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On coset vertex algebras with central charge 1

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Abstract. We present a coset realization of the vertex operator algebra V_L^+ with central charge 1. We investigate the vertex operator algebra $V^+_{\mathbb{Z}\sqrt{2n}}$ (resp. $V^+_{2\mathbb{Z}\sqrt{2n+1}}$) as a vertex subalgebra of $L_{D_n^{(1)}}(\Lambda_0) \otimes L_{D_n^{(1)}}(\Lambda_0)$ (resp. $L_{B_n^{(1)}}(\Lambda_0) \otimes L_{B_n^{(1)}}(\Lambda_0)$). Our construction is based on the boson-fermion correspondence and certain conformal embeddings.
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1. Introduction

The construction and classification of rational vertex operator algebras (which correspond to rational conformal field theories) are one of the most important problems in the theory of vertex operator algebras. Representation-categories of rational vertex operator algebras are finite tensor categories (cf. [12, 23]), and characters of irreducible modules span a representation of a modular group (cf. [34]). One way of constructing new vertex operator algebras is the coset construction (cf. [22]). Let us recall the coset construction in the case of affine vertex algebras. Let $\mathfrak{g}$ be a simple Lie algebra, $\hat{\mathfrak{g}}$ the associated (untwisted) affine Lie algebra, and $L(k\Lambda_0)$ the simple vertex algebra associated to $\hat{\mathfrak{g}}$ of level $k \in \mathbb{C}$. For $k \in \mathbb{Z}_{>0}$, $L(k\Lambda_0)$ is a rational vertex operator algebra. For $k, m \in \mathbb{Z}_{>0}$, $L((k+m)\Lambda_0)$ is a subalgebra of $L(k\Lambda_0) \otimes L(m\Lambda_0)$, and one has the associated coset (or commutant) vertex operator algebra

$$\{v \in L(k\Lambda_0) \otimes L(m\Lambda_0) \mid u_n v = 0 \text{ for all } u \in L((k+m)\Lambda_0), n \geq 0\},$$

of central charge

$$\frac{k \dim \mathfrak{g}}{k+h^\vee} + \frac{m \dim \mathfrak{g}}{m+h^\vee} - \frac{(k+m)\dim \mathfrak{g}}{k+m+h^\vee},$$

with $h^\vee$ being the dual Coxeter number of $\hat{\mathfrak{g}}$. In physics literature, this coset vertex algebra is denoted by

$$\frac{\mathfrak{g}_k \times \mathfrak{g}_m}{\mathfrak{g}_{k+m}}.$$

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It is an important problem to describe generators of these vertex algebras and study their representation theory. There are no precise general results (to the best of our knowledge) about the structure of these cosets.

It is believed that these vertex algebras are finitely-generated, and that in the case $m = 1$, they are related to the W-algebras obtained from the Drinfeld-Sokolov reduction (cf. [13]). It is also expected that W-algebras constructed in this way are rational. So we believe that it is important to present some evidence for rationality of coset vertex algebras in special cases. We shall demonstrate that coset vertex algebras in some cases are isomorphic to certain rational vertex algebras constructed earlier by using completely different methods.

In this paper we shall consider the simplest case is when $k = m = 1$, i.e., the commutant of $L(2\Lambda_0)$ in $L(\Lambda_0) \otimes L(\Lambda_0)$. We consider this commutant, in the case when $\mathfrak{g}$ is a simple orthogonal Lie algebra (i.e., a simple Lie algebra of type D_n or B_n). These cases are interesting because the central charges of corresponding coset vertex operator algebras are equal to 1. These cosets have been extensively studied in physics literature. In [5], the authors calculated the characters of this coset and expressed it as a sum of irreducible Virasoro characters. Their result implies that the character of the coset coincides with the character of the vertex operator algebra V_L^+, where V_L is a vertex operator algebra associated to certain even rank one lattice L, and V_L^+ its $\mathbb{Z}_2$-orbifold. The rationality of these vertex operator algebras has been established in [2].

In this paper we propose an alternative approach for studying these cosets. We give an explicit realization of V_L^+ as a subalgebra of $L(\Lambda_0) \otimes L(\Lambda_0)$ which commutes with $L(2\Lambda_0)$. Using a version of the boson-fermion correspondence, we embed these vertex operator algebras into a vertex superalgebra $V_{\mathbb{Z}^N}$, where $N = 2n$ in the D-case and $N = 2n+1$ in the B-case. We introduce a certain order two automorphism θ of $V_{\mathbb{Z}^N}$ such that $L(\Lambda_0) \otimes L(\Lambda_0)$ is an even part of the subalgebra of θ-invariants. By using these concepts, we present a new vertex algebraic proof of the isomorphism:

$$\mathrm{Com}(L_{D_n^{(1)}}(2\Lambda_0), L_{D_n^{(1)}}(\Lambda_0) \otimes L_{D_n^{(1)}}(\Lambda_0)) \cong V^+_{\mathbb{Z}\sqrt{2n}}.$$

Another interesting feature of this case is an isomorphism of fusion algebras for vertex operator algebras $L(2\Lambda_0)$ in the D_n-case and $V^+_{\mathbb{Z}\sqrt{2n}}$ (cf. [1, 6]). The isomorphism between these fusion algebras was observed in physics literature (cf. [5, 31]). An explicit proof for this isomorphism was given in [6]. In our construction, we consider $L(2\Lambda_0)$ and $V^+_{\mathbb{Z}\sqrt{2n}}$ as $\mathbb{Z}_2$-orbifolds of vertex operator algebras $V_{A_{2n-1}}$ and $V_{\mathbb{Z}\sqrt{2n}}$ respectively, with respect to the same automorphism θ, where A_{2n-1} denotes the corresponding root lattice (for details see Section 3). Using the fusion rules for lattice vertex operator algebras (cf. [8]), one sees that vertex operator algebras $V_{A_{2n-1}}$ and $V_{\mathbb{Z}\sqrt{2n}}$ have isomorphic fusion algebras. We believe that this could help to explain the isomorphism of fusion algebras for $L(2\Lambda_0)$ and $V^+_{\mathbb{Z}\sqrt{2n}}$.

