Math. Commun., Vol. **15**, No. 1, pp. 129-138 (2010)

Multiple positive solutions of boundary value problems for systems of nonlinear second-order dynamic equations on time scales

Allaka Kameswararao[1,*] and Sabbavarapu Nageswararao[2]

[1] *Department of Applied Mathematics, Andhra University, Visakhapatnam-530 003, India*
[2] *Department of Mathematics, Sri Prakash College of Engineering, Tuni-533 401, India.*

Received July 10, 2009; accepted November 17, 2009

Abstract. This paper is concerned with boundary value problems for systems of nonlinear second order dynamic equations on time scales. Under the suitable conditions, the existence and multiplicity of positive solutions are established by using abstract fixed-point theorems.
AMS subject classifications: 34B15, 39B10, 34B18
Key words: boundary value problems, dynamic equations, Green's function, positive solutions, multiplicity, cones

1. Introduction

Recently, existence and multiplicity of solutions for boundary value problems of dynamic equations have been of great interest in mathematics and its applications to engineering sciences. To our knowledge, most existing results on this topic are concerned with single equation and simple boundary conditions.

It should be pointed out that Erbe and Wang [4] discussed the boundary value problem

$$-u'' = f(t,u), \quad \alpha u(0) - \beta u'(0) = 0, \quad \gamma u(1) + \delta u'(1) = 0.$$

By using Krasnosel'skii's fixed point theorem, the existence of solutions is obtained in the case when either f is superlinear or f is sublinear. Yang and Sun [13] considered the boundary value problem of the system of differential equations

$$-u'' = f(t,u), \quad -v'' = g(t,u), \quad u(0) = u(1) = 0, \quad v(0) = v(1) = 0.$$

By appealing to the degree theory, the existence of solutions is established. Note that there is only one differential equation in [4] and the BVP in [13] contains simple boundary conditions.

Motivated by the works of [4] and [13], this paper is concerned with the existence and multiplicity of positive solutions for a dynamic equation on time scales

$$\begin{aligned} u^{\Delta\Delta}(t) + f(t,v) = 0, \ \ t \in [a,b]_{\mathbb{T}}, \\ v^{\Delta\Delta}(t) + g(t,u) = 0, \ \ t \in [a,b]_{\mathbb{T}}, \end{aligned} \tag{1}$$

*Corresponding author. *Email addresses:* `kamesh_1724@yahoo.com` (A. Kameswararao), `sabbavarapu_nag@yahoo.co.in` (S. Nageswararao)

satisfying the boundary conditions

$$\begin{aligned} \alpha u(a)-\beta u^{\Delta}(a)=0=\gamma u(\sigma^2(b))+\delta u^{\Delta}(\sigma(b)),\\ \alpha v(a)-\beta v^{\Delta}(a)=0=\gamma v(\sigma^2(b))+\delta v^{\Delta}(\sigma(b)), \end{aligned} \tag{2}$$

where $f,g\in C([a,\sigma^2(b)]_{\mathbb{T}}\times\mathbb{R}^+,\mathbb{R}^+)$, $f(t,0)\equiv 0$, $g(t,0)\equiv 0$, $\alpha,\beta,\delta,\gamma\geq 0$, $d=\gamma\beta+\alpha\delta+\alpha\gamma(\sigma^2(b)-a)$.

Let $\mathbb{T}$ be a time scale with $a,\sigma^2(b)\in\mathbb{T}$. Given an interval J of $\mathbb{R}$, we will use the interval notation

$$J_{\mathbb{T}}=J\cap\mathbb{T}.$$

The arguments for establishing the existence of solutions of (1)-(2) involve properties of Green's function that play a key role in defining some cones. A fixed point theorem due to Krasnosel'skii [8] is applied to yield the existence of positive solutions of (1)-(2). Another fixed point theorem about multiplicity is applied to obtain the multiplicity of positive solutions of (1)-(2).

