *The parabola $\mathcal{P}$ given by equation (4) has the tangents $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ with the equations*

$$\begin{aligned} \mathcal{A} \ldots 2ay &= x + a^2, \\ \mathcal{B} \ldots 2by &= x + b^2, \\ \mathcal{C} \ldots 2cy &= x + c^2, \\ \mathcal{D} \ldots 2dy &= x + d^2 \end{aligned} \tag{6}$$

at the points A, B, C, D in (5), respectively.

Proof. If we insert, e.g. $x = 2ay - a^2$ from the first equation of (6) into equation (4), we get the equation $y^2 - 2ay + a^2 = 0$ having a double solution $y = a$. This means that the line $\mathcal{A}$ in (6) touches $\mathcal{P}$ at the point having a as its ordinate. Therefore this contact point is A. $\square$

Lemma 3. *The points of intersection of the lines $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ given in (6) are*

$$\begin{aligned} \mathcal{A} \cap \mathcal{B} &= (ab, \frac{a+b}{2}), \quad & \mathcal{C} \cap \mathcal{D} &= (cd, \frac{c+d}{2}), \\ \mathcal{A} \cap \mathcal{C} &= (ac, \frac{a+c}{2}), \quad & \mathcal{B} \cap \mathcal{D} &= (bd, \frac{b+d}{2}), \\ \mathcal{A} \cap \mathcal{D} &= (ad, \frac{a+d}{2}), \quad & \mathcal{B} \cap \mathcal{C} &= (bc, \frac{b+c}{2}). \end{aligned} \tag{7}$$

Proof. Since

$$2a \cdot \frac{1}{2}(a+b) = ab + a^2,$$

we have that the point $(ab, \frac{a+b}{2})$ lies on the line $\mathcal{A}$ from (6). Similarly, this holds for the line $\mathcal{B}$ as well. $\square$

Combining Lemmas 1-3 the following theorem is achieved:

Theorem 1. *For any non-tangential quadrilateral $\mathcal{ABCD}$ there exist four distinct real numbers a, b, c, d such that, in the defined canonical affine coordinate system, the sides are given by (6), the vertices have the form (7) and the inscribed parabola has the equation (4).*

For the quadrilateral $\mathcal{ABCD}$ from Theorem 1 it is said to be in a standard position or it is a standard quadrilateral. Due to Theorem 1 every non-tangential quadrilateral can be represented in the standard position. In order to prove the properties of any non-tangential quadrilateral, it is sufficient to prove the properties for the standard quadrilateral.

For a further study of non-tangential quadrilaterals the following symmetric functions of the numbers a, b, c, d will be useful:

$$\begin{aligned} s &= a + b + c + d, \\ q &= ab + ac + ad + bc + bd + cd, \\ r &= abc + abd + acd + bcd, \\ p &= abcd. \end{aligned} \tag{8}$$

Theorem 2. *The midpoints of the line segments connecting the pairs of opposite vertices of the non-tangential quadrilateral lie on the line $\mathcal{M}$, with equation*

$$\mathcal{M} \ldots y = \frac{s}{4}, \tag{9}$$

related to the standard quadrilateral $\mathcal{ABCD}$.

Proof. As the midpoint of the line segment connecting the points $\mathcal{A}\cap\mathcal{B}$ and $\mathcal{C}\cap\mathcal{D}$ from (7) has coordinates $(\frac{ab+cd}{2}, \frac{a+b+c+d}{4})$, this point is incident with the line (9). □

By analogy with the euclidean case the line $\mathcal{M}$ from Theorem 2 will be called the median of the quadrilateral $\mathcal{ABCD}$. Let us point out that this is an affine notion.