In this paper we also investigate the relation between the vertex operator superalgebra $V^+_{\mathbb{Z}\sqrt{2n+1}}$ and the vertex operator algebra $L(2\Lambda_0)$ associated to the affine Lie algebra $B_n^{(1)}$. These algebras are also realized as $\mathbb{Z}_2$-orbifolds of vertex operator algebras $V_{A_{2n}}$ and $V_{\mathbb{Z}\sqrt{2n+1}}$, with respect to the same automorphism θ (cf. Section 4). In Section 5, we consider the case $n = 1$ when the coset algebra is isomorphic to the simple Neveu-Schwarz vertex operator superalgebra with central charge 1.

2. Preliminaries

Let $V = V^{even} \oplus V^{odd}$ be a vertex superalgebra (cf. [13, 25, 27, 29]). If $V^{odd} = 0$, V is a vertex algebra (see [4, 18, 20, 28]). For a subalgebra U of V, denote by

$$\mathrm{Com}(U, V) = \{v \in V \mid u_n v = 0 \text{ for all } u \in U, n \geq 0\} \tag{1}$$

the commutant of U in V (cf. [21, 22, 28]). Then $\mathrm{Com}(U, V)$ is a subalgebra of V (also called coset vertex superalgebra).

Let L be an integral lattice with $\mathbb{Z}$–bilinear form $\langle \cdot, \cdot \rangle$. Let $\mathfrak{h} = L \otimes_{\mathbb{Z}} \mathbb{C}$ and extend the $\mathbb{Z}$–bilinear form to a $\mathbb{C}$–bilinear form on $\mathfrak{h}$. Let $\hat{\mathfrak{h}} = \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K$ be its affinization with the center K. Then the Fock space $M(1) = S(\mathfrak{h} \otimes t^{-1}\mathbb{C}[t^{-1}])$ is a simple vertex operator algebra.

Denote by V_L the associated vertex superalgebra (cf. [8, 13, 25]). As a vector space

$$V_L \cong M(1) \otimes \mathbb{C}[L], \tag{2}$$

where $\mathbb{C}[L]$ is a group algebra of L, with basis $\{e^\alpha \mid \alpha \in L\}$. Denote by L^0 the dual lattice of L. Then $V_{L+\gamma}$, for $\gamma \in L^0/L$ are all irreducible modules for V_L (cf. [8]).

Vertex (super)algebra V_L has an order 2 automorphism which is lifted from the -1 isometry of the lattice L. Denote by V_L^+ the subalgebra of invariants of that automorphism (cf. [10, 20]). If L is an even lattice, V_L and V_L^+ are vertex algebras (cf. [7, 20, 28]).

Assume that L is a rank one lattice with a $\mathbb{Z}$–bilinear form defined by $\langle \alpha, \alpha \rangle = 2\ell$ for a positive integer ℓ. Then $\mathfrak{h}$ is 1-dimensional, and V_L and V_L^+ are vertex operator algebras with central charge 1.

For an affine Lie algebra $\hat{\mathfrak{g}}$ of type $X_n^{(1)}$, denote by $L_{X_n^{(1)}}(\Lambda)$ the irreducible highest weight $\hat{\mathfrak{g}}$-module, for any weight Λ of $\hat{\mathfrak{g}}$. Denote by Λ_i, $i = 0, \ldots, n$ the fundamental weights of $\hat{\mathfrak{g}}$ (cf. [24]). Then, $L_{X_n^{(1)}}(k\Lambda_0)$ is a vertex algebra, for any $k \in \mathbb{C}$ (cf. [13, 21, 25, 29, 28]).

Let X_n be the root lattice of rank n and type $X = A$, D or E. Let P_n be the associated weight lattice. Then $V_{X_n} \cong L_{X_n^{(1)}}(\Lambda_0)$ as vertex algebras and $V_{\lambda+X_n}$, for $\lambda \in P_n/X_n$ are all irreducible modules for V_{X_n} (cf. [8, 19, 20, 25, 32]).

3. Fermionic vertex superalgebras, the boson-fermion correspondence and affine Lie algebra $D_n^{(1)}$

In this section we shall apply the fermionic construction of the affine Lie algebra $D_n^{(1)}$ and the boson fermion correspondence to the study of the coset vertex operator algebra $\mathrm{Com}(L_{D_n^{(1)}}(2\Lambda_0), L_{D_n^{(1)}}(\Lambda_0) \otimes L_{D_n^{(1)}}(\Lambda_0))$. Then we shall present a (new) vertex-algebraic proof of the fact that this coset is isomorphic to the vertex operator algebra V_L^+.

We shall first recall the basic fact on infinite-dimensional Clifford algebras and the associated vertex operator superalgebras. The Clifford algebra CL_{2n} is a complex associative algebra generated by

$$\Psi_i^{\pm}(r), \Phi_i^{\pm}(r), \quad r \in \tfrac{1}{2} + \mathbb{Z},\ 1 \le i \le n$$

and non-trivial relations

$$\{\Psi_i^{\pm}(r), \Psi_j^{\mp}(s)\} = \{\Phi_i^{\pm}(r), \Phi_j^{\mp}(s)\} = \delta_{r+s,0}\delta_{i,j}$$

where $r, s \in \frac{1}{2} + \mathbb{Z}$, $i, j \in \{1, \dots, n\}$.

Let F_{2n} be the irreducible CL_{2n}–module generated by the cyclic vector $\mathbf{1}$ such that

$$\Psi_i^{\pm}(r)\mathbf{1} = \Phi_i^{\pm}(r)\mathbf{1} = 0, \quad \text{for} \ \ r > 0,\ 1 \le i \le n.$$

Define the following fields on F_{2n}

$$\Psi_i^{\pm}(z) = \sum_{n \in \mathbb{Z}} \Psi_i^{\pm}(n + \tfrac{1}{2}) z^{-n-1}, \quad \Phi_i^{\pm}(z) = \sum_{n \in \mathbb{Z}} \Phi_i^{\pm}(n + \tfrac{1}{2}) z^{-n-1}.$$

The fields $\Psi_i^{\pm}(z)$, $\Phi_i^{\pm}(z)$, $i = 1, \dots, n$ generate on F_{2n} the unique structure of a simple vertex superalgebra (cf. [16, 25, 27, 29]).