The rest of this paper is organized as follows. In Section 2, we shall provide some properties of certain Green's functions and preliminaries which are needed later. For the sake of convenience, we also state Krasnosel'skii's fixed point theorem in a cone. In Section 3, we establish the existence and multiplicity of positive solutions of (1)-(2).

2. Preliminaries

In this section, we will give some lemmas which are useful in proving our main results.

To obtain solutions of (1)-(2), we let $G(t,s)$ be the Green's function for the boundary value problem

$$-y^{\Delta\Delta}=0, \tag{3}$$

$$\alpha y(a)-\beta y^{\Delta}(a)=0,\quad \gamma y(\sigma^2(b))+\delta y^{\Delta}(\sigma(b))=0, \tag{4}$$

which is given by

$$G(t,s)=\frac{1}{d}\begin{cases}[\alpha(t-a)+\beta]\left[\gamma(\sigma^2(b)-\sigma(s))+\delta\right]: \mathrm{t}\leq \mathrm{s}\\ [\alpha(\sigma(s)-a)+\beta]\left[\gamma(\sigma^2(b)-t)+\delta\right]: \sigma(\mathrm{s})\leq \mathrm{t},\end{cases} \tag{5}$$

where $\alpha,\beta,\gamma,\delta\geq 0$ and

$$d:=\gamma\beta+\alpha\delta+\alpha\gamma(\sigma^2(b)-a)>0.$$

One can easily check that

$$G(t,s)>0,\quad (t,s)\in(a,\sigma^2(b))_{\mathbb{T}}\times(a,\sigma(b))_{\mathbb{T}}.$$

Define

$$I=\left[\frac{3a+\sigma^2(b)}{4},\frac{a+3\sigma^2(b)}{4}\right]_{\mathbb{T}}.$$

Lemma 1. *The Green's function $G(t,s)$ satisfies*

(i) $G(t,s) \leq G(\sigma(s),s)$, *for* $t \in [a,\sigma^2(b)]_{\mathbb{T}}$, $s \in [a,\sigma(b)]_{\mathbb{T}}$,

(ii) $G(t,s) \geq kG(\sigma(s),s)$, *for* $t \in I$, $s \in [a,\sigma(b)]_{\mathbb{T}}$,

where

$$k = \min\left\{\frac{\gamma(\sigma^2(b)-a)+4\delta}{4(\gamma(\sigma^2(b)-a)+\delta)}, \frac{\alpha(\sigma^2(b)-a)+4\beta}{4(\alpha(\sigma^2(b)-a)+\beta)}\right\} \leq 1.$$

The proof of this lemma is standard and therefore omitted.

We note that a pair $(u(t),v(t))$ is a solution of (1)-(2) if and only if

$$u(t) = \int_a^{\sigma(b)} G(t,s)f\left(s, \int_a^{\sigma(b)} G(\sigma(s),r)g(r,u(r))\Delta r\right)\Delta s, \;\; a \leq t \leq \sigma^2(b),$$
$$v(t) = \int_a^{\sigma(b)} G(t,s))g(s,u(s))\Delta s, \;\; a \leq t \leq \sigma^2(b).$$

Assume throughout that $[a,\sigma^2(b)]_{\mathbb{T}}$ is such that

$$\xi = \min\left\{t \in \mathbb{T} \mid t \geq \frac{3a+\sigma^2(b)}{4}\right\},$$

and

$$\omega = \max\left\{t \in \mathbb{T} \mid t \leq \frac{a+3\sigma^2(b)}{4}\right\}$$

both exist and satisfy

$$\frac{3a+\sigma^2(b)}{4} \leq \xi < \omega \leq \frac{a+3\sigma^2(b)}{4}.$$

Next, let $\tau \in [\xi,\omega]_{\mathbb{T}}$ be defied by

$$\int_\xi^\omega G(\tau,s)\Delta s = \max_{t\in[a,\sigma^2(b)]_{\mathbb{T}}} \int_\xi^\omega G(t,s)\Delta s.$$