Corollary 1. *The standard quadrilateral $\mathcal{ABCD}$ has the median with the equation (9).*

Theorem 3. *The circumscribed circles of the four triangles formed by the three sides of the non-tangential quadrilateral are incident with the same point O, which coincides with the focus of the parabola inscribed in this quadrilateral.*

Proof. Let us observe the circle $\mathcal{K}_d$ with the equation

$$\mathcal{K}_d \ldots 2abcy = -x^2 + (ab+ac+bc)x. \tag{10}$$

It passes through the points $\mathcal{A}\cap\mathcal{B}$, $\mathcal{A}\cap\mathcal{C}$ and $\mathcal{B}\cap\mathcal{C}$ from (7), because, for example, the first one satisfies the equality

$$2abc\cdot\frac{1}{2}(a+b) = -a^2b^2 + (ab+ac+bc)ab.$$

Besides, $\mathcal{K}_d$ clearly passes through the point $O=(0,0)$ as well. □

The point O from Theorem 3 will be called the focus of the quadrilateral $\mathcal{ABCD}$.

Corollary 2. *The focus of the quadrilateral is the focus of its inscribed parabola. The standard quadrilateral $\mathcal{ABCD}$ has the focus $O=(0,0)$.*

Before the next theorem we need the following definition: if sides of the triangle are polars of their opposite vertices with respect to the circle, such a circle, if it exists, is the polar circle of the triangle.

Theorem 4. *The polar circles of the trilaterals $\mathcal{BCD}$, $\mathcal{ACD}$, $\mathcal{ABD}$, $\mathcal{ABC}$ belong to the same pencil of circles, and the radical axis (the potential line) of this pencil is the median of the quadrilateral $\mathcal{ABCD}$.*

Proof. Let us show for example that the circle $\mathcal{P}_d$ with the equation

$$4abcy = x^2 + abc(a+b+c) \tag{11}$$

is the polar circle of the trilateral $\mathcal{ABC}$. It is sufficient to prove that the point $\mathcal{B}\cap\mathcal{C}$ has the line $\mathcal{A}$ as its polar with respect to the circle $\mathcal{P}_d$. The equation of the polar of any point (x_0, y_0) with respect to the circle (11) is

$$2abc(y+y_0) = x_0x + abc(a+b+c). \tag{12}$$

For the point $\mathcal{B}\cap\mathcal{C} = (bc, \frac{1}{2}(b+c))$ we get

$$abc(2y+b+c) = bc\cdot x + abc(a+b+c),$$

that is, factoring bc out, the equation of the line $\mathcal{A}$. Analogously, the polar circle $\mathcal{P}_c$ of the trilateral $\mathcal{ABD}$ has the equation

$$4abdy = x^2 + abd(a + b + d). \tag{13}$$

Subtracting (13) from (11), there follows the equation

$$4ab(c - d)y = ab[(a + b)(c - d) + c^2 - d^2].$$

Leaving the factor $ab(c-d)$ out of the equation, $4y = a+b+c+d$ is obtained, which is equation (9) of the median. □

Summing (10) and (11) up we get the equation

$$6abcy = (ab + ac + bc)x + abc(a + b + c)$$

of the line $\mathcal{H}_D$ that is the orthic of the trilateral $\mathcal{ABC}$. Namely, the corresponding sides of the triangle and its orthic triangle intersect in three points which lie on the same line called the orthic. (The feet of altitudes of the triangle form its orthic triangle.) In [6] it is shown that the equation of the orthic is the arithmetic mean of equations of sides of the triangle. The equations of the orthics $\mathcal{H}_A, \mathcal{H}_B, \mathcal{H}_C$ of the trilaterals $\mathcal{BCD}$, $\mathcal{ACD}$, $\mathcal{ABD}$ look similar. As a matter of fact:

Theorem 5. *The orthics $\mathcal{H}_A, \mathcal{H}_B, \mathcal{H}_C, \mathcal{H}_D$ of the trilaterals $\mathcal{BCD}$, $\mathcal{ACD}$, $\mathcal{ABD}$, $\mathcal{ABC}$ of the standard quadrilateral $\mathcal{ABCD}$ have the equations*

$$\begin{aligned}
\mathcal{H}_A \dots y &= \frac{bc + bd + cd}{6bcd}x + \frac{1}{6}(b + c + d),\\
\mathcal{H}_B \dots y &= \frac{ac + ad + cd}{6acd}x + \frac{1}{6}(a + c + d),\\
\mathcal{H}_C \dots y &= \frac{ab + ad + bd}{6abd}x + \frac{1}{6}(a + b + d),\\
\mathcal{H}_D \dots y &= \frac{ab + ac + bc}{6abc}x + \frac{1}{6}(a + b + c).
\end{aligned} \tag{14}$$

If the lines $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ are written in an explicit form, that is

$$\begin{aligned}
\mathcal{A} \dots y &= \frac{1}{2a}x + \frac{a}{2},\\
\mathcal{B} \dots y &= \frac{1}{2b}x + \frac{b}{2},\\
\mathcal{C} \dots y &= \frac{1}{2c}x + \frac{c}{2},\\
\mathcal{D} \dots y &= \frac{1}{2d}x + \frac{d}{2},
\end{aligned} \tag{15}$$

then the equations (14) are the arithmetic means of the equations of the three corresponding sides in (15). This agrees with the results given in [6].

Theorem 6. *If the lines $\mathcal{H}_A, \mathcal{H}_B, \mathcal{H}_C, \mathcal{H}_D$ are the orthics of the trilaterals $\mathcal{BCD}$, $\mathcal{ACD}$, $\mathcal{ABD}$, $\mathcal{ABC}$ of the quadrilateral $\mathcal{ABCD}$, then points of intersection $\mathcal{A} \cap \mathcal{H}_A$, $\mathcal{B} \cap \mathcal{H}_B$, $\mathcal{C} \cap \mathcal{H}_C$, $\mathcal{D} \cap \mathcal{H}_D$ lie on the same line $\mathcal{H}$.*

Proof. Combining $\mathcal{A} + 3\mathcal{H}_A$ we get the equation

$$4y = (\frac{1}{2a} + \frac{1}{2b} + \frac{1}{2c} + \frac{1}{2d})x + \frac{1}{2}(a + b + c + d),$$

i. e. the equation of the line $\mathcal{H}$:

$$\mathcal{H} \ldots y = \frac{r}{8p}x + \frac{s}{8}. \tag{16}$$

Clearly the point $\mathcal{A} \cap \mathcal{H}_A$ lies on $\mathcal{H}$. Likewise, the claim is valid for the other three points of intersection. □

The line $\mathcal{H}$ from Theorem 6 will be called the orthic of the quadrilateral $\mathcal{ABCD}$.

Corollary 3. *The standard quadrilateral $\mathcal{ABCD}$ has the orthic with equation (16). This equation is the arithmetic mean of the equations (15) of the sides $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$.*

Theorem 7. *The points of intersection $\mathcal{S}_{AB} \cap \mathcal{S}_{CD}$; $\mathcal{S}_{AC} \cap \mathcal{S}_{BD}$; $\mathcal{S}_{AD} \cap \mathcal{S}_{BC}$ of the bisectors $\mathcal{S}_{AB}, \mathcal{S}_{CD}, \mathcal{S}_{AC}$, $\mathcal{S}_{BD}, \mathcal{S}_{AD}, \mathcal{S}_{BC}$ of the pairs of lines $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}, \mathcal{D}$; $\mathcal{A}, \mathcal{C}$ and $\mathcal{B}, \mathcal{D}$; $\mathcal{A}, \mathcal{D}$ and $\mathcal{B}, \mathcal{C}$ through the pairs of the opposite vertices of the quadrilateral $\mathcal{ABCD}$ are incident with the orthic of the quadrilateral.*

Proof. The equations of the bisectors $\mathcal{S}_{AB}$ and $\mathcal{S}_{CD}$ are the arithmetic means of the first two and the last two equations in (15), while (16) can be obtained as the arithmetic mean of these two equations. Hence, the point $\mathcal{S}_{AB} \cap \mathcal{S}_{CD}$ is incident with the line $\mathcal{H}$. □