Let F_n^{Ψ} (resp. F_n^{Φ}) be the subalgebra of F_{2n} generated by the fields $\Psi_i^{\pm}(z)$ (resp. $\Phi_i^{\pm}(z)$), $i = 1, \dots, n$.

The following result is well-known.

Theorem 1 (see [15])**.** *We have*

$$(F_n^{\Psi})^{even} \cong (F_n^{\Phi})^{even} \cong L_{D_n^{(1)}}(\Lambda_0).$$

Define the following lattice

$$R_{2n} = \mathbb{Z}x_1 + \cdots + \mathbb{Z}x_n + \mathbb{Z}y_1 + \cdots + \mathbb{Z}y_n,$$
$$\langle x_i, x_j \rangle = \langle y_i, y_j \rangle = \delta_{i,j} \quad \langle x_i, y_j \rangle = 0,$$

where $i, j \in \{1, \dots, n\}$. We set $z_{2k-1} = x_k$, $z_{2k} = y_k$, where $k \in \{1, \dots, n\}$.

Let $V_{R_{2n}}$ be the associated lattice vertex superalgebra (cf. [25]). Here we choose the 2–cocycle $\varepsilon : R_{2n} \times R_{2n} \to \{\pm 1\}$ from the definition of lattice vertex superalgebra $V_{R_{2n}}$ such that

$$\varepsilon(z_i, z_j) = \begin{cases} 1, & \text{if } i \le j \\ -1, & \text{if } i > j \end{cases} \tag{3}$$

We shall use the following (non-standard) version of the boson-fermion correspondence (see [14, 17, 25]):

Theorem 2. *There exists the vertex superalgebra isomorphism $\varphi_{2n} : F_{2n} \to V_{R_{2n}}$ such that*

$$\begin{aligned}
\Psi_k^+(-\tfrac{1}{2})\mathbf{1} &\mapsto \frac{1}{\sqrt{2}}\left(e^{x_k} + e^{-y_k}\right), \\
\Psi_k^-(-\tfrac{1}{2})\mathbf{1} &\mapsto \frac{1}{\sqrt{2}}\left(e^{-x_k} + e^{y_k}\right), \\
\Phi_k^+(-\tfrac{1}{2})\mathbf{1} &\mapsto \frac{i}{\sqrt{2}}\left(e^{x_k} - e^{-y_k}\right), \\
\Phi_k^-(-\tfrac{1}{2})\mathbf{1} &\mapsto \frac{-i}{\sqrt{2}}\left(e^{-x_k} - e^{y_k}\right),
\end{aligned}$$

for $k = 1, \dots, n$.

In what follows we shall identify $v \in F_{2n}$ with image $\varphi_{2n}(v) \in V_{R_{2n}}$. Now we consider two commuting subalgebras of $V_{R_{2n}}$:

Theorem 3. *It holds:*

(i) The subalgebra of $V_{R_{2n}}$ generated by

$$e^{\gamma} \quad \textit{and} \quad e^{-\gamma} \tag{4}$$

where

$$\gamma = x_1 + \cdots + x_n + y_1 + \cdots + y_n \tag{5}$$

is isomorphic to the rank-one lattice vertex operator algebra V_L such that $L = \mathbb{Z}\gamma$, $\langle \gamma, \gamma \rangle = 2n$.

(ii) The subalgebra of $V_{R_{2n}}$ generated by elements

$$\begin{aligned}
&e^{x_k - y_l} + e^{x_l - y_k}, \quad e^{x_k - x_l} + e^{y_l - y_k}, \quad e^{-x_k + y_l} + e^{-x_l + y_k}, \\
&\textit{for} \quad k, l \in \{1, \dots, n\}, k \neq l
\end{aligned} \tag{6}$$

is isomorphic to $L_{D_n^{(1)}}(2\Lambda_0)$.

(iii) The vertex operator algebra $V_L \otimes L_{D_n^{(1)}}(2\Lambda_0)$ is isomorphic to a subalgebra of $V_{R_{2n}}$ generated by elements (4) and (6).

Proof. Assertion (i) is obvious. Next we notice that elements

$$\Psi_k^{\pm}(-\tfrac{1}{2})\Psi_l^{\pm}(-\tfrac{1}{2})\mathbf{1} + \Phi_k^{\pm}(-\tfrac{1}{2})\Phi_l^{\pm}(-\tfrac{1}{2})\mathbf{1}, \tag{7}$$

$$\Psi_k^{\pm}(-\tfrac{1}{2})\Psi_l^{\mp}(-\tfrac{1}{2})\mathbf{1} + \Phi_k^{\pm}(-\tfrac{1}{2})\Phi_l^{\mp}(-\tfrac{1}{2})\mathbf{1}, \tag{8}$$

generate the subalgebra of F_{2n} isomorphic to $L_{D_n^{(1)}}(2\Lambda_0)$. Assertion (ii) follows from the fact that the boson-fermion correspondence $\varphi_{2n} : F_{2n} \to V_{R_{2n}}$ from Theorem 2 maps generators (7)-(8) to elements proportional to generators (6). Assertion (iii) easily follows from (i), (ii) and the fact that

$$\langle \gamma, x_k - x_l \rangle = \langle \gamma, y_k - y_l \rangle = \langle \gamma, x_k - y_l \rangle = 0,$$

for $k \neq l$. $\square$

Next we notice that the root lattice of the Lie algebra $sl(2n,\mathbb{C})$ can be realized as a sublattice of R_{2n}:

$$A_{2n-1} = \mathbb{Z}(x_1-x_2)+\cdots+\mathbb{Z}(x_{n-1}-x_n)+\mathbb{Z}(x_n-y_n)+\mathbb{Z}(y_n-y_{n-1})+\cdots+\mathbb{Z}(y_2-y_1).$$

It is clear that

$$A_{2n-1} = L^{\perp}.$$

Let $\lambda_0 = 0$ and for $1 \le i \le n$ we define

$$\begin{aligned} \lambda_i &= x_1 + \cdots + x_i, \\ \lambda_{n+i} &= x_1 + \cdots + x_n + y_n + \cdots + y_{n-i+1}. \end{aligned}$$

For $0 \le i \le 2n-1$ we have that $V_{\lambda_i + A_{2n-1}}$ is isomorphic to the level one $A_{2n-1}^{(1)}$–module $L_{A_{2n-1}^{(1)}}(\Lambda_i)$.