Finally, we define

$$l = \min_{s\in[a,\sigma(b)]_{\mathbb{T}}} \frac{G(\sigma(\omega),s)}{G(\sigma(s),s)},$$

and let

$$m = \min\{k,l\}.$$

Let $E = \{u : [a,\sigma^2(b)]_{\mathbb{T}} \to \mathbb{R}\}$ with supremum norm

$$\| u \| = \sup_{t\in[a,\sigma^2(b)]_{\mathbb{T}}} | u(t) | .$$

Then $(E, \| \,.\, \|)$ is a Banach space. Denote

$$P = \{u \in E \mid u(t) \geq 0, \;\; \min_{t\in[\xi,\omega]_{\mathbb{T}}} u(t) \geq m \| u \|\}.$$

It is obvious that P is a positive cone in E. Define

$$Tu(t)=\int_a^{\sigma(b)}G(t,s)f\left(s,\int_a^{\sigma(b)}G(\sigma(s),r)g(r,u(r))\Delta r\right)\Delta s,\quad u\in P. \tag{6}$$

Lemma 2. *If the operator T is defined as* (6)*, then $T:P\to P$ is completely continuous.*

Proof. From the continuity of f and g, we know $Tu\in E$ for each $u\in P$. It follows from Lemma 1 that for $u\in P$,

$$\begin{aligned}Tu(t)&=\int_a^{\sigma(b)}G(t,s)f\left(s,\int_a^{\sigma(b)}G(\sigma(s),r)g(r,u(r))\Delta r\right)\Delta s\\&\le\int_a^{\sigma(b)}G(\sigma(s),s)f\left(s,\int_a^{\sigma(b)}G(\sigma(s),r)g(r,u(r))\Delta r\right)\Delta s.\end{aligned}$$

Note that by the nonnegativity of f and g, one has

$$\| Tu \|\le\int_a^{\sigma(b)}G(\sigma(s),s)f\left(s,\int_a^{\sigma(b)}G(\sigma(s),r)g(r,u(r))\Delta r\right)\Delta s,$$

from which we have

$$\begin{aligned}\min_{t\in[\xi,\omega]_{\mathbb{T}}}Tu(t)&=\min_{t\in[\xi,\omega]_{\mathbb{T}}}\int_a^{\sigma(b)}G(t,s)f\left(s,\int_a^{\sigma(b)}G(\sigma(s),r)g(r,u(r))\Delta r\right)\Delta s\\&\ge m\int_a^{\sigma(b)}G(\sigma(s),s)f\left(s,\int_a^{\sigma(b)}G(\sigma(s),r)g(r,u(r))\Delta r\right)\Delta s\\&\ge m\| Tu \|,\quad u\in P.\end{aligned}$$

Therefore $T:P\to P$. Since $G(t,s)$, $f(t,u)$ and $g(t,u)$ are continuous, it is easily known that $T:P\to P$ is completely continuous. The proof is complete. □

From the above arguments, we know that the existence of positive solutions of (1)-(2) can be transferred to the existence of positive fixed points of the operator T.

Lemma 3 (see [3, 4, 8])**.** *Let $(E,\| \,.\, \|)$ be a Banach space, and let $P\subset E$ be a cone in E. Assume that Ω_1 and Ω_2 are bounded open subsets of E such that $0\in\Omega_1$, $\overline{\Omega}_1\subset\Omega_2$. If*

$$T:P\cap(\overline{\Omega}_2\backslash\Omega_1)\to P$$

is a completely continuous operator such that either

(i) $\| Tu \|\le\| u \|$, $u\in P\cap\partial\Omega_1$, *and* $\| Tu \|\ge\| u \|$, $u\in P\cap\partial\Omega_2$, *or*

(ii) $\| Tu \|\ge\| u \|$, $u\in P\cap\partial\Omega_1$, *and* $\| Tu \|\le\| u \|$, $u\in P\cap\partial\Omega_2$,

then T has a fixed point in $P\cap(\overline{\Omega}_2\backslash\Omega_1)$.