Theorem 8. *Let $\mathcal{ABCD}$ be a non-tangential quadrilateral and $\mathcal{A}', \mathcal{B}', \mathcal{C}', \mathcal{D}'$ the circles touching the bisectors of the angles of the trilaterals $\mathcal{BCD}$, $\mathcal{ACD}$, $\mathcal{ABD}$ and $\mathcal{ABC}$, respectively. These circles touch a line $\mathcal{O}$.*

Proof. By summing up the first two equations in (6), we get the equation of the bisector of the lines $\mathcal{A}$ and $\mathcal{B}$:

$$2(a + b)y = 2x + a^2 + b^2. \tag{17}$$

Let us observe the circle $\mathcal{D}'$ with equation

$$\begin{aligned} 8(a + b)(a + c)(b + c)y = {} & 4x^2 + 4(a^2 + b^2 + c^2 + ab + ac + bc)x \\ & + (a^2 + b^2 + c^2 + ab + ac + bc)^2. \end{aligned} \tag{18}$$

Out of (18) and (17) multiplied by $4(a + c)(b + c)$ the next equation

$$\begin{aligned} & 4x^2 + 4[a^2 + b^2 + (a + c)(b + c)]x + [a^2 + b^2 + (a + c)(b + c)]^2 \\ & \quad = 8(a + c)(b + c)x + 4(a^2 + b^2)(a + c)(b + c), \end{aligned}$$

is obtained. The latter equation can be rewritten as follows

$$4x^2 + 4[a^2 + b^2 - (a+c)(b+c)]x + [a^2 + b^2 - (a+c)(b+c)]^2 = 0.$$

This equation has a double solution $x = \frac{1}{2}[(a+c)(b+c) - a^2 - b^2]$. Therefore, the circle $\mathcal{D}'$ touches the bisector of the lines $\mathcal{A}$ and $\mathcal{B}$ at the point

$$\left(\frac{1}{2}[(a+c)(b+c) - a^2 - b^2], \frac{(a+c)(b+c)}{2(a+b)}\right).$$

Analogously, the bisectors of the pairs of lines $\mathcal{A}, \mathcal{C}$ and $\mathcal{B}, \mathcal{C}$ touch $\mathcal{D}'$ as well. Furthermore, the line $\mathcal{O}$ with equation $y = 0$ touches the circle $\mathcal{D}'$. Namely, out of (18) under $y = 0$, an equation with a double solution $x = -\frac{1}{2}(a^2 + b^2 + c^2 + ab + ac + bc)$ is obtained. The line $\mathcal{O}$ is tangent to the circles $\mathcal{A}', \mathcal{B}', \mathcal{C}'$, too. □

Due to the notion of the Euler's center of a quadrangle, the line $\mathcal{O}$ from Theorem 8 will be called Euler's axis of the quadrilateral $\mathcal{ABCD}$.

Corollary 4. *The Euler axis of the quadrilateral is parallel to its median and passes through its focus.*

Theorem 9. *The lines connecting the focus of a quadrilateral and two opposite vertices have the same bisector, hence in the case of a standard quadrilateral it has the equation*

$$y = \frac{r}{4p}x.$$

Proof. As the joint lines of O and $\mathcal{A} \cap \mathcal{B}$ and O and $\mathcal{C} \cap \mathcal{D}$ have slopes $\frac{a+b}{2ab}$ and $\frac{c+d}{2cd}$, the slope of their bisector is

$$\frac{1}{2}\left(\frac{a+b}{2ab} + \frac{c+d}{2cd}\right) = \frac{1}{4p}[cd(a+b) + ab(c+d)] = \frac{r}{4p}.$$