Then we have the following decomposition:

$$\begin{aligned} V_{R_{2n}} &= \bigoplus_{i=0}^{2n-1} V_{\lambda_i + L + A_{2n-1}} \\ &\cong \bigoplus_{i=0}^{2n-1} V_{L+\frac{i}{2n}\gamma} \otimes V_{\lambda_i + A_{2n-1}} \\ &\cong \bigoplus_{i=0}^{2n-1} V_{L+\frac{i}{2n}\gamma} \otimes L_{A_{2n-1}^{(1)}}(\Lambda_i). \end{aligned} \tag{9}$$

Remark 1. *Theorem 3 shows that the vertex operators $Y(v,z)$ where $v \in V_L$ commute with the level two action of the affine Lie algebra $D_n^{(1)}$ on $V_{R_{2n}}$. In particular, $L_{D_n^{(1)}}(2\Lambda_0) \subset V_{A_{2n-1}}$. In other words, we present a lattice realization of conformal embedding of the vertex operator algebra $L_{D_n^{(1)}}(2\Lambda_0)$ into $L_{A_{2n-1}^{(1)}}(\Lambda_0)$.*

The following proposition was proved by M. Wakimoto by using characters of integrable representations of affine Lie algebras.

Proposition 1 (see [33])**.** *We have:*

$$\begin{aligned} V_{A_{2n-1}} &\cong L_{D_n^{(1)}}(2\Lambda_0) \oplus L_{D_n^{(1)}}(2\Lambda_1), \\ V_{\lambda_1 + A_{2n-1}} &\cong L_{D_n^{(1)}}(\Lambda_0 + \Lambda_1), \\ V_{\lambda_i + A_{2n-1}} &\cong L_{D_n^{(1)}}(\Lambda_i) \quad (2 \le i \le n-2), \\ V_{\lambda_{n-1} + A_{2n-1}} &\cong L_{D_n^{(1)}}(\Lambda_{n-1} + \Lambda_n), \\ V_{\lambda_n + A_{2n-1}} &\cong L_{D_n^{(1)}}(2\Lambda_{n-1}) \oplus L_{D_n^{(1)}}(2\Lambda_n), \\ V_{\lambda_{n+1} + A_{2n-1}} &\cong L_{D_n^{(1)}}(\Lambda_{n-1} + \Lambda_n), \\ V_{\lambda_{n+i} + A_{2n-1}} &\cong L_{D_n^{(1)}}(\Lambda_{n-i}) \quad (2 \le i \le n-2), \\ V_{\lambda_{2n-1} + A_{2n-1}} &\cong L_{D_n^{(1)}}(\Lambda_0 + \Lambda_1). \end{aligned}$$

Remark 2. *Proposition 1 shows that $V_{A_{2n-1}}$ is an extension of the vertex operator algebra $L_{D_n^{(1)}}(2\Lambda_0)$ by its simple-current module $L_{D_n^{(1)}}(2\Lambda_1)$. This fact can be also directly proved by using theory of simple-current extensions of vertex operator algebras from [9] and [30]. This theory can be applied to relate the category of $L_{D_n^{(1)}}(2\Lambda_0)$–modules to the category of $V_{A_{2n-1}}$–modules.*

Let $\theta : V_{R_{2n}} \to V_{R_{2n}}$ be the automorphism of the vertex superalgebra $V_{R_{2n}}$ which is lifted from the lattice automorphism

$$x_i \mapsto -y_i, \quad y_i \mapsto -x_i \quad (1 \le i \le n).$$

Then θ is an automorphism of order two. If we have a subalgebra $U \subset V_{R_{2n}}$ which is θ-invariant, we define

$$U^0 = \{u \in U \mid \theta(u) = u\}, \quad U^1 = \{u \in U \mid \theta(u) = -u\},$$

We have:

Proposition 2. *It holds:*

(i) $(V_{R_{2n}})^0 = F_n^{\Psi} \otimes (F_n^{\Phi})^{even}$.

(ii) $(V_L)^0 = V_L^+, \quad (V_L)^1 = V_L^-$.

(iii) $(V_{A_{2n-1}})^0 = L_{D_n^{(1)}}(2\Lambda_0), \quad (V_{A_{2n-1}})^1 = L_{D_n^{(1)}}(2\Lambda_1)$.

Proof. Assertion (i) follows from formulas

$$\theta(\Psi_k^{\pm}(-\tfrac{1}{2})\mathbf{1}) = \Psi_k^{\pm}(-\tfrac{1}{2})\mathbf{1}, \quad \theta(\Phi_k^{\pm}(-\tfrac{1}{2})\mathbf{1}) = -\Phi_k^{\pm}(-\tfrac{1}{2})\mathbf{1},$$

for $k = 1, \dots, n$.

Assertion (ii) follows from the fact that

$$\theta(\alpha) = -\alpha, \quad \text{for} \quad \alpha \in L.$$

Finally, Proposition 1 implies (iii). □

Now we shall consider a subalgebra of $V_{R_{2n}}$ which is isomorphic to the tensor product $L_{D_n^{(1)}}(\Lambda_0) \otimes L_{D_n^{(1)}}(\Lambda_0)$. Clearly, $L_{D_n^{(1)}}(2\Lambda_0)$ is a subalgebra of $L_{D_n^{(1)}}(\Lambda_0) \otimes L_{D_n^{(1)}}(\Lambda_0)$.