Lemma 4 (see [3, 4, 8])**.** *Let $(E, \| \, . \, \|)$ be a Banach space, and let $P \subset E$ be a cone in E. Assume that Ω_1, Ω_2 and Ω_3 are bounded open subsets of E such that $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$, $\overline{\Omega}_2 \subset \Omega_3$. If*

$$T : P \cap (\overline{\Omega}_3 \backslash \Omega_1) \to P$$

is a completely continuous operator such that:

$$\begin{aligned} &\| Tu \| \geq \| u \|, \ \forall u \in P \cap \partial\Omega_1; \\ &\| Tu \| \leq \| u \|, \ Tu \neq u, \forall u \in P \cap \partial\Omega_2; \\ &\| Tu \| \geq \| u \|, \ \forall u \in P \cap \partial\Omega_3, \end{aligned}$$

then T has at least two fixed points x^*, x^{**} in $P \cap (\overline{\Omega}_3 \backslash \Omega_1)$, and furthermore $x^* \in P \cap (\Omega_2 \backslash \Omega_1)$, $x^{**} \in P \cap (\overline{\Omega}_3 \backslash \overline{\Omega}_2)$.

3. Main Results

First we give the following assumptions:

(A1) $\lim_{u \to 0^+} \sup_{t \in [a, \sigma^2(b)]} \frac{f(t,u)}{u} = 0, \qquad \lim_{u \to 0^+} \sup_{t \in [a, \sigma^2(b)]} \frac{g(t,u)}{u} = 0;$

(A2) $\lim_{u \to \infty} \inf_{t \in [a, \sigma^2(b)]} \frac{f(t,u)}{u} = \infty, \qquad \lim_{u \to \infty} \inf_{t \in [a, \sigma^2(b)]} \frac{g(t,u)}{u} = \infty;$

(A3) $\lim_{u \to 0^+} \inf_{t \in [a, \sigma^2(b)]} \frac{f(t,u)}{u} = \infty, \qquad \lim_{u \to 0^+} \inf_{t \in [a, \sigma^2(b)]} \frac{g(t,u)}{u} = \infty;$

(A4) $\lim_{u \to \infty} \sup_{t \in [a, \sigma^2(b)]} \frac{f(t,u)}{u} = 0, \qquad \lim_{u \to \infty} \sup_{t \in [a, \sigma^2(b)]} \frac{g(t,u)}{u} = 0;$

(A5) $f(t,u)$, $g(t,u)$ are increasing functions with respect to u and, there is a number $N > 0$, such that

$$f\left(t, \int_a^{\sigma(b)} N' g(s,N) \Delta s\right) < \frac{N}{N'(\sigma^2(b) - a)}, \quad \forall t \in [a, \sigma^2(b)]_{\mathbb{T}}, \ s \in [a, \sigma(b)]_{\mathbb{T}},$$

where $N' = \frac{[\alpha(\sigma^2(b)-a)+\beta][\gamma(\sigma^2(b)-a)+\delta]}{d}$.

Theorem 1. *If $(A1)$ and $(A2)$ are satisfied, then* (1)-(2) *have at least one positive solution $(u,v) \in C^2([a, \sigma^2(b)]_{\mathbb{T}}, \mathbb{R}^+) \times C^2([a, \sigma^2(b)]_{\mathbb{T}}, \mathbb{R}^+)$ satisfying $u(t) > 0$, $v(t) > 0$.*