□

Theorem 10. *Let the lines parallel to the side $\mathcal{D}$ of the quadrilateral $\mathcal{ABCD}$ be drawn through the vertices $\mathcal{B} \cap \mathcal{C}$, $\mathcal{A} \cap \mathcal{C}$, $\mathcal{A} \cap \mathcal{B}$ of the trilateral $\mathcal{ABC}$ intersecting its circumscribed circle $\mathcal{K}_d$ residually in the points A_d, B_d, C_d. The lines determined by pairs of points A_d, $\mathcal{A} \cap \mathcal{D}$; B_d, $\mathcal{B} \cap \mathcal{D}$; C_d, $\mathcal{C} \cap \mathcal{D}$ are incident with the focus of the quadrilateral $\mathcal{ABCD}$.*

Proof. Let

$$A_d = \left(\frac{a}{d}(bd + cd - bc), \frac{a+d}{2d^2}(bd + cd - bc)\right). \tag{19}$$

The joint line of A_d and $\mathcal{B} \cap \mathcal{C}$ is parallel to the line $\mathcal{D}$ since its slope is of the form

$$\frac{\frac{a+d}{2d^2}(bd + cd - bc) - \frac{b+c}{2}}{\frac{a}{d}(bd + cd - bc) - bc} = \frac{1}{2d}.$$

The point A_d lies on the circle $\mathcal{K}_d$ represented by (10). Leaving the factor $\frac{a}{d^2}(bd + cd - bc)$ out, the following equality

$$bc(a+d) = -a(bd+cd-bc) + d(ab+ac+bc)$$

is obtained. The points $\mathcal{A} \cap \mathcal{D}$ and A_d from (7) and (19) lie on the line having the equation

$$y = \frac{a+d}{2ad}x.$$

This line passes through the focus O. □

Two pairs of lines are called antiparallel if they have the same bisector.

Theorem 11. *The lines passing through the vertices of the quadrilateral, antiparallel to the median of the quadrilateral with respect to the sides passing through the same vertices, are incident with the focus of the quadrilateral.*

For the Euclidean version of this theorem see [3].

Proof. Due to antiparallel lines, e.g. taking sides $\mathcal{A}$ and $\mathcal{B}$, the equality

$$0 + k = \frac{1}{2a} + \frac{1}{2b}$$

must be valid, where k stands for the slope of the line we are looking for. Hence, $k = \frac{a+b}{2ab}$, and the equation of the line is $y = \frac{a+b}{2ab}x$.

Repeating the same as above for each vertex one gets that all the lines are incident with the origin, i. e. the focus of the quadrilateral. □

Theorem 12. *Let us consider the following four points: the intersection point of the line $\mathcal{D}$ and a tangent of the circle $\mathcal{K}_d$ at $\mathcal{A} \cap \mathcal{B}$, the intersection point of the line $\mathcal{C}$ and a tangent of the circle $\mathcal{K}_c$ at $\mathcal{A} \cap \mathcal{B}$, the intersection point of the line $\mathcal{A}$ and a tangent of the circle $\mathcal{K}_a$ at $\mathcal{C} \cap \mathcal{D}$, and the intersection point of the line $\mathcal{B}$ and a tangent of the circle $\mathcal{K}_b$ at $\mathcal{C} \cap \mathcal{D}$. Points $\mathcal{A} \cap \mathcal{B}$ and $\mathcal{C} \cap \mathcal{D}$ together with the described four points are concyclic. Analogously, for the pairs of vertices $\mathcal{A} \cap \mathcal{C}$, $\mathcal{B} \cap \mathcal{D}$ and $\mathcal{A} \cap \mathcal{D}$, $\mathcal{B} \cap \mathcal{C}$ two more circles are achieved. These three circles intersect each other in the focus of the quadrilateral.*

For the Euclidean version of this theorem see [4].