Lemma 1. *We have*

$$(e^{\gamma} + e^{-\gamma}) \in L_{D_n^{(1)}}(\Lambda_0) \otimes L_{D_n^{(1)}}(\Lambda_0) \cong (F_n^{\Psi})^{even} \otimes (F_n^{\Phi})^{even}.$$

Proof. The assertion follows from the fact that

$$(e^{\gamma} + e^{-\gamma}) \in (V_{R_{2n}}^0)^{even} \cong (F_n^{\Psi})^{even} \otimes (F_n^{\Phi})^{even}.$$

□

Since $(e^{\gamma}+e^{-\gamma})$ generates the subalgebra of V_L isomorphic to V_L^+ (see [10]), Lemma 1 and Theorem 3 imply the following:

Theorem 4. *The vertex operator algebra $L_{D_n^{(1)}}(\Lambda_0)\otimes L_{D_n^{(1)}}(\Lambda_0)$ contains a subalgebra isomorphic to*

$$V_L^+\otimes L_{D_n^{(1)}}(2\Lambda_0).$$

The decompositions from relation (9), Proposition 1 and the fact that

$$(V_{R_{2n}}^0)^{even}\cong L_{D_n^{(1)}}(\Lambda_0)\otimes L_{D_n^{(1)}}(\Lambda_0)$$

imply the following:

Corollary 1. $\mathrm{Com}(L_{D_n^{(1)}}(2\Lambda_0), L_{D_n^{(1)}}(\Lambda_0)\otimes L_{D_n^{(1)}}(\Lambda_0))\cong V_L^+$.

Remark 3. *For a rational vertex operator algebra V, let $\mathcal{E}(V)$ denote its fusion algebra (or Verlinde algebra). Note that*

$$\mathcal{E}(V_L)\cong\mathcal{E}(L_{A_{2n-1}^{(1)}}(\Lambda_0)),\quad \mathcal{E}(V_L^+)\cong\mathcal{E}(L_{D_n^{(1)}}(2\Lambda_0)).$$

The second isomorphism was proved rigorously in [6]. We believe that the reason for the second isomorphism is in the fact that V_L^+ and $L_{D_n^{(1)}}(2\Lambda_0)$ are $\mathbb{Z}_2$-orbifolds of vertex operator algebras which have identical fusion algebras.

4. Level two $B_n^{(1)}$–modules and vertex superalgebra $V^+_{\mathbb{Z}\sqrt{2n+1}}$

In this section we study coset vertex algebras associated to the affine Lie algebra of type $B_n^{(1)}$. We will identify the coset algebra $\mathrm{Com}(L_{B_n^{(1)}}(2\Lambda_0), L_{B_n^{(1)}}(\Lambda_0)\otimes L_{B_n^{(1)}}(\Lambda_0))$. The construction and proofs are similar to those in Section 3.

We consider the Clifford algebra CL_{2n+1} generated by

$$\Psi_i^{\pm}(r),\Phi_i^{\pm}(r),\Psi_{2n+1}(r),\Phi_{2n+1}(r)\quad r\in\tfrac{1}{2}+\mathbb{Z},\ 1\le i\le n$$

and non-trivial relations

$$\{\Psi_i^{\pm}(r),\Psi_j^{\mp}(s)\}=\{\Phi_i^{\pm}(r),\Phi_j^{\mp}(s)\}=\delta_{r+s,0}\delta_{i,j},$$
$$\{\Psi_{2n+1}(r),\Psi_{2n+1}(s)\}=\{\Phi_{2n+1}(r),\Phi_{2n+1}(s)\}=\delta_{r+s,0}$$

where $r,s\in\frac{1}{2}+\mathbb{Z}$, $i,j\in\{1,\dots,n\}$.

Let F_{2n+1} be the irreducible CL_{2n+1}–module generated by the cyclic vector $\mathbf{1}$ such that

$$\Psi_i^{\pm}(r)\mathbf{1}=\Phi_i^{\pm}(r)\mathbf{1}=\Psi_{2n+1}(r)\mathbf{1}=\Phi_{2n+1}(r)\mathbf{1}=0\quad\text{for}\ \ r>0,\ 1\le i\le n.$$

Define the following fields on F_{2n+1}

$$\Psi_i^{\pm}(z)=\sum_{n\in\mathbb{Z}}\Psi_i^{\pm}(n+\tfrac{1}{2})z^{-n-1},\quad \Phi_i^{\pm}(z)=\sum_{n\in\mathbb{Z}}\Phi_i^{\pm}(n+\tfrac{1}{2})z^{-n-1},$$
$$\Psi_{2n+1}(z)=\sum_{m\in\mathbb{Z}}\Psi_{2n+1}(m+\tfrac{1}{2})z^{-m-1},\quad \Phi_{2n+1}(z)=\sum_{m\in\mathbb{Z}}\Phi_{2n+1}(m+\tfrac{1}{2})z^{-m-1}.$$

The fields $\Psi_i^{\pm}(z)$, $\Phi_i^{\pm}(z)$, $i=1,\dots,n$, $\Psi_{2n+1}(z)$, $\Phi_{2n+1}(z)$ generate on F_{2n+1} the unique structure of a simple vertex superalgebra (cf. [16], [25], [27], [29]).

Let $F^{\Psi}_{n+1/2}$ (resp. $F^{\Phi}_{n+1/2}$) be the subalgebra of F_{2n+1} generated by the fields $\Psi_i^{\pm}(z)$, $\Psi_{2n+1}(z)$ (resp. $\Phi_i^{\pm}(z)$, $\Phi_{2n+1}(z)$), $i=1,\dots,n$.