Proof. From $(A1)$ there is a number $H_1 \in (0,1)$, such that for each $(t,u) \in [a, \sigma^2(b)]_{\mathbb{T}} \times (0, H_1)$, one has

$$f(t,u) \leq \eta u, \quad g(t,u) \leq \eta u,$$

where $\eta > 0$ satisfies

$$\eta \int_a^{\sigma(b)} G(\sigma(t), t) \Delta t \leq 1.$$

For every $u \in P$ and $\| u \| = H_1/2$, note that

$$\int_a^{\sigma(b)} G(\sigma(s), r) g(r, u(r)) \Delta r \leq \eta \int_a^{\sigma(b)} G(\sigma(r), r) u(r) \Delta r \leq \| u \| = \frac{H_1}{2} < H_1,$$

thus

$$\begin{aligned} Tu(t) &= \int_a^{\sigma(b)} G(t, s) f\left(s, \int_a^{\sigma(b)} G(\sigma(s), r) g(r, u(r)) \Delta r\right) \Delta s \\ &\leq \eta \int_a^{\sigma(b)} G(\sigma(s), s) \int_a^{\sigma(b)} G(\sigma(s), r) g(r, u(r)) \Delta r \Delta s \\ &\leq \eta^2 \int_a^{\sigma(b)} G(\sigma(s), s) \int_a^{\sigma(b)} G(\sigma(r), r) u(r) \Delta r \Delta s \\ &\leq \eta^2 \| u \| \int_a^{\sigma(b)} G(\sigma(s), s) \int_a^{\sigma(b)} G(\sigma(r), r) \Delta r \Delta s \\ &\leq \| u \| . \end{aligned}$$

So, $\| Tu \| \leq \| u \|$. If we set

$$\Omega_1 = \{u \in E : \| u \| < H_1/2\},$$

then

$$\| Tu \| \leq \| u \|, \text{ for } u \in P \cap \partial\Omega_1. \tag{7}$$

On the other hand, from $(A2)$ there exist four positive numbers μ, μ', C_1 and C_2 such that

$$\begin{aligned} f(t, u) &\geq \mu u - C_1, \quad \forall (t, u) \in [a, \sigma^2(b)]_{\mathbb{T}} \times \mathbb{R}^+, \\ g(t, u) &\geq \mu' u - C_2, \quad \forall (t, u) \in [a, \sigma^2(b)]_{\mathbb{T}} \times \mathbb{R}^+, \end{aligned}$$

where μ and μ' satisfy

$$\mu m \int_\xi^\omega G(\tau, s) \Delta s \geq 2, \quad \mu' m \int_\xi^\omega G(\sigma(s), s) \Delta s \geq 1.$$

For $u \in P$, we have

$$\begin{aligned} Tu(\tau) &= \int_a^{\sigma(b)} G(\tau, s) f\left(s, \int_a^{\sigma(b)} G(\sigma(s), r) g(r, u(r)) \Delta r\right) \Delta s \\ &\geq \int_a^{\sigma(b)} G(\tau, s) \left[\mu \int_a^{\sigma(b)} G(\sigma(s), r) g(r, u(r)) \Delta r - C_1\right] \Delta s \\ &\geq \mu \int_a^{\sigma(b)} G(\tau, s) \int_a^{\sigma(b)} G(\sigma(s), r) g(r, u(r)) \Delta r \Delta s - C_1 \int_a^{\sigma(b)} G(\tau, s) \Delta s \\ &\geq \mu \int_a^{\sigma(b)} G(\tau, s) \int_a^{\sigma(b)} G(\sigma(s), r)[\mu' u(r) - C_2] \Delta r \Delta s - C_1 \int_a^{\sigma(b)} G(\tau, s) \Delta s \\ &\geq \mu\mu' \int_a^{\sigma(b)} G(\tau, s) \int_a^{\sigma(b)} G(\sigma(s), r) u(r) \Delta r \Delta s - C_3(\tau), \end{aligned}$$

where

$$\begin{aligned} C_3(\tau) &= \mu C_2 \int_a^{\sigma(b)} G(\tau,s) \int_a^{\sigma(b)} G(\sigma(s),r)\Delta r \Delta s + C_1 \int_a^{\sigma(b)} G(\tau,s)\Delta s \\ &\leq \mu C_2 \int_a^{\sigma(b)} G(\tau,s) \int_a^{\sigma(b)} G(\sigma(r),r)\Delta r \Delta s + C_1 \int_a^{\sigma(b)} G(\tau,s)\Delta s \\ &= C_3. \end{aligned}$$