Proof. The tangent $\mathcal{T}_{\mathcal{A}\cap\mathcal{B}}$ to the circle $\mathcal{K}_d$ at the point $\mathcal{A} \cap \mathcal{B}$ has the equation

$$2abcy = (ac + bc - ab)x + a^2b^2, \tag{20}$$

(for the equation of $\mathcal{K}_d$ see (10), and for the coordinates of $\mathcal{A} \cap \mathcal{B}$ see (7)). Indeed, putting the coordinates $x_0 = ab, y_0 = \frac{a+b}{2}$ of the point $\mathcal{A} \cap \mathcal{B}$ into equation

$$2abc(y + y_0) = -2xx_0 + (ab + ac + bc)(x + x_0)$$

the upper equation of the tangent is obtained. The point of intersection of the tangent $\mathcal{T}_{\mathcal{A}\cap\mathcal{B}}$ and the line $\mathcal{D}$ having equations (20) and (6) is the point with coordinates

$$D' = \Big(\frac{abd(ab-cd)}{ab(c+d)-cd(a+b)}, \frac{a^2b^2+abd^2-acd^2-bcd^2}{2(ab(c+d)-cd(a+b))}\Big). \tag{21}$$

The points $\mathcal{A}\cap\mathcal{B}$ and D' given by (7) and (21) are lying on the circle with the equation

$$2abcd(ab-cd)y = [cd(a+b)-ab(c+d)]x^2 + [a^2b^2(c+d)-c^2d^2(a+b)]x.$$

Namely, leaving out the factors ab and $\dfrac{abd(ab-cd)}{ab(c+d)-cd(a+b)}$ the following equalities

$$\begin{aligned} cd(ab-cd)(a+b) &= [cd(a+b)-ab(c+d)]ab + a^2b^2(c+d) - c^2d^2(a+b), \\ c[a^2b^2+abd^2-acd^2-bcd^2] &= -abd(ab-cd) + a^2b^2(c+d) - c^2d^2(a+b) \end{aligned}$$

are valid. For the points $\mathcal{C}\cap\mathcal{D}$ and $O=(0,0)$ the same statement is valid as well. □

Theorem 13. *The joint lines of any point T to the points $\mathcal{B}\cap\mathcal{C}$, $\mathcal{A}\cap\mathcal{C}$, $\mathcal{A}\cap\mathcal{B}$ meet the circumscribed circles of the trilaterals $\mathcal{BCD}$, $\mathcal{ACD}$, $\mathcal{ABD}$ residually in the points A_D, B_D, C_D which lie on a circle $\mathcal{D}'$ passing through the focus O of the quadrilateral $\mathcal{ABCD}$ and through the point T.*

For the Euclidean version of this theorem see [9].

Proof. The equation of the circle $\mathcal{K}_a$ circumscribed to the trilateral $\mathcal{BCD}$ is

$$\mathcal{K}_a \dots 2bcdy = -x^2 + (bc+bd+cd)x,$$

see (10). Let $T=(x_0,y_0)$.

Then the line joining the points T and $\mathcal{B}\cap\mathcal{C}$ has the equation

$$2y(x_0-bc) = x(2y_0-b-c) + x_0(b+c) - 2bcy_0.$$

The points $\mathcal{B}\cap\mathcal{C}$ and T lie on it since

$$\begin{aligned} (b+c)(x_0-bc) &= bc(2y_0-b-c) + x_0(b+c) - 2bcy_0, \\ 2y_0(x_0-bc) &= x_0(2y_0-b-c) + x_0(b+c) - 2bcy_0, \end{aligned}$$

respectively. This line meets $\mathcal{K}_a$ in two points, the point $\mathcal{B}\cap\mathcal{C}$ and the point A_D having the coordinates

$$A_D = \Big(d\frac{(b+c)x_0-2bcy_0}{x_0-bc}, \frac{[(b+c)x_0-2bcy_0](x_0+2dy_0-bc-bd-cd)}{2(x_0-bc)^2}\Big).$$