Then, we have:

Theorem 5 (see [15])**.** *We have:*

$$(F^{\Psi}_{n+1/2})^{even} \cong (F^{\Phi}_{n+1/2})^{even} \cong L_{B_n^{(1)}}(\Lambda_0).$$

Define the following lattice

$$\begin{aligned}
R_{2n+1} &= \mathbb{Z}x_1+\cdots+\mathbb{Z}x_n+\mathbb{Z}y_1+\cdots+\mathbb{Z}y_n+\mathbb{Z}x,\\
\langle x_i,x_j\rangle &= \langle y_i,y_j\rangle=\delta_{i,j} \quad \langle x_i,y_j\rangle=0,\\
\langle x,x_i\rangle &= \langle x,y_i\rangle=0, \quad \langle x,x\rangle=1
\end{aligned} \tag{10}$$

where $i,j\in\{1,\dots,n\}$. We set $z_{2k-1}=x_k,\ z_{2k}=y_k, z_{2n+1}=x$, where $1\le k\le n$.

Let $V_{R_{2n+1}}$ be the associated lattice vertex superalgebra (cf. [25]). As before, we can choose the 2–cocycle $\varepsilon : R_{2n+1}\times R_{2n+1}\to\{\pm1\}$ in the definition of lattice vertex superalgebra $V_{R_{2n+1}}$ such that relation (3) holds.

As in Theorem 2, we have the following version of the boson-fermion correspondence:

Theorem 6. *There exists the vertex superalgebra isomorphism $\varphi_{2n+1} : F_{2n+1}\to V_{R_{2n+1}}$ such that*

$$\begin{aligned}
\Psi_k^+(-\tfrac12)\mathbf{1} &\mapsto \frac{1}{\sqrt2}\left(e^{x_k}+e^{-y_k}\right),\\
\Psi_k^-(-\tfrac12)\mathbf{1} &\mapsto \frac{1}{\sqrt2}\left(e^{-x_k}+e^{y_k}\right),\\
\Phi_k^+(-\tfrac12)\mathbf{1} &\mapsto \frac{i}{\sqrt2}\left(e^{x_k}-e^{-y_k}\right),\\
\Phi_k^-(-\tfrac12)\mathbf{1} &\mapsto \frac{-i}{\sqrt2}\left(e^{-x_k}-e^{y_k}\right),\\
\Psi_{2n+1}(-\tfrac12)\mathbf{1} &\mapsto \frac{1}{\sqrt2}\left(e^{x}+e^{-x}\right),\\
\Phi_{2n+1}(-\tfrac12)\mathbf{1} &\mapsto \frac{i}{\sqrt2}\left(e^{x}-e^{-x}\right),
\end{aligned}$$

for $k=1,\dots,n$.

In what follows we shall identify $v\in F_{2n+1}$ with image $\varphi_{2n+1}(v)\in V_{R_{2n+1}}$. As in Theorem 3, we consider two commuting subalgebras of $V_{R_{2n+1}}$:

Theorem 7. *It holds:*

(i) The subalgebra of $V_{R_{2n+1}}$ generated by

$$e^{\gamma} \quad \textit{and} \quad e^{-\gamma} \tag{11}$$

where

$$\gamma = x_1 + \cdots + x_n + y_1 + \cdots + y_n + x \tag{12}$$

is isomorphic to the rank-one lattice vertex operator superalgebra $V_{L'}$ such that $L' = \mathbb{Z}\gamma$, $\langle \gamma, \gamma \rangle = 2n + 1$.

(ii) The subalgebra of $V_{R_{2n+1}}$ generated by elements

$$\begin{aligned} &e^{x_k - y_l} + e^{x_l - y_k}, \quad e^{x_k - x_l} + e^{y_l - y_k}, \quad e^{-x_k + y_l} + e^{-x_l + y_k}, \\ &e^{x - y_k} + e^{x_k - x}, \quad e^{x - x_k} + e^{y_k - x}, \end{aligned} \tag{13}$$

for $k, l \in \{1, \ldots, n\}, k \neq l$, is isomorphic to $L_{B_n^{(1)}}(2\Lambda_0)$.

(iii) The vertex operator superalgebra $V_{L'} \otimes L_{B_n^{(1)}}(2\Lambda_0)$ is isomorphic to a subalgebra of $V_{R_{2n+1}}$ generated by elements (11) and (13).

Proof. Assertion (i) is obvious. Next we notice that elements

$$\Psi_k^{\pm}(-\tfrac{1}{2})\Psi_l^{\pm}(-\tfrac{1}{2})\mathbf{1} + \Phi_k^{\pm}(-\tfrac{1}{2})\Phi_l^{\pm}(-\tfrac{1}{2})\mathbf{1}, \tag{14}$$
$$\Psi_k^{\pm}(-\tfrac{1}{2})\Psi_l^{\mp}(-\tfrac{1}{2})\mathbf{1} + \Phi_k^{\pm}(-\tfrac{1}{2})\Phi_l^{\mp}(-\tfrac{1}{2})\mathbf{1}, \tag{15}$$
$$\Psi_k^{\pm}(-\tfrac{1}{2})\Psi_{2n+1}(-\tfrac{1}{2})\mathbf{1} + \Phi_k^{\pm}(-\tfrac{1}{2})\Phi_{2n+1}(-\tfrac{1}{2})\mathbf{1}, \tag{16}$$

generate the subalgebra of F_{2n+1} isomorphic to $L_{B_n^{(1)}}(2\Lambda_0)$. Assertion (ii) follows from the fact that the boson-fermion correspondence $\varphi_{2n+1} : F_{2n+1} \to V_{R_{2n+1}}$ maps generators (14)-(16) to elements proportional to generators (13). Assertion (iii) easily follows from (i), (ii) and the fact that

$$\langle \gamma, x_k - x_l \rangle = \langle \gamma, y_k - y_l \rangle = \langle \gamma, x_k - y_l \rangle = \langle \gamma, x - y_k \rangle = \langle \gamma, x_k - x \rangle = 0,$$

for $k \neq l$. □

Next we notice that the root lattice of the Lie algebra $sl(2n+1, \mathbb{C})$ can be realized as a sublattice of R_{2n+1}:

$$\begin{aligned} A_{2n} = &\ \mathbb{Z}(x_1 - x_2) + \cdots + \mathbb{Z}(x_{n-1} - x_n) + \mathbb{Z}(x_n - x) + \mathbb{Z}(x - y_n) + \mathbb{Z}(y_n - y_{n-1}) \\ &+ \cdots + \mathbb{Z}(y_2 - y_1). \end{aligned}$$

It is clear that

$$A_{2n} = L'^{\perp}.$$

Let $\lambda_0 = 0$ and for $1 \leq i \leq n$, we define

$$\begin{aligned} \lambda_i &= x_1 + \cdots + x_i, \\ \lambda_{n+i} &= x_1 + \cdots + x_n + x + y_n + \cdots + y_{n-i+2}. \end{aligned}$$

For $0 \le i \le 2n$ we have that $V_{\lambda_i + A_{2n}}$ is isomorphic to the level one $A_{2n}^{(1)}$–module $L_{A_{2n}^{(1)}}(\Lambda_i)$.