Therefore

$$Tu(\tau) \geq \mu \int_\xi^\omega G(\tau,s)\Delta s.\ m\mu' \int_\xi^\omega G(\sigma(r),r)u(r)\Delta r - C_3 \geq 2 \parallel u \parallel - C_3,$$

from which it follows that $\parallel Tu \parallel \geq Tu(\tau) \geq \parallel u \parallel$ as $\parallel u \parallel \to \infty$.

Let $\Omega_2 = \{u \in E : \parallel u \parallel < H_2\}$. Then for $u \in P$ and $\parallel u \parallel = H_2 > 0$ sufficient by large, we have

$$\parallel Tu \parallel \geq \parallel u \parallel, \quad \text{for} \quad u \in P \cap \partial\Omega_2. \tag{8}$$

Thus, from (7), (8) and Lemma 3, we know that the operator T has a fixed point in $P \cap (\overline{\Omega}_2 \backslash \Omega_1)$. The proof is complete. □

Theorem 2. *If* $(A3)$ *and* $(A4)$ *are satisfied, then* (1)-(2) *have at least one positive solution* $(u,v) \in C^2([a,\sigma^2(b)]_{\mathbb{T}}, \mathbb{R}^+) \times C^2([a,\sigma^2(b)]_{\mathbb{T}}, \mathbb{R}^+)$ *satisfying* $u(t) > 0$*,* $v(t) > 0$.

Proof. From $(A3)$ there is a number $\widehat{H}_3 \in (0,1)$ such that for each $(t,u) \in [a,\sigma^2(b)]_{\mathbb{T}} \times (0,\widehat{H}_3)$, one has

$$f(t,u) \geq \lambda u, \quad g(t,u) \geq \lambda' u,$$

where $\lambda >$ and λ' satisfy

$$\lambda m \int_\xi^\omega G(\tau,t)\Delta t \geq 1, \quad \lambda' m \int_\xi^\omega G(\sigma(s),s)\Delta s \geq 1.$$

From $g(t,0) \equiv 0$ and the continuity of $g(t,u)$, we know that there exists a number $H_3 \in (0,\widehat{H}_3)$ small enough such that

$$g(t,u) \leq \frac{\widehat{H}_3}{\int_a^{\sigma(b)} G(\sigma(t),t)\Delta t}, \quad \forall (t,u) \in [a,\sigma^2(b)]_{\mathbb{T}} \times (0,H_3).$$

For every $u \in P$ and $\parallel u \parallel = H_3$, note that

$$\int_a^{\sigma(b)} G(\sigma(s),r)g(r,u(r))\Delta r \leq \int_a^{\sigma(b)} G(\sigma(s),r)\frac{\widehat{H}_3}{\int_a^{\sigma(b)} G(\sigma(r),r)\Delta r}\Delta r \leq \widehat{H}_3,$$

thus

$$\begin{aligned}
Tu(\tau) &= \int_a^{\sigma(b)} G(\tau,s) f\left(s, \int_a^{\sigma(b)} G(\sigma(s),r) g(r,u(r))\Delta r\right)\Delta s\\
&\geq \int_\xi^\omega G(\tau,s)\lambda \int_\xi^\omega G(\sigma(s),r)\lambda' u(r)\Delta r\Delta s\\
&\geq \gamma^2 \parallel u \parallel \lambda \int_\xi^\omega G(\tau,s)\lambda' \int_\xi^\omega G(\sigma(r),r)\Delta r\Delta s\\
&\geq \parallel u \parallel .
\end{aligned}$$

