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Maps for which some power is a contraction

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Abstract. It is well known that if some power of a self map of a complete metric space is a contraction, then the map has a unique fixed point. It is natural to ask whether such a map is itself a contraction with respect to some related metric on the space. We show that this is indeed so and furthermore, if the map is uniformly continuous, then the related metric is complete. Also, we give an example to show that, if the map is not continuous, then the related metric need not be complete.

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In applying the Contraction Mapping Principle to the initial value problem in a "vertical strip", one usually introduces a self map in a space of continuous functions with uniform metric and then shows that some power of the self map is a contraction [2, pp.286–287], [4, pp.139–141]. It is an idea of Bielecki [1] that one can switch from the uniform metric to another complete metric and show that the self map itself is a contraction, without ever having to take account of the fact that some power of it is a contraction under the uniform metric. Details may also be found in [3, pp.93–94].

With this background in mind, we ask whether some power of a self map being a contraction implies that there exists a related metric in the same space such that the map itself is a contraction. In this article, we show that this is indeed so and that, furthermore, if the map is uniformly continuous and the original metric complete, then the related metric is also complete. This does not by any means generalize Bielecki's idea, because the fact that some power of the self map is a contraction has to be taken into account.

Theorem 1. *Let d be a metric on a space X and $T : X \to X$ a self map such that its power T^n is a contraction with contraction constant K. Then for any λ such that $K^{1/n} < \frac{1}{\lambda} < 1$, the equality*

$$d'(x, y) = d(x, y) + \lambda d(Tx, Ty) + \cdots + \lambda^{n-1} d(T^{n-1}x, T^{n-1}y)$$

defines a metric on X such that T is a contraction with contraction constant $\frac{1}{\lambda}$.

Proof. It is trivial that d' is a metric on X. By a simple computation,

$$d'(Tx, Ty) = \tfrac{1}{\lambda}[d'(x, y) - d(x, y)] + \lambda^{n-1} d(T^n x, T^n y).$$

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Since T^n is assumed to have contraction constant K, it follows that

$$\begin{aligned} d'(Tx,Ty) &\le \tfrac{1}{\lambda}[d'(x,y) - d(x,y)] + K\lambda^{n-1}d(x,y) \\ &= \tfrac{1}{\lambda}d'(x,y) + (K - \tfrac{1}{\lambda^n})\lambda^{n-1}d(x,y) \\ &\le \tfrac{1}{\lambda}d'(x,y) \end{aligned}$$

because λ is chosen so that $K^{1/n} < \frac{1}{\lambda}$. □

Remark 1. *Under the hypotheses of Theorem 1, the infinite series*

$$\sum_{j=0}^{\infty} \lambda^j d(T^j x, T^j y)$$

converges to a sum $d''(x,y)$ *such that*

$$d'(x,y) \le d''(x,y) \le d'(x,y) + \lambda^n K d'(x,y) + \lambda^{2n} K^2 d'(x,y) + \cdots = \tfrac{1}{1-\lambda^n K} d'(x,y).$$

It therefore defines a metric d'' *equivalent to* d'*. It may be noted that, regardless of whether the hypotheses of Theorem 1 are fulfilled, whenever the series happens to converge for some* $\lambda > 1$*, it defines a metric with respect to which* T *is a contraction with contraction constant* $\frac{1}{\lambda}$*.*

Theorem 2. *If in the above Theorem,* T *is uniformly continuous and the metric* d *is complete, then the metric* d' *is also complete.*

Proof. Since $d(x,y) \le d'(x,y)$ for any $x, y \in X$, it is clear that any d'-Cauchy sequence is d-Cauchy and any d'-convergent sequence is d-convergent. We need only to prove that when T is uniformly continuous, any d-convergent sequence is d'-convergent.

Let $A = \max\{\lambda, \dots, \lambda^{n-1}\} \ge \lambda > 1$ and $\{x_p\}$ be a d-convergent sequence with limit ξ. Note that powers of T are also uniformly continuous. For any $\varepsilon > 0$, there exists $\eta > 0$ such that

$$d(x,y) < \eta \Rightarrow d(T^j x, T^j y) < \varepsilon/An \text{ for } 0 \le j \le n-1.$$

Choose a positive integer N such that $p \ge N \Rightarrow d(x_p, \xi) < \eta$. Then

$$\begin{aligned} p \ge N &\Rightarrow d(T^j x_p, T^j \xi) < \varepsilon/An, \text{ for } 0 \le j \le n-1 \\ &\Rightarrow d'(x_p, \xi) < (\varepsilon/n)[1/A + \lambda/A + \cdots + \lambda^{n-1}/A] < \varepsilon. \end{aligned}$$

Thus $\{x_p\}$ converges to ξ also with respect to the metric d'. □

Remark 2. *A slight modification of the above argument demonstrates something more that is unrelated to contractions: Given a uniformly continuous self map* T *in a space* X *with metric* d *and any* $\lambda > 0$*, the metric* d' *defined on* X *in the manner of Theorem 1 is complete whenever* d *is. All one has to do is to take* $A > \max\{1, \lambda, \dots, \lambda^{n-1}\}$.

The example below shows that unless one assumes at least pointwise continuity in Theorem 2, the conclusion can fail. The question remains whether the hypothesis of uniform continuity can be weakened to pointwise continuity in the Theorem and/or in the remark of the preceding paragraph. A related question would be whether a continuous mapping, some power of which is a contraction, must be uniformly continuous. The author does not have the answers to these questions.

Example 1. *We consider a well known discontinuous map whose square is a constant [2, p.286]. Define* $T : [0,3] \to [0,3]$ *as* $Tx = 1$ *if* $0 \leq x \leq 2$ *and* $Tx = 2$ *if* $2 < x \leq 3$*. Then* T *is discontinuous at* 2*. Besides,* $T^2 x = 1$ *for all* $x \in [0,3]$*, so that* T^2 *is a contraction with unique fixed point* 1*. Any* K *with* $0 < K < 1$ *serves as the contraction constant. Therefore Theorem 1 is applicable with any* λ *satisfying* $0 < \frac{1}{\lambda} < 1$*. This means we may take* λ *to be any number greater than 1. Then the metric* d' *of Theorem 1 is given by*

$$d'(x,y) = \begin{cases} |x-y|, & \text{if } x, y \leq 2 \text{ or } x, y > 2 \\ |x-y| + \lambda, & \text{otherwise.} \end{cases}$$

This metric on $[0,3]$ *is not complete because the sequence* $\{x_p\}$ *given by* $x_p = 2 + \frac{1}{p}$ *is* d'*-Cauchy but does not converge; no number other than 2 can be its limit, while* $d'(x_p, 2) = \frac{1}{p} + \lambda > \lambda$ *for all* p*.*

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