Then we have the following decomposition

$$
\begin{aligned}
V_{R_{2n+1}} &= \bigoplus_{i=0}^{2n} V_{\lambda_i + L' + A_{2n}} \\
&\cong \bigoplus_{i=0}^{2n} V_{L' + \frac{i}{2n+1}\gamma} \otimes V_{\lambda_i + A_{2n}} \\
&\cong \bigoplus_{i=0}^{2n} V_{L' + \frac{i}{2n+1}\gamma} \otimes L_{A_{2n}^{(1)}}(\Lambda_i). \qquad (17)
\end{aligned}
$$

Remark 4. *Theorem 7 provides a lattice realization of the conformal embedding of the vertex operator algebra $L_{B_n^{(1)}}(2\Lambda_0)$ into $L_{A_{2n}^{(1)}}(\Lambda_0)$.*

Proposition 3 (see [33])**.** *We have:*

$$
\begin{aligned}
V_{A_{2n}} &\cong L_{B_n^{(1)}}(2\Lambda_0) \oplus L_{B_n^{(1)}}(2\Lambda_1), \\
V_{\lambda_1 + A_{2n}} &\cong L_{B_n^{(1)}}(\Lambda_0 + \Lambda_1), \\
V_{\lambda_i + A_{2n}} &\cong L_{B_n^{(1)}}(\Lambda_i) \quad (2 \le i \le n-1), \\
V_{\lambda_n + A_{2n}} &\cong L_{B_n^{(1)}}(2\Lambda_n), \\
V_{\lambda_{n+1} + A_{2n}} &\cong L_{B_n^{(1)}}(2\Lambda_n), \\
V_{\lambda_{n+i} + A_{2n}} &\cong L_{B_n^{(1)}}(\Lambda_{n+1-i}) \quad (2 \le i \le n-1), \\
V_{\lambda_{2n} + A_{2n}} &\cong L_{B_n^{(1)}}(\Lambda_0 + \Lambda_1).
\end{aligned}
$$

Remark 5. *As in the case of affine Lie algebra $D_n^{(1)}$, the previous proposition was proved by M. Wakimoto by using characters. On the other hand, this result can be also proved by using the theory of simple current extension of vertex operator algebras. Since $V_{A_{2n}}$ is a simple current extension of $L_{B_n^{(1)}}(2\Lambda_0)$, we have that every integrable $B_n^{(1)}$–module of level two can be constructed from twisted or untwisted $V_{A_{2n}}$–modules.*

Let $\theta : V_{R_{2n+1}} \to V_{R_{2n+1}}$ be the automorphism of the vertex superalgebra $V_{R_{2n+1}}$ which is lifted from the lattice automorphism

$$x_i \mapsto -y_i, \quad y_i \mapsto -x_i \quad (1 \le i \le n), \quad x \mapsto -x.$$

Then θ is an automorphism of order two. If we have a subalgebra $U \subset V_{R_{2n+1}}$ which is θ-invariant, we define

$$U^0 = \{u \in U \mid \theta(u) = u\}, \quad U^1 = \{u \in U \mid \theta(u) = -u\},$$

We have:

Proposition 4. *It holds:*

(i) $(V_{R_{2n+1}})^0 = F^{\Psi}_{n+1/2} \otimes (F^{\Phi}_{n+1/2})^{even}$.

(ii) $(V_{L'})^0 = V^+_{L'}, \quad (V_{L'})^1 = V^-_{L'}$.

(iii) $(V_{A_{2n}})^0 = L_{B_n^{(1)}}(2\Lambda_0), \quad (V_{A_{2n}})^1 = L_{B_n^{(1)}}(2\Lambda_1)$.

Proof. Assertion (i) follows from formulas

$$\theta(\Psi^{\pm}_k(-\tfrac{1}{2})\mathbf{1}) = \Psi^{\pm}_k(-\tfrac{1}{2})\mathbf{1}, \quad \theta(\Phi^{\pm}_k(-\tfrac{1}{2})\mathbf{1}) = -\Phi^{\pm}_k(-\tfrac{1}{2})\mathbf{1},$$

for $k = 1, \ldots, n$ and

$$\theta(\Psi_{2n+1}(-\tfrac{1}{2})\mathbf{1}) = \Psi_{2n+1}(-\tfrac{1}{2})\mathbf{1}, \quad \theta(\Phi_{2n+1}(-\tfrac{1}{2})\mathbf{1}) = -\Phi_{2n+1}(-\tfrac{1}{2})\mathbf{1}.$$

Assertion (ii) follows from the fact that

$$\theta(\alpha) = -\alpha, \text{ for } \alpha \in L'.$$

Finally, Proposition 3 implies (iii). $\square$

Now we shall consider a subalgebra of $V_{R_{2n+1}}$ which is isomorphic to the tensor product $L_{B_n^{(1)}}(\Lambda_0) \otimes L_{B_n^{(1)}}(\Lambda_0)$. Clearly, $L_{B_n^{(1)}}(2\Lambda_0)$ is a subalgebra of $L_{B_n^{(1)}}(\Lambda_0) \otimes L_{B_n^{(1)}}(\Lambda_0)$.

Lemma 2. *We have:*

(i) $(e^{\gamma} + e^{-\gamma}) \in V^0_{R_{2n+1}}$.