So, $\parallel Tu \parallel \geq \parallel u \parallel$. If we set

$$\Omega_3 = \{u \in E : \parallel u \parallel < H_3\},$$

then

$$\parallel Tu \parallel \geq \parallel u \parallel, \quad \text{for} \quad u \in P \cap \partial\Omega_3. \tag{9}$$

On the other hand, we know from $(A4)$ that there exist three positive numbers η', C_4, and C_5 such that for every $(t,u) \in [a, \sigma^2(b)]_{\mathbb{T}} \times \mathbb{R}^+$,

$$f(t,u) \leq \eta' u + C_4, \quad g(t,u) \leq \eta' u + C_5,$$

where

$$\eta' \int_a^{\sigma(b)} G(\sigma(t),t)\Delta t \leq \frac{1}{2}.$$

Thus we have

$$\begin{aligned}
Tu(t) &= \int_a^{\sigma(b)} G(t,s) f\left(s, \int_a^{\sigma(b)} G(\sigma(s),r) g(r,u(r))\Delta r\right)\Delta s\\
&\leq \int_a^{\sigma(b)} G(\sigma(s),s)\left[\eta' \int_a^{\sigma(b)} G(\sigma(s),r) g(r,u(r))\Delta r + C_4\right]\Delta s\\
&\leq \eta' \int_a^{\sigma(b)} G(\sigma(s),s)\Delta s \int_a^{\sigma(b)} G(\sigma(r),r)[\eta' u(r) + C_5]\Delta r\Delta s\\
&\quad + C_4 \int_a^{\sigma(b)} G(\sigma(s),s)\Delta s\\
&\leq \frac{1}{4} \parallel u \parallel + C_6,
\end{aligned}$$

where

$$C_6 = C_5\eta' \int_a^{\sigma(b)} G(\sigma(s),s)\Delta s \int_a^{\sigma(b)} G(\sigma(r),r)\Delta r + C_4 \int_a^{\sigma(b)} G(\sigma(s),s)\Delta s,$$

from which it follows that $Tu(t) \leq \parallel u \parallel$ as $\parallel u \parallel \to \infty$. Let $\Omega_4 = \{u \in E : \parallel u \parallel < H_4\}$. For each $u \in P$ and $\parallel u \parallel = H_4 > 0$ large enough, we have

$$\parallel Tu \parallel \leq \parallel u \parallel, \text{ for } u \in P \cap \partial\Omega_4. \tag{10}$$

From (9), (10) and Lemma 3, we know that the operator T has a fixed point in $P \cap (\overline{\Omega}_4 \backslash \Omega_3)$. The proof is complete. $\square$

Theorem 3. *If* $(A2)$, $(A3)$ *and* $(A5)$ *are satisfied, then* (1)-(2) *have at least two distinct positive solutions* $(u_1, v_1), (u_2, v_2) \in C^2([a, \sigma^2(b)]_{\mathbb{T}}, \mathbb{R}^+) \times C^2([a, \sigma^2(b)]_{\mathbb{T}}, \mathbb{R}^+)$ *satisfying* $u_i(t) > 0$, $v_i(t) > 0$ $(i = 1, 2)$.

Proof. Note that $G(t,s) \leq \frac{[\alpha(\sigma^2(b)-a)+\beta][\gamma(\sigma^2(b)-a)+\delta]}{d} = N'$. Let $B_N = \{u \in E : \| u \| < N\}$. Then from $(A5)$, for every $u \in \partial B_N \cap P$, $t \in [a, \sigma^2(b)]_{\mathbb{T}}$, we have

$$\begin{aligned} Tu(t) &= \int_a^{\sigma(b)} G(t,s) f\left(s, \int_a^{\sigma(b)} G(\sigma(s), r) g(r, u(r)) \Delta r\right) \Delta s \\ &\leq N' \int_a^{\sigma(b)} f\left(s, \int_a^{\sigma(b)} N' g(r, N) \Delta r\right) \Delta s \\ &< N' \int_a^{\sigma(b)} \frac{N}{N'(\sigma^2(b) - a)} \leq N. \end{aligned}$$