Since

$$\begin{aligned} &[(b+c)x_0-2bcy_0](x_0+2dy_0-bc-bd-cd) \\ &\quad = d[(b+c)x_0-2bcy_0](2y_0-b-c) + (x_0-bc)[(b+c)x_0-2bcy_0], \end{aligned}$$

it follows that A_D lies on the line joining T and $\mathcal{B}\cap\mathcal{C}$. That A_D lies on the circle $\mathcal{K}_a$ follows from the equality

$$\begin{aligned} &bcd[(b+c)x_0 - 2bcy_0](x_0 + 2dy_0 - bc - bd - cd) \\ &\quad = -d^2[(b+c)x_0 - 2bcy_0]^2 + d(bc+bd+cd)[(b+c)x_0 - 2bcy_0](x_0 - bc). \end{aligned}$$

Points A_D, T and the focus O are incident with the circle $\mathcal{D}'$ having equation

$$\mathcal{D}' \ldots 2dx_0 y = -x^2 + (x_0 + 2dy_0)x.$$

Obviously, the coordinates of the points O and T satisfy the equation of the circle $\mathcal{D}'$. By the equality

$$x_0(x_0 + 2dy_0 - bc - bd - cd) = -d[(b+c)x_0 - 2bcy_0] + (x_0 - bc)(x_0 + 2dy_0),$$

we show that the point A_D fulfils it as well. The same statement can be proved for the points B_D and C_D . □

As the circles $\mathcal{A}', \mathcal{B}', \mathcal{C}'$ and $\mathcal{D}'$ are incident with the points O and T, the joint line OT having equation $y = y_0 x/x_0$ is the radical axis of these circles. Before the following theorem, we need the definition: two lines are reciprocal with respect to the triangle if they intersect each side of the triangle in two points whose midpoint coincides with the midpoint of this side.

Theorem 14. *Let $\mathcal{ABCD}$ be a standard quadrilateral. The median of the quadrilateral is parallel to the lines $\mathcal{A}', \mathcal{B}', \mathcal{C}', \mathcal{D}'$, where $\mathcal{A}'$ represents a line reciprocal to the line $\mathcal{A}$ with respect to the trilateral $\mathcal{BCD}$ (by analogy for $\mathcal{B}', \mathcal{C}', \mathcal{D}'$).*

For the Euclidean version of this theorem see [5].

Proof. The line $\mathcal{A}'$ having the equation

$$\mathcal{A}' \ldots y = \frac{b+c+d-a}{2}$$

is reciprocal to the line $\mathcal{A}$ with respect to the trilateral $\mathcal{BCD}$. Indeed, the point $B_a = (b(c+d-a), (b+c+d-a)/2)$ obviously lies on the line $\mathcal{A}'$ and on the line $\mathcal{B}$; for the equations of $\mathcal{A}$ and $\mathcal{B}$ see (6). Furthermore, the points $\mathcal{A}\cap\mathcal{B}$, B_a and the points $\mathcal{B}\cap\mathcal{C}$, $\mathcal{B}\cap\mathcal{D}$ have the same midpoint because of

$$ab + b(c+d-a) = bc + bd.$$

It is analogously valid for the points of intersection of the lines $\mathcal{A}$, $\mathcal{A}'$ with the line $\mathcal{C}$ and with the line $\mathcal{D}$. The line $\mathcal{A}'$ is obviously parallel to the median (9). □

Out of symmetry, the equations of the lines $\mathcal{B}', \mathcal{C}', \mathcal{D}'$ are

$$\begin{aligned} \mathcal{B}' \ldots y &= \frac{a+c+d-b}{2} \\ \mathcal{C}' \ldots y &= \frac{a+b+d-c}{2} \\ \mathcal{D}' \ldots y &= \frac{a+b+c-d}{2}. \end{aligned}$$

Corollary 5. *The median of the quadrilateral $\mathcal{ABCD}$ is the orthic of the quadrilateral $\mathcal{A}'\mathcal{B}'\mathcal{C}'\mathcal{D}'$.*

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