(ii) $(e^{2\gamma} + e^{-2\gamma}) \in L_{B_n^{(1)}}(\Lambda_0) \otimes L_{B_n^{(1)}}(\Lambda_0) \cong (F^{\Psi}_{n+1/2})^{even} \otimes (F^{\Phi}_{n+1/2})^{even}$.

Proof. Assertion (i) follows from the fact that $(e^{\gamma} + e^{-\gamma})$ is θ-invariant. Since $(V^0_{R_{2n+1}})^{even} \cong L_{B_n^{(1)}}(\Lambda_0) \otimes L_{B_n^{(1)}}(\Lambda_0)$ and $e^{2\gamma} + e^{-2\gamma}$ is an even vector, we get assertion (ii). $\square$

Since $(e^{\gamma} + e^{-\gamma})$ generates the subalgebra of $V_{L'}$ isomorphic to $V^+_{L'}$, Lemma 2 and Theorem 3 imply the following:

Theorem 8. *It holds:*

(i) The vertex operator superalgebra $V^0_{R_{2n+1}}$ contains a subalgebra isomorphic to

$$V^+_{L'} \otimes L_{B_n^{(1)}}(2\Lambda_0).$$

(ii) The vertex operator algebra $L_{B_n^{(1)}}(\Lambda_0) \otimes L_{B_n^{(1)}}(\Lambda_0)$ contains a subalgebra isomorphic to

$$V^+_{2L'} \otimes L_{B_n^{(1)}}(2\Lambda_0).$$

The decomposition from relation (17), Proposition 3 and the fact that

$$(V^0_{R_{2n+1}})^{even} \cong L_{B_n^{(1)}}(\Lambda_0) \otimes L_{B_n^{(1)}}(\Lambda_0)$$

imply the following:

Corollary 2. *We have:*

(i) $\mathrm{Com}(L_{B_n^{(1)}}(2\Lambda_0), V^0_{R_{2n+1}}) \cong V^+_{L'}$.

(ii) $\mathrm{Com}(L_{B_n^{(1)}}(2\Lambda_0), L_{B_n^{(1)}}(\Lambda_0) \otimes L_{B_n^{(1)}}(\Lambda_0)) \cong V^+_{2L'}$.

Remark 6. *It is natural to investigate fusion algebras* $\mathcal{E}(L_{B_n^{(1)}}(2\Lambda_0))$ *and* $\mathcal{E}(V^+_{L'})$. *It this case, fusion algebras are not isomorphic.*

5. Level four $A_1^{(1)}$–modules and vertex superalgebra $L^{ns}(1,0)$

In this section we shall extend the results from Section 4 to the case $n = 1$. Let $L^{ns}(c,h)$ denote the irreducible highest weight module for the Neveu-Schwarz algebra with central charge c and highest weight h (cf. [3]). When $n = 1$, the coset vertex superalgebra from Section 4 is in fact a minimal Neveu-Schwarz vertex superalgebra $L^{ns}(1,0)$.

The following tensor product can be decomposed as a module for $L^{ns}(1,0) \otimes L_{A_1^{(1)}}(4\Lambda_0)$ (cf. [3, 26]):

$$\begin{aligned} L_{A_1^{(1)}}(2\Lambda_0) \otimes (L_{A_1^{(1)}}(2\Lambda_0) \oplus L_{A_1^{(1)}}(2\Lambda_1)) = {} & L^{ns}(1,0) \otimes L_{A_1^{(1)}}(4\Lambda_0) \oplus L^{ns}(1,1/6) \\ & \otimes L_{A_1^{(1)}}(2\Lambda_0 + 2\Lambda_1) \oplus \quad L^{ns}(1,1) \\ & \otimes L_{A_1^{(1)}}(4\Lambda_1). \end{aligned} \tag{18}$$

Proposition 5. *It holds:*

(1)

$$F^{\Psi}_{3/2} \cong F^{\Phi}_{3/2} = L_{A_1^{(1)}}(2\Lambda_0) \oplus L_{A_1^{(1)}}(2\Lambda_1).$$

(2)

$$(F^{\Psi}_{3/2})^{even} \cong (F^{\Phi}_{3/2})^{even} \cong L_{A_1^{(1)}}(2\Lambda_0); \quad (F^{\Psi}_{3/2})^{odd} \cong (F^{\Phi}_{3/2})^{odd} \cong L_{A_1^{(1)}}(2\Lambda_1).$$

(3)

$$V^0_{R_3} \cong L_{A_1^{(1)}}(2\Lambda_0) \otimes (L_{A_1^{(1)}}(2\Lambda_0) \oplus L_{A_1^{(1)}}(2\Lambda_1)). \tag{19}$$

Proof. Assertions (1) and (2) are well-known (cf. [22, 27, 3]). Applying the same construction as in Section 4, we get (3). □

By using decomposition (17) in the case $n = 1$, the conformal embedding of $L_{A_1^{(1)}}(4\Lambda_0)$ into $V_{A_2} \cong L_{A_2^{(1)}}(\Lambda_0)$ (cf. [11, 33]) we get:

Proposition 6. *It holds:*

(1) $V^0_{A_2} \cong L_{A_1^{(1)}}(4\Lambda_0), \quad V^1_{A_2} \cong L_{A_1^{(1)}}(4\Lambda_1)$,

(2) $V_{\lambda_1+A_2} \cong V_{\lambda_2+A_2} \cong L_{A_1^{(1)}}(2\Lambda_0+2\Lambda_1)$.

By combining Propositions 5 and 6, and decomposition (19) we obtain:

Corollary 3. *We have:*

(i) $\mathrm{Com}(L_{A_1^{(1)}}(4\Lambda_0), V^0_{R_3}) \cong V^+_{L'} \cong L^{ns}(1,0)$.

(ii) $\mathrm{Com}(L_{A_1^{(1)}}(4\Lambda_0), L_{A_1^{(1)}}(2\Lambda_0) \otimes L_{A_1^{(1)}}(2\Lambda_0)) \cong V^+_{2L'} \cong L^{ns}(1,0)^{even}$.

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