Thus

$$\| Tu \| \leq \| u \|, \text{ for } u \in P \cap \partial B_N. \tag{11}$$

And from $(A2)$ and $(A3)$ we have

$$\| Tu \| \geq \| u \|, \text{ for } u \in P \cap \partial \Omega_2, \tag{12}$$

$$\| Tu \| \geq \| u \|, \text{ for } u \in P \cap \partial \Omega_3. \tag{13}$$

We can choose H_2, H_3 and N such that $H_3 \leq N \leq H_2$ and (11)-(13) are satisfied. From Lemma 4, T has at least two fixed points in $P \cap (\overline{\Omega}_2 \setminus B_N)$ and $P \cap (\overline{B}_N \setminus \Omega_2)$, respectively. The proof is complete. □

4. Examples

Some examples are given to illustrate our main results.

(i) Let $f(x, v) = v^2$ and $g(x, u) = u^3$, then conditions of Theorem 1 are satisfied. From Theorem 1 BVP (1)-(2) have at least one positive solution.

(ii) Let $f(x, v) = v^{1/2}$ and $g(x, u) = u^{1/2}$, then conditions of Theorem 2 are satisfied. From Theorem 2 BVP (1)-(2) have at least one positive solution.

(iii) Let $f(x, v) = \frac{v^{1/2}+v^2}{12}$, $g(x, u) = u^{1/2} + u^2$, $\alpha = \beta = \gamma = \delta = 1$ and $a = 0$, $b = 1$. Then $N' = \frac{4}{3}$. We can choose $N = 1$, then conditions of Theorem 3 are satisfied. From Theorem 3 BVP (1)-(2) have at least two positive solutions.

Acknowledgement

The authors express their gratitude to the excellent guidance of Prof. K. Rajendra Prasad and the referee for valuable comments and suggestions.

References

[1] R. P. Agarwal, D. O'regan, P. J. Y. Wong, *Positive Solutions of Differential, Difference and Integral Equations*, Kluwer Academic, Dordrecht, 1999.

[2] M. Bohner, A. Peterson, *Dynamic Equations on Time Scales: An Introduction with Applications*, Birkhäuser, Boston, 2001.

[3] K. Deimling, *Nonlinear Functional Analysis*, Springer-Verlag, Berlin, 1985.

[4] L. H. Erbe, H. Wang, *On the existence of positive solutions of ordinary differential equations*, Proc. Amer. Math. Soc. **120**(1994), 743–748.

[5] J. R. Greaf, B. Yang, *Boundary value problems for second order nonlinear ordinary differential equations*, Comm. Applied Anal. **6**(2002), 273–288.

[6] D. J. Guo, V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, Academic Press, New York, 1988.

[7] L. Hu, L. Wang, *Multiple positive solutions of boundary value problems for systems of nonlinear second-order differential equations*, J. Math. Anal. Appl. **335**(2007), 1052–1060.

[8] M. A. Krasnosel'skii, *Positive solutions of operator equations*, P. Noordhoff Ltd, Groningen, 1964.

[9] W. T. Li, H. R. Sun, *Multiple positive solutions for nonlinear dynamical systems on a measure chain*, J. Comp. Appl. Math. **162**(2004), 421–430.

[10] R. Y. Ma, *Multiple nonnegative solutions of second-order systems of boundary value problems*, Nonlinear Anal. (TMA) **42**(2000), 1003–1010.

[11] H. R. Sun, *Existence of positive solutions to second-order time scale systems*, Comp. Math. Appl. **49**(2005), 131–145.

[12] H. Wang, *On the number of positive solutions of nonlinear systems*, J. Math. Anal. Appl. **281**(2003), 287–306.

[13] Z. L. Yang, J. X. Sun, *Positive solutions of boundary value problems for system of nonlinear second order ordinary differential equations*, Acta. Math. Sinica. **47**(2004), 111–118.